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ON GRÜSS TYPE INTEGRAL INEQUALITIES

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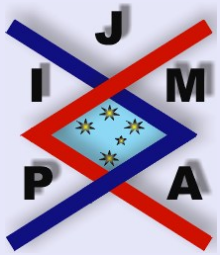

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Abstract

In this paper we establish two new integral inequalities similar to that of the Grüss inequality by using a fairly elementary analysis.

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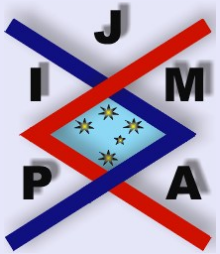
Key words: Grüss type, Integral Inequalities, Absolutely Continuous, Integral Identity.

Contents



1. Introduction

In 1935 (see [4, p. 296]), G. Grüss proved the following integral inequality which gives an estimation for the integral of a product in terms of the product of integrals:

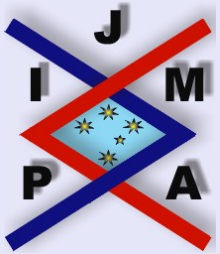
$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(M-m)(N-n),$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

for all $x \in [a,b]$, where m, M, n, N are given real constants.

A great deal of attention has been given to the above inequality and many papers dealing with various generalizations, extensions and variants have appeared in the literature, see [1] – [6] and the references cited therein. The main purpose of the present paper is to establish two new integral inequalities similar to that of the Grüss inequality involving functions and their higher order derivatives. The analysis used in the proof is elementary and our results provide new estimates on inequalities of this type.



2. Statement of Results

In this section, we state our results to be proved in this paper. In what follows, we denote by $\mathbb{R}$, the set of real numbers and $[a,b] \subset \mathbb{R}$, $a < b$.

Our main results are given in the following theorems.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be functions such that $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)}, g^{(n)} \in L_\infty [a,b]$. Then*

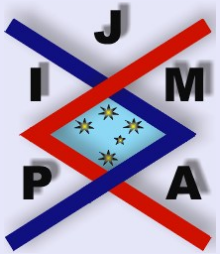
(2.1)
$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) - \frac{1}{2(b-a)^2}\int_a^b \left[\left(\sum_{k=1}^{n-1} F_k(x)\right) g(x) + \left(\sum_{k=1}^{n-1} G_k(x)\right) f(x)\right] dx \right| \leq \frac{1}{2(b-a)^2}\int_a^b \left(|g(x)|\left\|f^{(n)}\right\|_\infty + |f(x)|\left\|g^{(n)}\right\|_\infty\right) A_n(x)\,dx,$$

where

$$F_k(x) = \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x), \tag{2.2}$$

$$G_k(x) = \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] g^{(k)}(x), \tag{2.3}$$

$$A_n(x) = \int_a^b |K_n(x,t)|\,dt, \tag{2.4}$$



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in which $K_n : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K_n(x,t) = \begin{cases} \dfrac{(t-a)^n}{n!} & \text{if} \quad t \in [a,x] \\ \dfrac{(t-b)^n}{n!} & \text{if} \quad t \in (x,b] \end{cases} \tag{2.5}$$

and

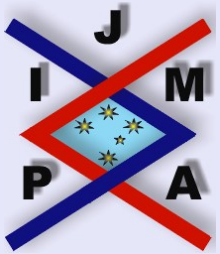
$$\begin{aligned} \left\| f^{(n)} \right\|_\infty &= \sup_{t\in[a,b]} \left| f^{(n)}(t) \right| < \infty, \\ \left\| g^{(n)} \right\|_\infty &= \sup_{t\in[a,b]} \left| g^{(n)}(t) \right| < \infty, \end{aligned}$$

for $x \in [a,b]$ *and* $n \geq 1$ *a natural number.*

Theorem 2.2. *Let* $p, q : [a,b] \to \mathbb{R}$ *be functions such that* $p^{(n-1)}, q^{(n-1)}$ *are absolutely continuous on* $[a,b]$ *and* $p^{(n)}, q^{(n)} \in L_\infty[a,b]$. *Then*

(2.6)

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b p(x)\,q(x)\,dx - n\left(\frac{1}{b-a}\int_a^b p(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b q(x)\,dx\right) \right. \\ &\quad \left. + \frac{1}{2(b-a)}\int_a^b \left[\left(\sum_{k=1}^{n-1} P_k(x)\right) q(x) + \left(\sum_{k=1}^{n-1} Q_k(x)\right) p(x)\right] dx \right| \\ &\leq \frac{1}{2(n-1)!\,(b-a)^2}\int_a^b \left(|q(x)| \left\|p^{(n)}\right\|_\infty + |p(x)| \left\|q^{(n)}\right\|_\infty \right) B_n(x)\,dx, \end{aligned}$$



where

$$(2.7)\quad P_k(x) = \frac{(n-k)}{k!}\cdot\frac{p^{(k-1)}(a)(x-a)^k - p^{(k-1)}(b)(x-b)^k}{b-a},$$

$$(2.8)\quad Q_k(x) = \frac{(n-k)}{k!}\cdot\frac{q^{(k-1)}(a)(x-a)^k - q^{(k-1)}(b)(x-b)^k}{b-a},$$

$$(2.9)\quad B_n(x) = \int_a^b \left|(x-t)^{n-1} r(t,x)\right| dt,$$

in which

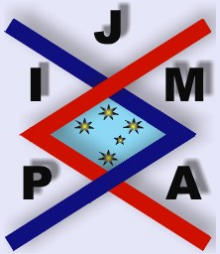
$$r(t,x) = \begin{cases} t-a, & \text{if } a \le t \le x \le b, \\ t-b, & \text{if } a \le x < t \le b, \end{cases}$$

and

$$\left\|p^{(n)}\right\|_\infty = \sup_{t\in[a,b]}\left|p^{(n)}(t)\right| < \infty,$$

$$\left\|q^{(n)}\right\|_\infty = \sup_{t\in[a,b]}\left|q^{(n)}(t)\right| < \infty,$$

for $x \in [a,b]$ and $n \ge 1$ is a natural number.



3. Proof of Theorem 2.1

From the hypotheses on f, we have the following integral identity (see [1, p. 52]):

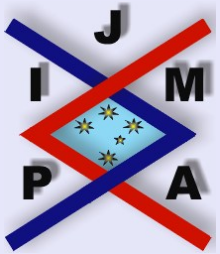
$$(3.1)\quad \int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt,$$

for $x \in [a,b]$. In [1] the identity (3.1) is proved by mathematical induction. For a different proof, see [6]. The identity (3.1) can be rewritten as

$$(3.2)\quad f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{b-a}\sum_{k=1}^{n-1} F_k(x) - \frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\, f^{(n)}(t)\,dt.$$

Similarly, from the hypotheses on g we have the identity

$$(3.3)\quad g(x) = \frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\sum_{k=1}^{n-1} G_k(x) - \frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\, g^{(n)}(t)\,dt.$$

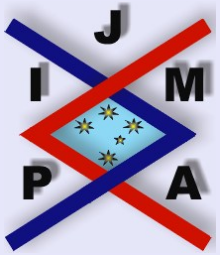


Multiplying (3.2) by $g(x)$ and (3.3) by $f(x)$ and summing the resulting identities and integrating from a to b and rewriting we have

$$
\begin{aligned}
(3.4)\quad \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &= \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \\
&\quad - \frac{1}{2(b-a)^2}\int_a^b \left[\left(\sum_{k=1}^{n-1} F_k(x)\right) g(x) + \left(\sum_{k=1}^{n-1} G_k(x)\right) f(x)\right] dx \\
&\quad - \frac{1}{2(b-a)^2}\left[\int_a^b g(x)\left\{\frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt\right\} dx\right. \\
&\quad \left. + \int_a^b f(x)\left\{\frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\,g^{(n)}(t)\,dt\right\} dx\right].
\end{aligned}
$$

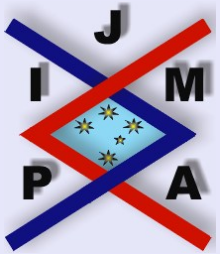
From (3.4) we observe that

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right. \\
&\quad \left. - \frac{1}{2(b-a)^2}\int_a^b \left[\left(\sum_{k=1}^{n-1} F_k(x)\right) g(x) + \left(\sum_{k=1}^{n-1} G_k(x)\right) f(x)\right] dx\right| \\
&\leq \frac{1}{2(b-a)^2}\int_a^b \left(|g(x)|\left(\int_a^b |K_n(x,t)|\left|f^{(n)}(t)\right| dt\right)\right. \\
&\quad \left. + |f(x)|\left(\int_a^b |K_n(x,t)|\left|g^{(n)}(t)\right| dt\right)\right) dx
\end{aligned}
$$



$$\leq \frac{1}{2(b-a)^2} \int_a^b \left(|g(x)| \left\| f^{(n)} \right\|_\infty + |f(x)| \left\| g^{(n)} \right\|_\infty \right) A_n(x),$$

which is the required inequality in (2.1). The proof is complete.



4. Proof of Theorem 2.2

From the hypotheses on p we have the following integral identity (see [2, p. 291]):

$$(4.1)\quad \frac{1}{n}\left(p(x)+\sum_{k=1}^{n-1}P_k(x)\right)-\frac{1}{b-a}\int_a^b p(y)\,dy = \frac{1}{n!\,(b-a)}\int_a^b (x-t)^{n-1}\,r(t,x)\,p^{(n)}(t)\,dt,$$

for $x\in[a,b]$. The identity (4.1) can be rewritten as

$$(4.2)\quad p(x)=\frac{n}{b-a}\int_a^b p(x)\,dx-\sum_{k=1}^{n-1}P_k(x) + \frac{1}{(n-1)!\,(b-a)}\int_a^b (x-t)^{n-1}\,r(t,x)\,p^{(n)}(t)\,dt.$$

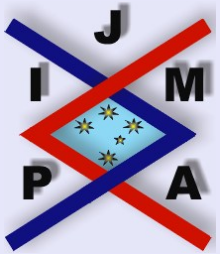
Similarly, from the hypotheses on q we have the identity

$$(4.3)\quad q(x)=\frac{n}{b-a}\int_a^b q(x)\,dx-\sum_{k=1}^{n-1}Q_k(x) + \frac{1}{(n-1)!\,(b-a)}\int_a^b (x-t)^{n-1}\,r(t,x)\,q^{(n)}(t)\,dt.$$

Multiplying (4.2) by $q(x)$ and (4.3) by $p(x)$ and summing the resulting identities and integrating from a to b and rewriting we have

(4.4)
$$\begin{aligned}\frac{1}{b-a}\int_a^b p(x)\,q(x)\,dx &= n\left(\frac{1}{b-a}\int_a^b p(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b q(x)\,dx\right)\\ &\quad-\frac{1}{2(b-a)}\int_a^b\left[\left(\sum_{k=1}^{n-1}P_k(x)\right)q(x)+\left(\sum_{k=1}^{n-1}Q_k(x)\right)p(x)\right]dx\\ &\quad+\frac{1}{2(n-1)!(b-a)^2}\left[\int_a^b q(x)\left\{\int_a^b (x-t)^{n-1}r(t,x)\,p^{(n)}(t)\,dt\right\}dx\right.\\ &\quad\left.+\int_a^b p(x)\left\{\int_a^b (x-t)^{n-1}r(t,x)\,q^{(n)}(t)\,dt\right\}dx\right].\end{aligned}$$

From (4.4) and following the similar arguments as in the last part of the proof of Theorem 2.1, we get the desired inequality in (2.6). The proof is complete.



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