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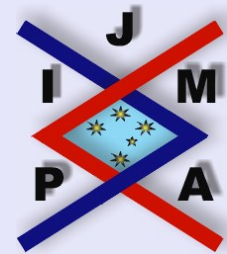
## DIFFERENTIAL ESTIMATE FOR $n$ -ARY FORMS ON CLOSED ORTHANTS

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## Abstract

We prove a differential inequality for real forms of arbitrary degree, the problem being considered on closed orthants  $H \subset \mathbb{R}^n$ . A sufficient positivity criterion is derived. Our results allow computer implementation and contain enough information to imply the fundamental  $AM - GM$  inequality.

*2000 Mathematics Subject Classification:* 26D15, 26D05.

*Key words:* Homogeneous polynomial, Symmetric polynomial, Estimate, Positivity.

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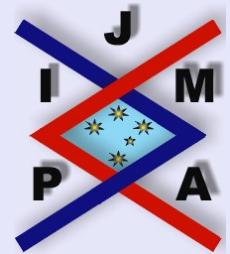
# 1. Introduction and Notations

Positivity criteria for real  $n$ -ary  $d$ -forms ( $d$ -homogeneous polynomials  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ) are of practical significance, but effective results exist for lower degree only.

- For quadratic forms, Sylvester's criterion characterizes strict positivity on  $\mathbb{R}^n \setminus \{0_n\}$ .
- For symmetric cubics ( $d = 3$ ), positivity on the first orthant  $\mathbb{R}_+^n$  is reduced in [2] to a finite number of tests (see Theorem 1.1 below), which are the same for all cubics.
- For symmetric quartics ( $d = 4$ ) on  $\mathbb{R}_+^n$  or  $\mathbb{R}^n$ , and for symmetric quintics ( $d = 5$ ) on  $\mathbb{R}_+^n$ , positivity is expressed in [11] in terms of finite test-sets depending on the symmetric form. For quartics, explicit discriminants and effective related algorithms (Maple worksheets) are derived in [12].

All mentioned results provide equivalent conditions and allow computer implementation. For arbitrary degree, it is of interest to find "reasonable" sufficient conditions for positivity on closed orthants in  $\mathbb{R}^n$ . In our entire discussion we require that  $n \in \mathbb{N}$ ,  $n \geq 2$ .

For every  $k \in \{1, \dots, n\}$ , write  $0_k := (0, \dots, 0) \in \mathbb{R}^k$ ,  $1_k := (1, \dots, 1) \in \mathbb{R}^k$ , and set  $\epsilon_k := (1_k, 0_{n-k}) \in \mathbb{R}^n$ ,  $\bar{\epsilon}_k := k^{-1}\epsilon_k$ . For  $x \in \mathbb{R}^n$ , it is convenient to write  $x_k$  for its  $k$ th component. Therefore, we avoid denoting vectors with symbols with lower indexes (upper indexes will be allowed) and  $0_k$ ,  $1_k$ ,  $\epsilon_k$ ,  $\bar{\epsilon}_k$  are *the only exceptions* to this rule.



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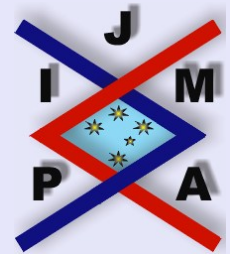
For every  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , set

$$\text{supp}(x) := \{j \in \{1, \dots, n\} \mid x_j \neq 0\}, \quad \|x\| := \sum_{j=1}^n |x_j|.$$

We need the following theorem, which is known in the context of even symmetric sextics (for the original statement see [2, Th. 3.7, p. 567]).

**Theorem 1.1.** *Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a symmetric cubic. Then*

$$g \geq 0 \text{ on } \mathbb{R}_+^n \iff g(\epsilon_k) \geq 0 \text{ for every } k \in \{1, \dots, n\}.$$



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## 2. Main Results

Let us consider an arbitrary  $d$ -form  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $d \in \mathbb{N}^*$ .

For ease of exposition, let us define the order relation “ $\ll$ ” on  $\mathbb{R}^n$  by

$$u \ll v \stackrel{\text{def.}}{\iff} u_j \in \{0, v_j\} \text{ for every } j \in \{1, \dots, n\}$$

$$\iff u_j = v_j \text{ for every } j \in \text{supp}(u).$$

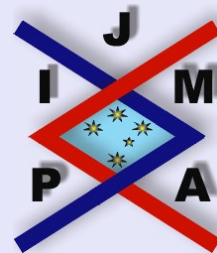
**Remark 2.1.** *The following properties of “ $\ll$ ” are immediate.*

- 1)  $0_n \ll u$  for every  $u \in \mathbb{R}^n$ .
- 2)  $u \ll 1_n \iff u \in \{0, 1\}^n$ .
- 3) For every  $u \in \mathbb{R}^n$ , the set  $\{x \in \mathbb{R}^n \mid x \ll u\}$  is finite.
- 4) If  $H$  is a closed orthant in  $\mathbb{R}^n$  and if  $u \ll v \in H$ , then  $u \in H$ .
- 5) If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a diagonal linear isomorphism or a permutation of coordinates or a composition of finitely many such operators, then

$$u \ll v \iff Tu \ll Tv.$$

The following result provides a lower estimate<sup>1</sup> for  $f$  in terms of its  $d$ -th differential  $f^{(d)}$ . As we shall see, its symmetric variant (Theorem 2.2) contains enough information to imply the fundamental  $AM - GM$  inequality.

<sup>1</sup>Replacing  $f$  by  $-f$  leads to the corresponding upper estimate.



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**Theorem 2.1.** Let  $H \subset \mathbb{R}^n$  be a closed orthant. There exist  $u^1, \dots, u^d \in \{-1, 0, 1\}^n \cap H$ , such that  $0_n \neq u^1 \ll \dots \ll u^d$  and

$$(2.1) \quad f(x) \geq \frac{\|x\|^d}{d!} \cdot \frac{f^{(d)}(0_n)(u^1, \dots, u^d)}{\|u^1\| \dots \|u^d\|} \text{ for every } x \in H.$$

In particular, if  $f^{(d)}$  satisfies the inequality

$$(2.2) \quad f^{(d)}(0_n)(u^1, \dots, u^d) \geq 0$$

for all  $0_n \neq u^1 \ll \dots \ll u^d \in \{-1, 0, 1\}^n \cap H$ , then  $f \geq 0$  on  $H$ .

**Theorem 2.2.** Assume  $f$  to be symmetric, with  $d \geq 4$ . Then there exist in  $\{1, \dots, n\}$  integers  $k_1 \geq \dots \geq k_{d-3} \geq k$ , such that

$$(2.3) \quad f(x) \geq \frac{6\|x\|^d}{d!} f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}})$$

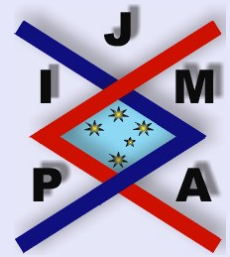
$$(2.4) \quad = \frac{\|x\|^d}{d!} f^{(d)}(0_n)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}, \bar{\epsilon}_k, \bar{\epsilon}_k, \bar{\epsilon}_k) \text{ for every } x \in \mathbb{R}_+^n.$$

In particular, we have (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c), where:

(a)  $f^{(d)}(0_n)(\epsilon_{k_1}, \dots, \epsilon_{k_d}) \geq 0$  for all  $k_1 \geq \dots \geq k_d$ ,

(b)  $f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) \geq 0$  for all  $k_1 \geq \dots \geq k_{d-3} \geq k$ ,

(c)  $f \geq 0$  on  $\mathbb{R}_+^n$ .



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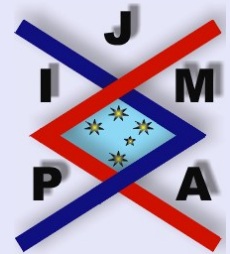
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Computer implementation of Theorems 2.1 and 2.2 is possible. The presence of  $f^{(d)}$  in the above statements poses no serious computation problem, since we have the identity

$$f^{(d)}(0_n)(u^1, \dots, u^d) = \sum_{J \subset \{1, \dots, d\}} (-1)^{\text{card}(J)} f \left( - \sum_{j \in J} u^j \right) \text{ for all } u^1, \dots, u^d \in \mathbb{R}^n.$$




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## 3. Proofs

### 3.1. The Non-symmetric Case

We first need the following lemma:

**Lemma 3.1.** *There exist  $u \in \{0, 1\}^n$  and  $y \in \mathbb{R}_+^n$ , such that  $\|y\| = 1$ ,  $\text{supp}(y) = \text{supp}(u)$ , and*

$$(3.1) \quad f(x) \geq \frac{f'(y)(u)}{d\|u\|} \cdot \|x\|^d \text{ for every } x \in \mathbb{R}_+^n.$$

*If  $f$  is symmetric, we can find  $y$  such that  $y_1 \geq \dots \geq y_n$ . In this case, we have  $u = \epsilon_k$  for some  $k \in \{1, \dots, n\}$ .*

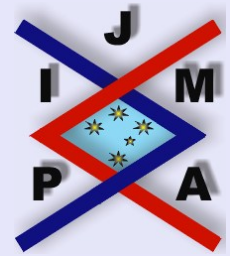
*Proof.* Let us first observe that inequality (3.1) is  $d$ -homogeneous in  $x$ . Consider the compact set  $K := \{x \in \mathbb{R}_+^n \mid \|x\| = 1\}$  and choose  $y \in K$ , such that  $f(y) = \min(f|_K)$ . Assume for simplicity that  $y_1 \geq \dots \geq y_n$  (if  $f$  is symmetric, we can find  $y$  with this property). It follows that  $\text{supp}(y) = \{1, \dots, k\}$  for some  $k \leq n$ . We claim that

$$(3.2) \quad kdf(y) = f'(y)(\epsilon_k).$$

By Euler's theorem on homogeneous functions we get

$$(3.3) \quad df(y) = \sum_{j=1}^k y_j \frac{\partial f}{\partial x_j}(y) = f'(y)(y).$$

We need to consider two cases:



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- i) If  $k = 1$ , then  $y = \epsilon_1$  and (3.3) obviously reduces to (3.2).
- ii) If  $k \geq 2$ , then  $y' := (y_1, \dots, y_k)$  is a global minimum for the restriction of the map  $]0, \infty[^k \ni z \mapsto f(z, 0_{n-k}) \in \mathbb{R}$  to the subset  $\{z \in ]0, \infty[^k \mid \|z\| = 1\}$ . Thus, applying the method of Lagrange multipliers shows that

$$(3.4) \quad \frac{\partial f}{\partial x_1}(y) = \frac{\partial f}{\partial x_2}(y) = \dots = \frac{\partial f}{\partial x_k}(y) = \lambda$$

for some  $\lambda \in \mathbb{R}$ . Now (3.3) and (3.4) yield  $\lambda = df(y)$ . Using this in (3.4) leads to

$$f'(y)(\epsilon_k) = \sum_{j=1}^k \frac{\partial f}{\partial x_j}(y) = k\lambda = kdf(y).$$

Our claim is proved. For every  $x \in \mathbb{R}_+^n \setminus \{0_n\}$  we have  $\|x\|^{-1}x \in K$ , and so

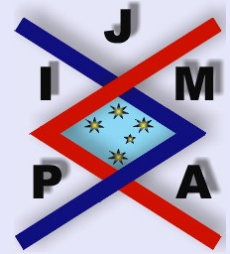
$$\frac{f(x)}{\|x\|^d} = f(\|x\|^{-1}x) \geq f(y) = \frac{f'(y)(\epsilon_k)}{kd} = \frac{f'(y)(\epsilon_k)}{d\|\epsilon_k\|},$$

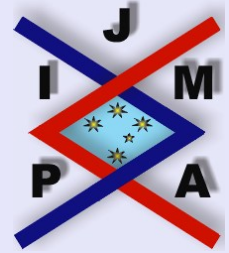
which proves (3.1) for  $u = \epsilon_k$ . □

*Proof of Theorem 2.1.*

**Step 1.** We first consider the particular case  $H = \mathbb{R}_+^n$ . Let us show by induction that for every  $i \in \{1, \dots, d\}$ , there exist  $y^i \in \mathbb{R}_+^n$  and  $u^1, \dots, u^i \in \{0, 1\}^n$ , such that  $\|y^i\| = 1$ ,  $\text{supp}(y^i) = \text{supp}(u^i)$ ,  $u^i \ll \dots \ll u^1$ , and

$$(3.5) \quad f(x) \geq \frac{(d-i)! \|x\|^d}{d! \|u^1\| \dots \|u^i\|} \cdot f^{(i)}(y^i)(u^1, \dots, u^i) \text{ for every } x \in \mathbb{R}_+^n.$$





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For  $i = 1$  this is clear, by Lemma 3.1. Assuming the statement to hold for some  $i < d$ , we will prove it for  $i + 1$ . For simplicity, we shall assume that  $y_1^i \geq \dots \geq y_n^i$  (if  $f$  is symmetric, we can find such  $y^i$  at each step of our induction). Consequently, we have  $\text{supp}(y^i) = \{1, \dots, k\}$  and  $u^i = \epsilon_k$  for some  $k \in \{1, \dots, n\}$ . Let us observe that the map  $\mathbb{R}^k \ni z \mapsto f^{(i)}(z, 0_{n-k})(u^1, \dots, u^i) \in \mathbb{R}$  is a  $(d-i)$ -form. According to Lemma 3.1, there exist  $y^{i+1} = (\zeta, 0_{n-k}) \in \mathbb{R}_+^k \times \mathbb{R}_+^{n-k}$  and  $u^{i+1} = (v, 0_{n-k}) \in \{0, 1\}^n$ , such that  $\|y^{i+1}\| = 1$ ,  $\text{supp}(y^{i+1}) = \text{supp}(u^{i+1})$ , and

$$f^{(i)}(z, 0_{n-k})(u^1, \dots, u^i) \geq \frac{f^{(i+1)}(y^{i+1})(u^1, \dots, u^i, u^{i+1})}{(d-i)\|u^{i+1}\|} \cdot \|z\|^{d-i}$$

for every  $z \in \mathbb{R}_+^k$ .

For  $z = (y_1^i, \dots, y_k^i)$  we have  $(z, 0_{n-k}) = y^i$ ,  $\|z\| = \|y^i\| = 1$ , and consequently

$$(3.6) \quad f^{(i)}(y^i)(u^1, \dots, u^i) \geq \frac{f^{(i+1)}(y^{i+1})(u^1, \dots, u^{i+1})}{(d-i)\|u^{i+1}\|}.$$

We also have  $0_n \neq u^{i+1} \ll \epsilon_k = u^i$ . Now combining (3.5) with (3.6) completes our induction, as well as the proof for  $H = \mathbb{R}_+^n$ , since  $f^{(d)}$  is constant,  $f^{(d)}(y^d) = f^{(d)}(0_n)$ .

**Step 2.** We next turn to the general case. Clearly, we can find  $\delta_1, \dots, \delta_n \in \{-1, 1\}$ , such that the linear isomorphism  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $Tx = (\delta_1 x_1, \dots, \delta_n x_n)$ , maps  $\mathbb{R}_+^n$  onto  $H$ , that is,  $T(\mathbb{R}_+^n) = H$ . Applying the conclusion of Step 1 to the  $d$ -form  $f \circ T$  yields the existence of  $d$  vectors  $v^1, \dots, v^d \in \mathbb{R}_+^n$ , such that

$0_n \neq v^d \ll \dots \ll v^1 \ll 1_n$  and

$$f(Tx) \geq \frac{\|x\|^d}{d!} \cdot \frac{(f \circ T)^{(d)}(0_n)(v^1, \dots, v^d)}{\|v^1\| \dots \|v^d\|} \text{ for every } x \in \mathbb{R}_+^n.$$

Let us observe that  $\|Tx\| = \|x\|$  for every  $x \in \mathbb{R}^n$ , and that

$$(f \circ T)^{(d)}(0_n)(v^1, \dots, v^d) = f^{(d)}(0_n)(Tv^1, \dots, Tv^d),$$

$$0_n \neq Tv^1 \ll \dots \ll Tv^d \ll T1_n \in \{-1, 1\}^n.$$

It follows that the vectors  $u^i := Tv^i$  are all in  $\{-1, 0, 1\}^n \cap H$ , and that (2.1) holds.  $\square$

### 3.2. The Symmetric Case

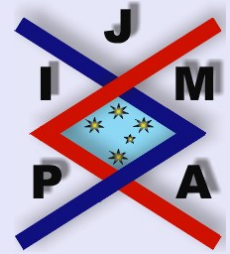
The following needed lemma is a slight generalization of Theorem 1.1.

**Lemma 3.2.** *Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a symmetric polynomial with  $\deg(g) \leq 3$ . Then for every  $\sigma > 0$  we have*

$$\min\{g(x) \mid x \in \mathbb{R}_+^n, \|x\| = \sigma\} = \min_{1 \leq k \leq n} g(\sigma \bar{e}_k).$$

*Proof.* Fix  $\sigma > 0$  and set  $\alpha := \min_{1 \leq k \leq n} g(\sigma \bar{e}_k)$ ,  $K := \{x \in \mathbb{R}_+^n \mid \|x\| = \sigma\}$ . Hence,  $\alpha = g(\sigma \bar{e}_p)$  for some  $p \in \{1, \dots, n\}$ . We have for  $g$  the decomposition  $g = \sum_{i=0}^3 g_i$ , with  $g_i$  symmetric  $i$ -form for every  $i \in \{0, 1, 2, 3\}$ . Now define the symmetric cubic

$$h : \mathbb{R}^n \rightarrow \mathbb{R}, \quad h = \sum_{i=0}^3 S^{3-i} g_i - \alpha S^3,$$



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where  $S(x) := \sigma^{-1} \sum_{j=1}^n x_j$ . Obviously,  $h|_K = g|_K - \alpha$ . By Theorem 1.1, we have  $h \geq 0$  on  $\mathbb{R}_+^n$ , and so  $g|_K \geq \alpha$ . Since  $\sigma\bar{\epsilon}_p \in K$  and  $g(\sigma\bar{\epsilon}_p) = \alpha$ , we get  $\alpha = \min_{x \in K} g(x)$ .  $\square$

*Proof of Theorem 2.2.* As in the proof of Theorem 2.1 (Step 1), by the same induction based on Lemma 3.1, we get  $y \in \mathbb{R}_+^n$  and  $u^1 = \epsilon_{k_1}, \dots, u^{d-3} = \epsilon_{k_{d-3}}$ , such that  $\|y\| = 1$ ,  $k_1 \geq \dots \geq k_{d-3}$ ,  $\text{supp}(y) = \text{supp}(\epsilon_{k_{d-3}}) = \{1, \dots, k_{d-3}\}$ , and

$$(3.7) \quad f(x) \geq \frac{6\|x\|^d}{d!} \cdot f^{(d-3)}(y)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) \text{ for every } x \in \mathbb{R}_+^n.$$

Note that the above inequality corresponds to (3.5) for  $i = d-3$ . Since  $\|y\| = 1$ ,  $\text{supp}(y) = \{1, \dots, k_{d-3}\}$ , and the polynomial map

$$\mathbb{R}^{k_{d-3}} \ni z \mapsto f^{(d-3)}(z, 0_{n-k_{d-3}})(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) \in \mathbb{R}$$

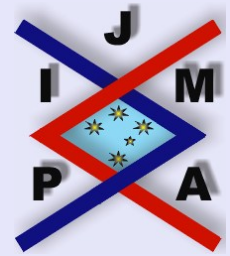
is a symmetric 3-form, applying Lemma 3.2 shows that

$$(3.8) \quad f^{(d-3)}(y)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) \geq f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}})$$

for some  $k \leq k_{d-3}$ . Now combining (3.7) with (3.8) yields (2.3). As the map

$$g : \mathbb{R}^k \rightarrow \mathbb{R}, \quad g(z) = f^{(d-3)}(z, 0_{n-k})(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}})$$

is a 3-form, we have  $6g(z) = g'''(z)(z, z, z) = g'''(0_k)(z, z, z)$  for every  $z \in \mathbb{R}^k$ .



This gives

$$\begin{aligned} f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) &= g(1_k) \\ &= \frac{g'''(0_k)(1_k, 1_k, 1_k)}{6} \\ &= \frac{f^{(d)}(0_n)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}, \bar{\epsilon}_k, \bar{\epsilon}_k, \bar{\epsilon}_k)}{6}, \end{aligned}$$

which proves (2.4). □

**Example 3.1.** *Proof of the fundamental AM – GM inequality by verifying condition (a) from Theorem 2.2.*

*Proof.* Let  $n \in \mathbb{N}^*$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f(x) = \sum_{i=1}^n x_i^n - n \prod_{i=1}^n x_i$ . We shall prove that  $f \geq 0$  on  $\mathbb{R}_+^n$ . Since Theorem 1.1 shows this for  $n \leq 3$ , assume that  $n \geq 4$ . For all  $u^1, \dots, u^n \in \mathbb{R}^n$ , we have

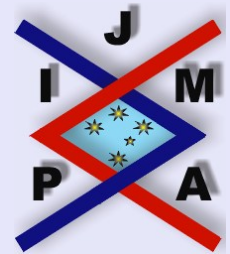
$$(3.9) \quad f^{(n)}(0_n)(u^1, \dots, u^n) = \sum_{i_1, \dots, i_n=1}^n \frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}(0_n) p_{i_1}(u^1) \cdots p_{i_n}(u^n),$$

where  $p_1, \dots, p_n : \mathbb{R}^n \rightarrow \mathbb{R}$  are the standard linear projections. For ease of exposition, let us define the map

$$v : \mathbb{R}^n \rightarrow \{1, \dots, n\}, \quad v(x) = \text{card}(\{x_1, \dots, x_n\}),$$

and consider the set  $A := \{1, \dots, n\}^n$ . For every  $i = (i_1, \dots, i_n) \in A$ , set

$$(3.10) \quad \frac{\partial^n f}{\partial x^i} := \frac{\partial^n f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} \equiv \begin{cases} n!, & v(i) = 1 \\ -n, & v(i) = n \\ 0, & 1 < v(i) < n \end{cases}$$



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the last equality being easily checked. Now fix  $k_1 \geq \dots \geq k_n$  in  $\{1, \dots, n\}$ , and let

$$B := \{i \in A \mid i_1 \leq k_1, i_2 \leq k_2, \dots, i_n \leq k_n\}.$$

By (3.9) and (3.10) we get

$$(3.11) \quad f^{(n)}(0_n)(\epsilon_{k_1}, \dots, \epsilon_{k_n}) = \sum_{i \in B} \frac{\partial^n f}{\partial x^i}(0_n) = n! \cdot \text{card}(B_1) - n \cdot \text{card}(B_n),$$

where  $B_1 := \{i \in B \mid v(i) = 1\}$  and  $B_n := \{i \in B \mid v(i) = n\}$ . Since obviously

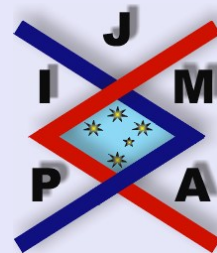
$$B_1 = \{\epsilon_n, 2\epsilon_n, \dots, k_n \epsilon_n\}, \quad B_n \subset \{i \in A \mid i_n \leq k_n, v(i) = n\} =: E,$$

we have  $\text{card}(B_1) = k_n$  and  $\text{card}(B_n) \leq \text{card}(E) = k_n(n-1)!$ . To prove the last equality, let us observe that every element  $i \in E$  can be obtained by selecting  $i_n \in \{1, \dots, k_n\}$  (there are  $k_n$  possibilities), and then choosing pairwise distinct  $i_1, \dots, i_{n-1} \in \{1, \dots, n\} \setminus \{i_n\}$  (that is, a permutation of this set). By (3.11) we get

$$f^{(n)}(0_n)(\epsilon_{k_1}, \dots, \epsilon_{k_n}) \geq 0.$$

The conclusion follows by Theorem 2.2.  $\square$

For Pólya's general result on strictly positive forms, we refer the reader to [3]. Bounds for the exponent from Pólya's theorem are given in [5, 8]. Various symmetric inequalities can be found especially in [3, 6], but also in [1, 4, 7, 9]. Some general results on symmetric inequalities can be found in [10] and [11].



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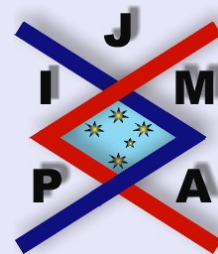
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