

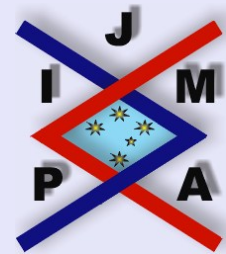
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## ON LANDAU TYPE INEQUALITIES FOR FUNCTIONS WITH HÖLDER CONTINUOUS DERIVATIVES

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## Abstract

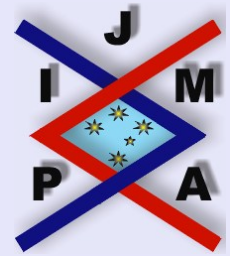
An inequality of Landau type for functions whose derivatives satisfy Hölder's condition is studied.

*2000 Mathematics Subject Classification:* 26D15

*Key words:* Landau inequality, Hölder continuity

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# 1. Introduction

S.S. Dragomir and C.I. Preda have proved the following theorem (see [1]):

**Theorem A.** *Let  $I$  be an interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  locally absolutely continuous function on  $I$ . If  $f \in L_\infty(I)$  and the derivative  $f' : I \rightarrow \mathbb{R}$  satisfies Hölder's condition*

$$(1.1) \quad |f'(t) - f'(s)| \leq H \cdot |t - s|^\alpha \quad \text{for any } t, s \in I,$$

where  $H > 0$  and  $\alpha \in (0, 1]$  are given, then  $f' \in L_\infty(I)$  and one has the inequalities:

$$(1.2) \quad \|f'\| \leq \begin{cases} \left[ 2 \left( 1 + \frac{1}{\alpha} \right) \right]^{\frac{\alpha}{\alpha+1}} \cdot \|f\|_{\frac{\alpha}{\alpha+1}}^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} & \text{if } m(I) \geq 2^{\frac{\alpha+2}{\alpha+1}} \left( \frac{\|f\|}{H} \right)^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)^{\frac{1}{\alpha+1}}; \\ \frac{4 \cdot \|f\|}{m(I)} + \frac{H}{2^{\alpha(\alpha+1)}} [m(I)]^\alpha & \text{if } 0 < m(I) \leq 2^{\frac{\alpha+2}{\alpha+1}} \left( \frac{\|f\|}{H} \right)^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)^{\frac{1}{\alpha+1}}, \end{cases}$$

where  $\|\cdot\|$  is the  $\infty$ -norm on the interval  $I$ , and  $m(I)$  is the length of  $I$ .

In our paper we shall give an improvement of this theorem.



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## 2. Main Results

**Theorem 2.1.** Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  function on  $I$  satisfying conditions of Theorem A. Then  $f' \in L_\infty(I)$  and the following inequalities hold:

$$(2.1) \quad \|f'\| \leq \begin{cases} \left[ 2 \left( 1 + \frac{1}{\alpha} \right) \right]^{\frac{\alpha}{\alpha+1}} \cdot \|f\|^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} \\ \quad \text{if } m(I) \geq 2^{\frac{1}{\alpha+1}} \left( \frac{\|f\|}{H} \right)^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)^{\frac{1}{\alpha+1}} ; \\ \frac{2\|f\|}{m(I)} + \frac{H}{\alpha+1} [m(I)]^\alpha \\ \quad \text{if } 0 < m(I) \leq 2^{\frac{1}{\alpha+1}} \left( \frac{\|f\|}{H} \right)^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)^{\frac{1}{\alpha+1}} , \end{cases}$$

where  $\|\cdot\|$  is the  $\infty$ -norm on the interval  $I$ , and  $m(I)$  is the length of  $I$ .

In our proof and in the subsequent discussion we use three lemmas.

**Lemma 2.2.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $\alpha \in (0, 1]$ . Then the following inequality holds:

$$(2.2) \quad (b-x)^{\alpha+1} + (x-a)^{\alpha+1} \leq (b-a)^{\alpha+1}, \quad \forall x \in [a, b].$$

*Proof.* Consider the function  $y : [a, b] \rightarrow \mathbb{R}$  given by:

$$y(x) = (b-x)^{\alpha+1} + (x-a)^{\alpha+1}.$$

We observe that the unique solution of the equation

$$y'(x) = (\alpha+1)[(x-a)^\alpha - (b-x)^\alpha] = 0$$



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is  $x_0 = \frac{a+b}{2} \in [a, b]$ . The function  $y'(x)$  is decreasing on  $(a, x_0)$  and increasing on  $(x_0, b)$ . Thus, the maximal values for  $y(x)$  are attained on the boundary of  $[a, b]$ :  $y(a) = y(b) = (b-a)^{\alpha+1}$ , which proves the lemma.  $\square$

A generalization of the following lemma is proved in [1]:

**Lemma 2.3.** *Let  $A, B > 0$  and  $\alpha \in (0, 1]$ . Consider the function  $g_\alpha : (0, \infty) \rightarrow \mathbb{R}$  given by:*

$$(2.3) \quad g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha.$$

Define  $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0, \infty)$ . Then for  $\lambda_1 \in (0, \infty)$  we have the bound

$$(2.4) \quad \inf_{\lambda \in (0, \lambda_1]} g_\alpha(\lambda) = \begin{cases} \frac{A}{\lambda_1} + B \cdot \lambda_1^\alpha & \text{if } 0 < \lambda_1 < \lambda_0 \\ (\alpha + 1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}} & \text{if } \lambda_1 \geq \lambda_0. \end{cases}$$

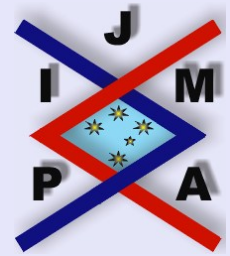
*Proof.* We have:

$$g'_\alpha(\lambda) = -\frac{A}{\lambda^2} + \alpha \cdot B \cdot \lambda^{\alpha-1}.$$

The unique solution of the equation  $g'_\alpha(\lambda) = 0$ ,  $\lambda \in (0, \infty)$ , is  $\lambda_0 = \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0, \infty)$ . The function  $g_\alpha(\lambda)$  is decreasing on  $(0, \lambda_0)$  and increasing on  $(\lambda_0, \infty)$ . The global minimum for  $g_\alpha(\lambda)$  on  $(0, \infty)$  is:

$$(2.5) \quad g_\alpha(\lambda_0) = A \left(\frac{\alpha B}{A}\right)^{\frac{1}{\alpha+1}} + B \left(\frac{A}{\alpha B}\right)^{\frac{\alpha}{\alpha+1}} = (\alpha + 1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}},$$

which proves (2.4).  $\square$



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**Lemma 2.4.** Let  $A, B > 0$  and  $\alpha \in (0, 1]$ . Consider the functions  $g_\alpha : (0, \infty) \rightarrow \mathbb{R}$  and  $h_\alpha : (0, \infty) \rightarrow \mathbb{R}$  defined by:

$$(2.6) \quad \begin{cases} g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha \\ h_\alpha(\lambda) = \frac{2A}{\lambda} + \frac{B}{2^\alpha} \lambda^\alpha. \end{cases}$$

Define  $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0, \infty)$ . Then for  $\lambda_1 \in (0, \infty)$  we have:

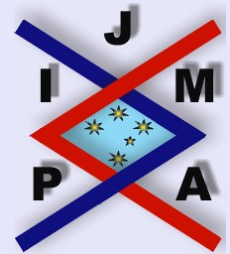
$$(2.7) \quad \begin{cases} \inf_{\lambda \in (0, \lambda_1]} g_\alpha(\lambda) < \inf_{\lambda \in (0, \lambda_1]} h_\alpha(\lambda) & \text{if } 0 < \lambda_1 < 2\lambda_0 \\ \inf_{\lambda \in (0, \lambda_1]} g_\alpha(\lambda) = \inf_{\lambda \in (0, \lambda_1]} h_\alpha(\lambda) & \text{if } \lambda_1 \geq 2\lambda_0. \end{cases}$$

*Proof.* In Lemma 2.3, we found that the global minimum for  $g_\alpha(\lambda)$  is obtained for  $\lambda = \lambda_0$ . Similarly we find that the global minimum for  $h_\alpha(\lambda)$  is obtained for  $\lambda = 2\lambda_0$ , and its value is equal to the minimal value of  $g_\alpha(\lambda)$ , i.e.  $h_\alpha(2\lambda_0) = g_\alpha(\lambda_0)$ .

The only solution of equation  $g_\alpha(\lambda) = h_\alpha(\lambda)$ ,  $\lambda \in (0, \infty)$ , is:

$$\lambda_S = \left[ \frac{A}{B(1 - 2^{-\alpha})} \right]^{\frac{1}{\alpha+1}},$$

and we can easily check that  $\lambda_0 < \lambda_S < 2\lambda_0$ . Thus, for  $\lambda_1 < \lambda_0$  we have  $g_\alpha(\lambda_1) < h_\alpha(\lambda_1)$  and  $\inf_{\lambda \in (0, \lambda_1]} g_\alpha(\lambda) < \inf_{\lambda \in (0, \lambda_1]} h_\alpha(\lambda)$ , and the rest of the proof is obvious. □




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*Proof of Theorem 2.1.* Now we start proving our theorem using the identity:

$$(2.8) \quad f(x) = f(a) + (x - a)f'(a) + \int_a^x [f'(s) - f'(a)]ds; \quad a, x \in I$$

or, by changing  $x$  with  $a$  and  $a$  with  $x$ :

$$(2.9) \quad f(a) = f(x) + (a - x)f'(x) + \int_x^a [f'(s) - f'(x)]ds; \quad a, x \in I.$$

Analogously, we have for  $b \in I$ :

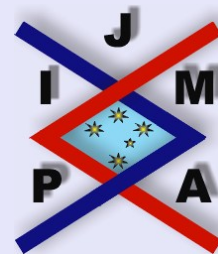
$$(2.10) \quad f(b) = f(x) + (b - x)f'(x) + \int_x^b [f'(s) - f'(x)]ds; \quad b, x \in I.$$

From (2.9) and (2.10) we obtain:

$$(2.11) \quad f(b) - f(a) = (b - a)f'(x) + \int_x^b [f'(s) - f'(x)]ds \\ + \int_a^x [f'(s) - f'(x)]ds; \quad a, b, x \in I$$

and

$$(2.12) \quad f'(x) = \frac{f(b) - f(a)}{b - a} - \frac{1}{b - a} \int_x^b [f'(s) - f'(x)]ds \\ - \frac{1}{b - a} \int_a^x [f'(s) - f'(x)]ds.$$



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Assuming that  $b > a$  we have the inequality:

$$(2.13) \quad |f'(x)| \leq \frac{|f(b) - f(a)|}{b - a} + \frac{1}{b - a} \left| \int_x^b |f'(s) - f'(x)| ds \right| + \frac{1}{b - a} \left| \int_a^x |f'(s) - f'(x)| ds \right|.$$

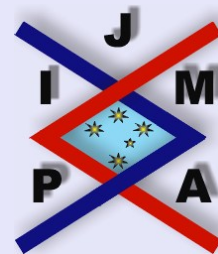
Since  $f'$  is of  $\alpha - H$  Hölder type, then:

$$(2.14) \quad \left| \int_x^b |f'(s) - f'(x)| ds \right| \leq H \cdot \left| \int_x^b |s - x|^\alpha ds \right| = H \int_x^b (s - x)^\alpha ds = \frac{H}{\alpha + 1} (b - x)^{\alpha+1}; \quad b, x \in I, b > x$$

$$(2.15) \quad \left| \int_a^x |f'(s) - f'(x)| ds \right| \leq H \cdot \left| \int_a^x |s - x|^\alpha ds \right| = H \int_a^x (x - s)^\alpha ds = \frac{H}{\alpha + 1} (x - a)^{\alpha+1}; \quad a, x \in I, a < x.$$

From (2.13), (2.14) and (2.15) we deduce:

$$(2.16) \quad |f'(x)| \leq \frac{|f(b) - f(a)|}{b - a}$$



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$$+ \frac{H}{(b-a)(\alpha+1)} [(b-x)^{\alpha+1} + (x-a)^{\alpha+1}];$$

$$a, b, x \in I, a < x < b.$$

Since  $f \in L_\infty(I)$  then  $|f(b) - f(a)| \leq 2 \cdot \|f\|$ . Using Lemma 2.2 we obviously get that:

$$(2.17) \quad |f'(x)| \leq \frac{2\|f\|}{b-a} + \frac{H}{\alpha+1}(b-a)^\alpha; \quad a, b, x \in I, a < x < b.$$

Denote  $b - a = \lambda$ . Since  $a, b \in I, b > a$ , we have  $\lambda \in (0, m(I))$ , and we can analyze the right-hand side of the inequality (2.17) as a function of variable  $\lambda$ . Thus we obtain:

$$(2.18) \quad |f'(x)| \leq \frac{2\|f\|}{\lambda} + \frac{H}{\alpha+1}\lambda^\alpha = g_\alpha(\lambda)$$

for  $x \in I$  and for every  $\lambda \in (0, m(I))$ .

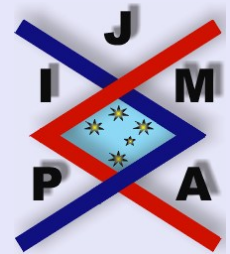
Taking the infimum over  $\lambda \in (0, m(I))$  in (2.18), we get:

$$(2.19) \quad |f'(x)| \leq \inf_{\lambda \in (0, m(I))} g_\alpha(\lambda).$$

If we take the supremum over  $x \in I$  in (2.19) we conclude that

$$(2.20) \quad \sup_{x \in I} |f'(x)| = \|f'\| \leq \inf_{\lambda \in (0, m(I))} g_\alpha(\lambda).$$

Making use of Lemma 2.3 we obtain the desired result (2.1). □



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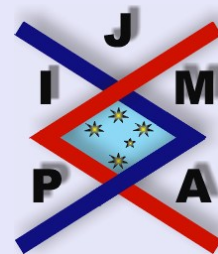


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**Remark 2.1.** Denote  $\lambda_0 = \left[2 \left(1 + \frac{1}{\alpha}\right) \frac{\|f\|}{H}\right]^{\frac{1}{\alpha+1}}$ . Comparing the results of Theorem A and Theorem 2.1 we can see that in the case of  $m(I) \geq 2\lambda_0$  the estimated values for  $\|f'\|$  in both theorems coincide. If  $0 < m(I) < 2\lambda_0$  the estimated value for  $\|f'\|$  given by (2.1) is better than the one given by (1.2). Namely, using Lemma 2.4 we have:

$$(2.21) \quad \frac{2\|f\|}{m(I)} + \frac{H}{\alpha+1} [m(I)]^\alpha < \frac{4\|f\|}{m(I)} + \frac{H}{2^\alpha(\alpha+1)} [m(I)]^\alpha; \quad m(I) \in (0, \lambda_0]$$

and

$$(2.22) \quad \left[2 \left(1 + \frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot \|f\|^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} < \frac{4\|f\|}{m(I)} + \frac{H}{2^\alpha(\alpha+1)} [m(I)]^\alpha; \quad m(I) \in [\lambda_0, 2\lambda_0).$$

**Remark 2.2.** Let the conditions of Theorem 2.1 be fulfilled. Then a simple consequence of (2.11) is the following inequality:

$$|(b-a)f'(x) - f(b) + f(a)| \leq \frac{H}{\alpha+1} [(b-x)^{\alpha+1} + (x-a)^{\alpha+1}];$$

$$a, b, x \in I, \quad a < x < b.$$

This result is an extension of the result obtained by V.G. Avakumović and S. Aljančić in [2] (see also [3]).

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