



# A SIMPLE PROOF OF THE GEOMETRIC-ARITHMETIC MEAN INEQUALITY

**YASUHARU UCHIDA**

Kurashiki Kojyoike High School  
Okayama Pref., Japan  
EMail: [haru@sqr.or.jp](mailto:haru@sqr.or.jp)

*Received:* 13 March, 2008

*Accepted:* 06 May, 2008

*Communicated by:* **P.S. Bullen**

*2000 AMS Sub. Class.:* 26D99.

*Key words:* Arithmetic mean, Geometric mean, Inequality.

*Abstract:* In this short note, we give another proof of the Geometric-Arithmetic Mean inequality.

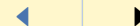
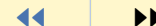
**Geometric-Arithmetic  
Mean Inequality**

Yasuharu Uchida

vol. 9, iss. 2, art. 56, 2008

[Title Page](#)

[Contents](#)



Page 1 of 5

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



Title Page

Contents



Page 2 of 5

Go Back

Full Screen

Close

Various proofs of the Geometric-Arithmetic Mean inequality are known in the literature, for example, see [1]. In this note, we give yet another proof and show that the G-A Mean inequality is merely a result of simple iteration of a well-known lemma.

The following theorem holds.

**Theorem 1 (Geometric-Arithmetic Mean Inequality).** For arbitrary positive numbers  $A_1, A_2, \dots, A_n$ , the inequality

$$(1) \quad \frac{A_1 + A_2 + \dots + A_n}{n} \geq \sqrt[n]{A_1 A_2 \dots A_n}$$

holds, with equality if and only if  $A_1 = A_2 = \dots = A_n$ .

Letting  $a_i = \sqrt[n]{A_i}$  ( $i = 1, 2, \dots, n$ ) and multiplying both sides by  $n$ , we have an equivalent Theorem 2.

**Theorem 2.** For arbitrary positive numbers  $a_1, a_2, \dots, a_n$ , the inequality

$$(2) \quad a_1^n + a_2^n + \dots + a_n^n \geq n a_1 a_2 \dots a_n$$

holds, with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

To prove Theorem 2, we use the following lemma.

**Lemma 3.** If  $a_1 \geq a_2$ ,  $b_1 \geq b_2$ , then

$$(3) \quad a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1.$$

*Proof.* Quite simply, we have

$$\begin{aligned} a_1 b_1 + a_2 b_2 - (a_1 b_2 + a_2 b_1) &= a_1(b_1 - b_2) - a_2(b_1 - b_2) \\ &= (a_1 - a_2)(b_1 - b_2) \geq 0. \end{aligned}$$





Title Page

Contents



Page 3 of 5

Go Back

Full Screen

Close

Iterating Lemma 3, we naturally obtain Theorem 2.

*Proof of Theorem 2 by induction on n.* Without loss of generality, we can assume that the terms are in decreasing order.

1. When  $n = 1$ , the theorem is trivial since  $a_1^1 \geq 1 \cdot a_1$ .
2. If Theorem 2 is true when  $n = k$ , then, for arbitrary positive numbers  $a_1, a_2, \dots, a_k$ ,

$$(4) \quad a_1^k + a_2^k + \dots + a_k^k \geq k a_1 a_2 \dots a_k.$$

Now assume that  $a_1 \geq a_2 \geq \dots \geq a_k \geq a_{k+1} > 0$ .

Exchanging factors  $a_{k+1}$  and  $a_i$  ( $i = k, k-1, \dots, 2, 1$ ) between the last term and the other sequentially, by Lemma 3, we obtain the following inequalities

$$\begin{aligned} & a_1^{k+1} + a_2^{k+1} + \dots + a_k^{k+1} + a_{k+1}^{k+1} \\ &= a_1^{k+1} + a_2^{k+1} + \dots + a_{k-1}^{k+1} + a_k^k \cdot \underline{a_k} + a_{k+1}^k \cdot \underline{a_{k+1}} \\ &\geq a_1^{k+1} + a_2^{k+1} + \dots + a_{k-1}^{k+1} + a_k^k \cdot \underline{a_{k+1}} + a_{k+1}^k \cdot \underline{a_k} \end{aligned}$$

.....

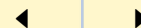
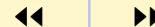
$$\begin{aligned} &\geq a_1^{k+1} + a_2^{k+1} + \dots + a_{i-1}^{k+1} + a_i^k \cdot \underline{a_i} + \dots \\ &\quad + a_k^k a_{k+1} + a_{k+1}^i a_{i+1} a_{i+2} \dots a_k \cdot \underline{a_{k+1}}. \end{aligned}$$

As  $a_i^k \geq a_{k+1}^i a_{i+1} a_{i+2} \dots a_k$ ,  $a_i \geq a_{k+1}$ , we can apply Lemma 3 so that



Title Page

Contents



Page 4 of 5

Go Back

Full Screen

Close

$$\begin{aligned}
 & a_1^{k+1} + a_2^{k+1} + \cdots + a_{i-1}^{k+1} + a_i^k \cdot \underline{a_i} + \cdots \\
 & \quad + a_k^k a_{k+1} + a_{k+1}^i a_{i+1} a_{i+2} \cdots a_k \cdot \underline{a_{k+1}} \\
 & \geq a_1^{k+1} + a_2^{k+1} + \cdots + a_{i-1}^{k+1} + a_i^k \cdot \underline{a_{k+1}} + \cdots + a_k^k a_{k+1} \\
 & \quad + a_{k+1}^i a_{i+1} a_{i+2} \cdots a_k \cdot \underline{a_i} \\
 & \quad \dots\dots\dots \\
 & \geq a_1^k a_{k+1} + a_2^k a_{k+1} + \cdots + a_k^k a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
 & = (a_1^k + a_2^k + \cdots + a_k^k) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1}.
 \end{aligned}$$

By assumption of induction (4), we have

$$\begin{aligned}
 & (a_1^k + a_2^k + \cdots + a_k^k) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
 & \geq (k a_1 a_2 \cdots a_k) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
 & = (k + 1) a_1 a_2 \cdots a_k a_{k+1}.
 \end{aligned}$$

From the same proof of Lemma 3,

$$\text{if } a_1 > a_2, b_1 > b_2, \quad \text{then } a_1 b_1 + a_2 b_2 > a_1 b_2 + a_2 b_1.$$

Thus, in the above sequence of inequalities, if the relationship  $a_i \geq a_{k+1}$  is replaced by  $a_i > a_{k+1}$  for some  $i$ , the inequality sign  $\geq$  also has to be replaced by  $>$  at the conclusion. We have the equality if and only if  $a_1 = a_2 = \cdots = a_n$ .

□

## References

- [1] P.S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Acad. Publ., Dordrecht, 2003.



---

**Geometric-Arithmetic  
Mean Inequality**

Yasuharu Uchida

**vol. 9, iss. 2, art. 56, 2008**

---

Title Page

Contents



Page 5 of 5

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756