A SIMPLE PROOF OF THE GEOMETRIC-ARITHMETIC MEAN INEQUALITY

YASUHARU UCHIDA
Kurashiki Kojyoike High School
Okayama Pref., Japan
EMail: haru@sqr.or.jp

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Abstract: In this short note, we give another proof of the Geometric-Arithmetic Mean inequality.

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Various proofs of the Geometric-Arithmetic Mean inequality are known in the literature, for example, see [1]. In this note, we give yet another proof and show that the G-A Mean inequality is merely a result of simple iteration of a well-known lemma.

The following theorem holds.

Theorem 1 (Geometric-Arithmetic Mean Inequality). *For arbitrary positive numbers $A_1, A_2, \dots, A_n$, the inequality*

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \geq \sqrt[n]{A_1 A_2 \cdots A_n} \tag{1}$$

holds, with equality if and only if $A_1 = A_2 = \cdots = A_n$.

Letting $a_i = \sqrt[n]{A_i}$ $(i = 1, 2, \dots, n)$ and multiplying both sides by n, we have an equivalent Theorem 2.

Theorem 2. *For arbitrary positive numbers $a_1, a_2, \dots, a_n$, the inequality*

$$a_1{}^n + a_2{}^n + \cdots + a_n{}^n \geq n a_1 a_2 \cdots a_n \tag{2}$$

holds, with equality if and only if $a_1 = a_2 = \cdots = a_n$.

To prove Theorem 2, we use the following lemma.

Lemma 3. *If $a_1 \geq a_2$, $b_1 \geq b_2$, then*

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1. \tag{3}$$

Proof. Quite simply, we have

$$\begin{aligned} a_1 b_1 + a_2 b_2 - (a_1 b_2 + a_2 b_1) &= a_1(b_1 - b_2) - a_2(b_1 - b_2) \\ &= (a_1 - a_2)(b_1 - b_2) \geq 0. \end{aligned}$$

□

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Iterating Lemma 3, we naturally obtain Theorem 2.

Proof of Theorem 2 by induction on n. Without loss of generality, we can assume that the terms are in decreasing order.

1. When $n = 1$, the theorem is trivial since ${a_1}^1 \geq 1 \cdot a_1$.

2. If Theorem 2 is true when $n = k$, then, for arbitrary positive numbers $a_1, a_2, \dots, a_k$,

$$ {a_1}^k + {a_2}^k + \cdots + {a_k}^k \geq k a_1 a_2 \cdots a_k. \tag{4} $$

   Now assume that $a_1 \geq a_2 \geq \cdots \geq a_k \geq a_{k+1} > 0$.
   Exchanging factors a_{k+1} and a_i $(i = k, k-1, \dots, 2, 1)$ between the last term and the other sequentially, by Lemma 3, we obtain the following inequalities

$$
\begin{aligned}
&{a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_k}^{k+1} + {a_{k+1}}^{k+1} \\
&\qquad = {a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_{k-1}}^{k+1} + {a_k}^k \cdot \underline{a_k} + {a_{k+1}}^k \cdot \underline{a_{k+1}} \\
&\qquad \geq {a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_{k-1}}^{k+1} + {a_k}^k \cdot \underline{a_{k+1}} + {a_{k+1}}^k \cdot \underline{a_k} \\
&\qquad\qquad \cdots\cdots \\
&\qquad \geq {a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_{i-1}}^{k+1} + {a_i}^k \cdot \underline{a_i} + \cdots \\
&\qquad\qquad + {a_k}^k a_{k+1} + {a_{k+1}}^i a_{i+1} a_{i+2} \cdots a_k \cdot \underline{a_{k+1}}.
\end{aligned}
$$

   As $a_i^k \geq a_{k+1}^i a_{i+1} a_{i+2} \dots a_k,\ a_i \geq a_{k+1},$ we can apply Lemma 3 so that

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$$
\begin{aligned}
{a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_{i-1}}^{k+1} + {a_i}^{k} \cdot \underline{a_i} + \cdots \\
+ {a_k}^{k} a_{k+1} + {a_{k+1}}^{i} a_{i+1} a_{i+2} \cdots a_k \cdot \underline{a_{k+1}} \\
\geq {a_1}^{k+1} + {a_2}^{k+1} + \cdots + {a_{i-1}}^{k+1} + {a_i}^{k} \cdot \underline{a_{k+1}} + \cdots + {a_k}^{k} a_{k+1} \\
+ {a_{k+1}}^{i} a_{i+1} a_{i+2} \cdots a_k \cdot \underline{a_i}
\end{aligned}
$$

$$\cdots\cdots$$

$$
\begin{aligned}
&\geq {a_1}^{k} a_{k+1} + {a_2}^{k} a_{k+1} + \cdots + {a_k}^{k} a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
&= ({a_1}^{k} + {a_2}^{k} + \cdots + {a_k}^{k}) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1}.
\end{aligned}
$$

By assumption of induction (4), we have

$$
\begin{aligned}
({a_1}^{k} + {a_2}^{k} + \cdots + {a_k}^{k}) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
\geq (k a_1 a_2 \cdots a_k) a_{k+1} + a_1 a_2 a_3 \cdots a_{k+1} \\
= (k+1) a_1 a_2 \cdots a_k a_{k+1}.
\end{aligned}
$$

From the same proof of Lemma 3,

$$\text{if } \ a_1 > a_2, b_1 > b_2, \quad \text{then } \ a_1 b_1 + a_2 b_2 > a_1 b_2 + a_2 b_1.$$

Thus, in the above sequence of inequalities, if the relationship $a_i \geq a_{k+1}$ is replaced by $a_i > a_{k+1}$ for some i, the inequality sign $\geq$ also has to be replaced by $>$ at the conclusion. We have the equality if and only if $a_1 = a_2 = \cdots = a_n$.

□

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References

[1] P.S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Acad. Publ., Dordrecht, 2003.

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