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AN INEQUALITY OF THE $1-D$ KLEIN-GORDON EQUATION WITH A TIME-VARYING PARAMETER

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ABSTRACT. In this paper we present an inequality concerning the regularity of the solution of the Klein-Gordon equation with a time-varying parameter. In particular, we present an inequality comparing the norm of the initial state and the norm of the solution on some part of the boundary.

Key words and phrases: Partial differential equation, Klein-Gordon, Inequality, Controllability.

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1. INTRODUCTION

Let $\Omega = (0,1)$ and α be a smooth function. We consider the following initial value problem

$$v_{tt} - v_{xx} + \alpha(t)v = 0 \text{ in } \Omega \times (0,T), \tag{1.1}$$

$$v(0,t) = 0 = v(1,t), \tag{1.2}$$

$$v(x,0) = v_0(x) \in H_0^1(\Omega), \quad \text{and} \quad v_t(x,0) = v_1(x) \in L^2(\Omega). \tag{1.3}$$

The smoothness of α guarantees the existence of the above system. In this paper we are interested with the following function

$$z(t) = v_x(1,t). \tag{1.4}$$

In particular, we establish some conditions on α that guarantee the existence of T and positive constants k_T, K_T that depend only on $T > 0$ such that the $L^2(0,T)$-norm of z and the $H_0^1(\Omega) \times$

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$L^2(\Omega)$-norm of the initial state (v_0, v_1) are equivalent, i.e.

$$k_T\|(v_0,v_1)\|^2_{H^1_0\times L^2} \le \int_0^T z^2(t)\,dt \le K_T\|(v_0,v_1)\|^2_{H^1_0\times L^2} \tag{1.5}$$

for every $(v_0, v_1) \in H^1_0 \times L^2$. The upper estimate means that z belongs to $L^2(0,T)$, and the lower one means that the topology induced by the $L^2(0,T)$-norm of z is stronger than the $H^1_0 \times L^2$-norm.

We establish the inequality by the multiplier method, described by Komornik [3].

The particular inequality above is crucial in studying the exact controllability of distributed parameter systems. One can observe that by the inequality, one may regard the $L^2(0,T)$-norm of $v_x(x,t)$ evaluated only at $x = 1$ as the equivalent norm of the initial condition. This fact is fully utilized in solving the exact controllablity of the systems. For detailed explanations on the exact controllability concepts and properties, readers should consult Lions [4].

In the higher dimensional case, it is proved in Pranoto [6] that the inequality is true. In the case $\alpha \equiv 0$, one can obtain the inequality and it is sharp in the sense that the time T must be > 2 and cannot be smaller. For example, one can consult Bardos et al. in [2] for this result. They use micro local analysis to obtain the inequality. In the case $\alpha \equiv 1$, we are able to compute numerically the exact control. Please consult Pranoto [5] for its numerical scheme. It uses the Galerkin method. A more recent result on a different type of system is given in Avalos et al. [1].

2. MAIN RESULTS

Let $S = H^1_0(\Omega) \times L^2(\Omega)$ and $|\cdot|$ be the L^2-norm. In S, we define

$$\|(v_0,v_1)\|_S = \sqrt{|\partial_x v_0|^2 + |v_1|^2} \tag{2.1}$$

for every $(v_0, v_1) \in S$. Here, ∂_x denotes the partial derivative on variable x. With this norm, we define *energy* E of the solution at time t, that is $(v(\cdot,t), v_t(\cdot,t))$, as

$$E(t) = \frac{1}{2}\left(\|(v,v_t)\|^2_S + \alpha(t)|v|^2\right). \tag{2.2}$$

Proposition 2.1. *If α satisfies the following conditions:*

A1: *There is an $\epsilon \in (0,1)$, such that $|\alpha(t)| \le (1-\epsilon)\pi^2$ for every t;*

A2: $\mathrm{Var}(\alpha) < \infty$*, where* $\mathrm{Var}(\alpha)$ *denotes the variation of the function α on $(0,\infty)$,*

then there exist $T > 0$ and $k_T, K_T > 0$ such that

$$k_T\|(v_0,v_1)\|^2_{H^1_0\times L^2} \le \int_0^T z^2(t)\,dt \le K_T\|(v_0,v_1)\|^2_{H^1_0\times L^2} \tag{2.3}$$

for every $(v_0, v_1) \in H^1_0 \times L^2$.

Proof. By the assumption (A1), the energy $E(t)$ is non-negative. If we differentiate it, we obtain

$$E'(t) = \frac{\alpha'(t)}{2}|v|^2. \tag{2.4}$$

One can show that $|v|^2 \le 2\left[1 + \frac{1-\epsilon}{\pi^2}\right]E(t)$, by (A1). It then implies that

$$|E'(t)| \le 2|\alpha'(t)|\left[1 + \frac{1-\epsilon}{\pi^2}\right]E(t).$$

Thus, we obtain

$$\frac{E(0)}{M(T)} \le E(t) \le M(T)\, E(0), \tag{2.5}$$

where $M(T) = \exp\left(2\left[1+\frac{1-\epsilon}{\pi^2}\right]\int_0^T |\alpha'(s)|ds\right)$.

Because we assume (A2), the following inequality holds for every $t>0$

$$\frac{E(0)}{\mu} \le E(t) \le \mu\, E(0), \tag{2.6}$$

where $\mu = \exp\left(2\left[1+\frac{1-\epsilon}{\pi^2}\right]\text{Var}(\alpha)\right)$. If α is constant, $E(t)=E(0)$ for every $t>0$. It means the energy is conserved.

After multiplying both sides of (1.1) by xv_x and integrating it over $\Omega\times(0,T)$, we obtain

$$\int_0^T \int_\Omega (v_{tt} - v_{xx} + \alpha(t)v)\, xv_x\, dx\, dt = 0. \tag{2.7}$$

Thus we have the following identity

$$\int_0^T \int_\Omega v_{tt}\, xv_x\, dx\, dt - \int_0^T \int_\Omega v_{xx}\, xv_x\, dx\, dt + \int_0^T \int_\Omega \alpha(t)v\, xv_x\, dx\, dt = 0. \tag{2.8}$$

Next, we evaluate the first term on the left hand side, and we obtain

$$\begin{aligned}\int_0^T \int_\Omega v_{tt}\, xv_x\, dx\, dt &= \rho(T) - \rho(0) \\ &\quad - \left(\int_0^T \left[\frac{x}{2}(v_t(x,t))^2\right]_{x=0}^1 dt - \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, dx\, dt, \right).\end{aligned} \tag{2.9}$$

By the boundary condition, the above equation becomes

$$\int_0^T \int_\Omega v_{tt}\, xv_x\, dx\, dt = \rho(T) - \rho(0) + \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, dx\, dt, \tag{2.10}$$

where $\rho(t) = \int_0^1 v_t(x,t)\; x\; v_x(x,t)\; dx$. Next, we evaluate the second term on the left hand side of (2.8) and obtain

$$-\int_0^T \int_\Omega v_{xx}\, xv_x\, dx\, dt = \frac{1}{2}\int_0^T \int_\Omega |v_x|^2\, dx\, dt - \frac{1}{2}\int_0^T |v_x(1,t)|^2\, dt. \tag{2.11}$$

After that, we evaluate the last term on the left hand side of (2.8). By the boundary condition, we obtain

$$\int_0^T \int_\Omega \alpha(t)v\, xv_x\, dx\, dt = -\frac{1}{2}\int_0^T \int_\Omega \alpha(t)|v|^2\, dx\, dt. \tag{2.12}$$

If one adds the equations (2.10-2.12) altogether, one obtains

$$0 = \rho(T) - \rho(0) + \frac{1}{2}\int_0^T \int_\Omega \left(|v_t|^2 + |v_x|^2 - \alpha(t)|v|^2\right)\, dx\, dt - \frac{1}{2}\int_0^T |v_x(1,t)|^2\, dt. \tag{2.13}$$

The value of the function ρ can be estimated by

$$|\rho(t)| \le \left(\int_0^1 |v_t(x,t)|^2\, dx\right)^{\frac{1}{2}} \left(\int_0^1 |v_x(x,t)|^2\, dx\right)^{\frac{1}{2}}. \tag{2.14}$$

Thus, we have

$$|\rho(t)| \leq \int_0^1 |v_t(x,t)|^2\,dx + \int_0^1 |v_x(x,t)|^2\,dx \leq D\,E(t), \tag{2.15}$$

where $D = \frac{2}{\epsilon}$.

Next, we estimate the term

$$\int\limits_0^T \int\limits_\Omega \left(|v_t|^2 + |v_x|^2 - \alpha(t)|v|^2\right)\,dx\,dt$$

$$\geq \int\limits_0^T \int\limits_\Omega \left(|v_t|^2 + (1-\epsilon)|v_x|^2 + \epsilon|v_x|^2 - (1-\epsilon)\pi^2|v|^2\right)\,dx\,dt \tag{2.16}$$

$$\geq \epsilon \int\limits_0^T \int\limits_\Omega \left(|v_t|^2 + |v_x|^2\right)\,dx\,dt \tag{2.17}$$

$$\geq \frac{\epsilon}{2} \int\limits_0^T \int\limits_\Omega \left(|v_t|^2 + |v_x|^2 + \pi^2|v|^2\right)\,dx\,dt \tag{2.18}$$

$$\geq \frac{\epsilon}{2} \int\limits_0^T \int\limits_\Omega \left(|v_t|^2 + |v_x|^2 + \alpha(t)|v|^2\right)\,dx\,dt. \tag{2.19}$$

We then apply the above estimates into (2.13), and we obtain

$$\frac{1}{2}\int_0^T |v_x(1,t)|^2\,dt \geq -D(E(T)+E(0)) + \frac{\epsilon}{2}\int_0^T E(t)\,dt. \tag{2.20}$$

By the estimate on the energy growth (2.5), this implies

$$\begin{aligned}\frac{1}{2}\int_0^T |v_x(1,t)|^2\,dt &\geq -D(\mu+1)E(0) + \frac{\epsilon}{2}\int_0^T \frac{E(0)}{\mu}\,dt \\ &= \left(-D(\mu+1) + \frac{1}{D}\,\frac{T}{\mu}\right)\,E(0).\end{aligned} \tag{2.21}$$

We then choose any $T > \mu(\mu+1)D^2$. So, $\left(-D(\mu+1) + \frac{1}{D}\,\frac{T}{\mu}\right) > 0$. In order to simplify the notation, we write the same T to denote this particular T. We then let

$$c_T = \left(-D(\mu+1) + \frac{1}{D}\,\frac{T}{\mu}\right). \tag{2.22}$$

Next, we know that

$$E(0) = \frac{1}{2}\|(v_0,v_1)\|_S^2 + \frac{\alpha(0)}{2}\int_\Omega |v_0(x)|^2\,dx.$$

Moreover, by (A1), we have

$$\int_\Omega \left(|v_x(x,t)|^2 + \alpha(t)|v(x,t)|^2\right)\,dx \geq \epsilon \int_\Omega |v_x(x,t)|^2\,dx. \tag{2.23}$$

Thus, we obtain

$$\int_0^T |z(t)|^2\,dt \geq k_T\|(v_0,v_1)\|_S^2, \tag{2.24}$$

where $k_T = 2\epsilon c_T = -4(\mu+1) + \epsilon^2 \frac{T}{\mu}$.

Next, we need to prove the upper estimate of (2.3). In order to do this, we start from the identity (2.13). Using (A1), one then obtains the following inequality

$$\begin{aligned}
\frac{1}{2}\int_0^T |v_x(1,t)|^2\,dt &\le |\rho(T)| + |\rho(0)| \\
&\qquad + \frac{1}{2}\left|\int_0^T\int_\Omega \left(|v_t|^2 + |v_x|^2 - \alpha(t)|v|^2\right)\,dx\,dt\right| \\
&\le D(\mu+1)E(0) \\
&\qquad + \frac{1}{2}\left|\int_0^T\int_\Omega \left(|v_t|^2 + |v_x|^2 + \alpha(t)|v|^2 + 2\pi^2|v|^2\right)\,dx\,dt\right| \qquad (2.25)\\
&\le D(\mu+1)E(0) + C\int_0^T E(t)\,dt \qquad (2.26)\\
&\le \left(D(\mu+1) + C\mu\, T\right)E(0), \qquad (2.27)
\end{aligned}$$

where $C = 1 + \frac{2}{\epsilon}$. The inequalities (2.25) and (2.26) are obtained by the fact that

$$\begin{aligned}
\|v_x\|_{L^2}^2 - \alpha(t)\|v\|_{L^2}^2 &\le \|v_x\|_{L^2}^2 + 2\pi^2\|v\|_{L^2}^2 + \alpha(t)\|v\|_{L^2}^2 \qquad (2.28)\\
&\le C\left(\|v_x\|_{L^2}^2 + \alpha(t)\|v\|_{L^2}^2\right).
\end{aligned}$$

Since $E(0) \le \frac{2-\epsilon}{2}\|(v_0,v_1)\|_S^2$, we then obtain

$$\int_0^T |z(t)|^2\,dt \le K_T\|(v_0,v_1)\|_S^2, \qquad (2.29)$$

where the constant K_T equals $\frac{4-2\epsilon}{\epsilon}(\mu+1) + \left(\frac{4}{\epsilon} - \epsilon\right)\mu\, T$. This completes the proof. □

The above proposition implies that the norm $\|(v_0,v_1)\|_S$ of the initial state is equivalent to the $L^2(0,T)$-norm of the function z. It is interesting to note that $z(t)$ which is equal to $v_x(1,t)$ is evaluated at $x = 1$ only. Roughly speaking, it means that one can observe the dynamic of the system via the information of z only. Thus, if we have two initial conditions that give the same z's, then those two initial conditions are equal. This property is related to the Holmgren Uniqueness Theorem in the study of PDE's.

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