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A GEOMETRICAL PROOF OF A NEW INEQUALITY FOR THE GAMMA FUNCTION

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Abstract

Using the inclusions between the unit balls for the p-norms, we obtain a new inequality for the gamma function.

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Since the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

is one of the most important functions in Mathematics, there exists an extensive literature on its inequalities (see [1], [2]).

Our aim here is to present and prove the inequalities

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1 \quad x \in [0,1], \quad n \in \mathbb{N}.$$

As we will show the above inequalities follow immediately from a key geometrical argument. From now on for any $r > 0$, $p, n \geq 1$ we will consider the notation:

$$D^{n,r}_{\|\cdot\|_p} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,/ \|(x_1, \ldots, x_n)\|_p < r\}$$

for the n-ball of radius r for the p-norm $\|(x_1,\dots,x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$.

To this end, we need to prove the following:

Lemma 1. *For all n in $\mathbb{N}$, $p \geq 1$ and $r > 0$ we have:*

$$\text{(1)} \qquad \textit{Volume}\left(D^{n,r}_{\|\cdot\|_p}\right) = 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} r^n.$$

Proof. For $n = 1$, $D^{1,r}_{\|\cdot\|_p}$ is the interval $(-r, r)$, whose measure is $2r$, i.e.,

$$2r = 2\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} r$$

and (1) holds. By induction, let us assume that (1) holds for $n-1$. Then we note that $|x_1|^p + \cdots + |x_n|^p < r^p$ is equivalent to $|x_1|^p + \cdots + |x_{n-1}|^p < r^p - |x_n|^p$ and by virtue of the induction hypothesis we have

$$\begin{aligned}\text{Volume}\left(D^{n,r}_{\|\cdot\|_p}\right) &= \int_{D^{n,r}_{\|\cdot\|_p}} dx_1 \dots dx_n \\ &= 2\int_0^r \left(\int_{D^{n-1,(r^p-|x_n|^p)^{1/p}}_{\|\cdot\|_p}} dx_1 \dots dx_{n-1}\right) dx_n\end{aligned}$$

$$= 2\int_0^r 2^{n-1}\frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)}(r^p - x_n^p)^{\frac{n-1}{p}}\,dx_n$$

$$= 2^n\frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)}r^n\int_0^1 (1-z^p)^{\frac{n-1}{p}}\,dz,$$

where $z = x_n/r$.

If we consider $F(a,b,c,z)$ the first hypergeometric function (see [3]), then

$$\int (1-z^p)^{\frac{n-1}{p}}\,dz = zF\left(\frac{1}{p}, -\frac{n-1}{p}, 1+\frac{1}{p}, z^n\right)$$

and by well-known properties of the hypergeometric function we deduce:

$$\text{Volume}\ \left(D^{n,r}_{\|\cdot\|_p}\right) = 2^n\frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)}r^n F\left(\frac{1}{p}, -\frac{n-1}{p}, 1+\frac{1}{p}, 1\right)$$

$$= 2^n\frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)}r^n\frac{\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1+\frac{n-1}{p}\right)}{\Gamma\left(1+\frac{n}{p}\right)}$$

$$= 2^n\frac{\Gamma\left(1+\frac{1}{p}\right)^{n}}{\Gamma\left(1+\frac{n}{p}\right)}r^n.$$

□

Therefore we have

Theorem 2. *For all $n \in \mathbb{N}$ and x in $(0,1)$ we have*

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

Proof. For all n in $\mathbb{N}$ and $p \geq 1$, from the inclusions

$$D^{n,1}_{\|\cdot\|_1} \subseteq D^{n,1}_{\|\cdot\|_p} \subseteq D^{n,1}_{\|\cdot\|_\infty},$$

we deduce

$$\text{Volume}\,\left(D^{n,1}_{\|\cdot\|_1}\right) \leq \text{Volume}\,\left(D^{n,1}_{\|\cdot\|_p}\right) \leq \text{Volume}\,\left(D^{n,1}_{\|\cdot\|_\infty}\right),$$

so by Lemma 1:

$$2^n \frac{\Gamma(2)^n}{\Gamma(n+1)} \leq 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \leq 2^n$$

and with $1/p = x$, bearing in mind that $\Gamma(2) = 1$, $\Gamma(n+1) = n!$,

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

□

From this it follows immediately that the function $\Gamma(1+x)^n/\Gamma(1+nx)$ is strictly decreasing on $(0,1]$.

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