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ON THE STABILITY OF A CLASS OF FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we study the Baker's superstability for the following functional equation

$$(E(K)) \qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|f(x)f(y), \ \ x,y\in G$$

where G is a locally compact group, K is a compact subgroup of G, ω_K is the normalized Haar measure of K, Φ is a finite group of K-invariant morphisms of G and f is a continuous complex-valued function on G satisfying the Kannappan type condition, for all $x,y,z\in G$

$$(*) \quad \int_K\int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyk^{-1}hxh^{-1})d\omega_K(k)d\omega_K(h).$$

We treat examples and give some applications.

Key words and phrases: Functional equation, Stability, Superstability, Central function, Gelfand pairs.

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1. INTRODUCTION, NOTATIONS AND PRELIMINARIES

Let G be a locally compact group. Let K be a compact subgroup of G. Let ω_K be the normalized Haar measure of K. A mapping $\varphi : G \longmapsto G$ is a morphism of G if φ is a homeomorphism of G onto itself which is either a group-homorphism, i.e ($\varphi(xy) = \varphi(x)\varphi(y), x, y \in G$), or a group-antihomorphism, i.e ($\varphi(xy) = \varphi(y)\varphi(x), x, y \in G$). We denote by $Mor(G)$ the group of morphisms of G and Φ a finite subgroup of $Mor(G)$ of a K-invariant morphisms of G (i.e $\varphi(K) \subset K$). The number of elements of a finite group Φ will be designated by $|\Phi|$. The Banach algebra of bounded measures on G with complex values is denoted by $M(G)$ and the Banach space of all complex measurable and essentially bounded functions on G by $L_\infty(G)$. $\mathcal{C}(G)$ designates the Banach space of all continuous complex valued functions on G. We say that a

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function f is a K-central function on G if

$$f(kx) = f(xk), \quad x \in G,\ k \in K. \tag{1.1}$$

In the case where $G = K$, a function f is central if

$$f(xy) = f(yx) \quad x, y \in G. \tag{1.2}$$

See [2] for more information.

In this note, we are going to generalize the results obtained by J.A. Baker in [8] and [9]. As applications, we discuss the following cases:

- **a)** $K \subset Z(G)$, ($Z(G)$ is the center of G).
- **b)** $f(hxk) = f(x),\ h, k \in K,\ x \in G$ (i.e. f is bi-K-invariant (see [3] and [6])).
- **c)** $f(hxk) = \chi(k)f(x)\chi(h),\ x \in G,\ k, h \in K$ (χ is a unitary character of K) (see [11]).
- **d)** (G, K) is a Gelfand pair (see [3], [6] and [11]).
- **e)** $G = K$ (see [2]).

In the next section, we note some results for later use.

2. General Properties

In what follows, we study general properties. Let G, K and Φ be given as above.

Proposition 2.1. *For an arbitrary fixed $\tau \in \Phi$, the mapping*

$$\Phi \longrightarrow \Phi,$$
$$\varphi \longrightarrow \varphi \circ \tau$$

is a bijection.

Proof. Follows from the fact that Φ is a finite group. □

Proposition 2.2. *Let $\varphi \in \Phi$ and $f \in \mathcal{C}(G)$, then we have:*

i) $\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) = \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k), \quad x, y \in G,\ h \in K.$

ii) *If f satisfy (*), the for all $z, y, x \in G$, we have*

$$\int_K \int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) = \int_K \int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k).$$

Proof. i) Let $\varphi \in \Phi$ and let $x, y \in G$, $h \in K$, then we have

Case 1: If φ is a group-homomorphism, we obtain, by replacing k by $k\varphi(h)^{-1}$

$$\begin{aligned}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) &= \int_K f(xk\varphi(h)\varphi(y)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(y)\varphi(h)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k).\end{aligned}$$

Case 2: if φ is a group-antihomomorphism, we have, by replacing k by $k\varphi(h)$

$$\begin{aligned}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) &= \int_K f(xk\varphi(y)\varphi(h)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(h)\varphi(y)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k).\end{aligned}$$

ii) Follows by simple computation.

□

Proposition 2.3. *For each $\tau \in \Phi$ and $x, y \in G$, we have*

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) = \sum_{\psi\in\Phi}\int_K f(xk\psi(y)k^{-1})d\omega_K(k). \tag{2.1}$$

Proof. By applying Proposition 2.1, we get that when φ is iterated over Φ, the morphism of the form $\varphi \circ \tau$ annihilates all the elements of Φ. □

3. The Main Results

Theorem 3.1. *Let G be a locally compact group; let K be a compact subgroup of G with the normalized Haar measure ω_K and let Φ given as above.*

Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)f(y)\right| \le \delta,\ \ x, y \in G. \tag{3.1}$$

Then one of the assertions is satisfied:

(a) *If f is bounded, then*

$$|f(x)| \le \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}. \tag{3.2}$$

(b) *If f is unbounded, then*
  i) *f is K-central,*
  ii) *$f \circ \tau = f$, for all $\tau \in \Phi$,*
  iii) *$\int_K f(xkyk^{-1})d\omega_K(k) = \int_K f(ykxk^{-1})d\omega_K(k), \quad x, y \in G.$*

Proof.

a) Let $X = \sup|f|$, then we get for all $x \in G$

$$|\Phi||f(x)f(x)| \le |\Phi|X + \delta,$$

from which we obtain that

$$|\Phi|X^2 - |\Phi|X - \delta \le 0,$$

such that

$$X \le \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}.$$

b) i) Let $x, y \in G,\ h \in K$, then by using Proposition 2.2, we find

$$\begin{aligned}|\Phi||f(x)||f(hy) - f(yh)| &= ||\Phi|f(x)f(hy) - |\Phi|f(x)f(yh)| \\ &\le \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) - |\Phi|f(x)f(hy)\right| \\ &\quad + \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(yh)k^{-1})d\omega_K(k) - |\Phi|f(x)f(yh)\right| \\ &\le 2\delta.\end{aligned}$$

Since f is unbounded it follows that $f(yh) = f(hy)$, for all $h \in K, y \in G$.

ii) Let $\tau \in \Phi$, by using Proposition 2.3, we get for all $x, y \in G$

$$
\begin{aligned}
|\Phi||f(x)||f\circ\tau(y) - f(y)| &= ||\Phi|f(x)f(\tau(y)) - |\Phi|f(x)f(y)| \\
&\leq \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) - |\Phi|\,f(x)f(\tau(y))\right| \\
&\quad + \left|\sum_{\psi\in\Phi}\int_K f(xk\psi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)f(y)\right| \\
&\leq 2\delta.
\end{aligned}
$$

Since f is unbounded it follows that $f\circ\tau = f$, for all $\tau \in \Phi$.
iii) Let f be an unbounded solution of the functional inequality (3.1), such that f satisfies the condition (*), then, for all $x, y \in G$, we obtain, by using Part i) of Proposition 2.2:

$$
\begin{aligned}
&|\Phi||f(z)|\left|\int_K f(xkyk^{-1})d\omega_K(k) - \int_K f(ykxk^{-1})d\omega_K(k)\right| \\
&\quad = \Bigg||\Phi|\int_K f(z)f(xkyk^{-1})d\omega_K(k) \\
&\qquad - |\Phi|\int_K f(z)f(ykxk^{-1})d\omega_K(k)\Bigg| \\
&\quad \leq \Bigg|\int_K \Sigma_{\varphi\in\Phi}\int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) \\
&\qquad - |\Phi|\int_K f(z)f(xkyk^{-1})d\omega_K(k)\Bigg| \\
&\qquad + \Bigg|\int_K \Sigma_{\varphi\in\Phi}\int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) \\
&\qquad -|\Phi|\int_K f(z)f(ykxk^{-1})d\omega_K(k)\Bigg| \\
&\quad \leq 2\delta.
\end{aligned}
$$

Since f is unbounded we get

$$\int_K f(xkyk^{-1})d\omega_K(k) = \int_K f(ykxk^{-1})d\omega_K(k), \quad x, y \in G.$$

□

The main result is the following theorem.

Theorem 3.2. *Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality*

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)f(y)\right| \leq \delta, \quad x, y \in G. \tag{3.3}$$

Then either

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G, \tag{3.4}$$

or

$$(E\,(K))\qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|f(x)f(y),\;\; x,y\in G.$$

Proof. The idea is inspired by the paper [1].

If f is bounded, by using Theorem 3.1, we obtain the first case of the theorem.

Now let f be an unbounded solution of the functional inequality (3.3), then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in G such that $f(z_n)\neq 0$ and $\lim_n|f(z_n)| = +\infty$.

For the second case we will use the following lemma.

Lemma 3.3. *Let f be an unbounded solution of the functional inequality (3.3) satisfying the condition (*) and let $(z_n)_{n\in\mathbb{N}}$ be a sequence in G such that $f(z_n)\neq 0$ and $\lim_n|f(z_n)| = +\infty$. It follows that the convergence of the sequences of functions:*

i)

$$(3.5)\qquad x\longmapsto \frac{\sum_{\varphi\in\Phi}\int_K f(z_nk\varphi(x)k^{-1})d\omega_K(k)}{f(z_n)},\quad n\in\mathbb{N},$$

to the function

$$x\longmapsto |\Phi|f(x).$$

ii)

$$(3.6)\quad x\longmapsto \frac{\sum_{\varphi\in\Phi}\int_K f(z_nh\varphi(xk\varphi(\tau(y))k^{-1})h^{-1})d\omega_K(h)}{f(z_n)},\quad n\in\mathbb{N},\ \tau\in\Phi,\ k\in K,\ y\in G$$

to the function

$$x\longmapsto |\Phi|f(xk\tau(y)k^{-1})\quad \tau\in\Phi,\ k\in K,\ y\in G,$$

is uniform.

By inequality (3.1), we have

$$\left|\frac{\sum_{\varphi\in\Phi}\int_K f(z_nk\varphi(y)k^{-1})d\omega_K(k)}{f(z_n)} - |\Phi|f(y)\right| \le \frac{\delta}{|f(z_n)|},$$

then we have, by letting $n\longmapsto +\infty$, that

$$\lim_n \frac{\sum_{\varphi\in\Phi}\int_K f(z_nk\varphi(y)k^{-1})d\omega_K(k)}{f(z_n)} = |\Phi|f(y),$$

and

$$\lim_n \frac{\sum_{\varphi\in\Phi}\int_K f\left(z_nh\varphi\left(xk\varphi\left(\tau(y)\right)k^{-1}\right)h^{-1}\right)d\omega_K(h)}{f(z_n)} = |\Phi|f(xk\tau(y)k^{-1}).$$

Since by Proposition 2.3, we have

$$\sum_{\tau\in\Phi}\int_K \frac{\sum_{\varphi\in\Phi}\int_K f(z_nh\varphi(x)k\varphi(\tau(y))k^{-1}h^{-1})d\omega_K(h)}{f(z_n)}d\omega_K(k)$$
$$= \sum_{\psi\in\Phi}\int_K \frac{\sum_{\varphi\in\Phi}\int_K f(z_nh\varphi(x)k\psi(y)k^{-1}h^{-1})d\omega_K(h)}{f(z_n)}d\omega_K(k),$$

combining this and the fact that f satisfies the condition (*), we obtain

$$\left|\sum_{\tau\in\Phi}\int_K \frac{\sum_{\varphi\in\Phi}\int_K f(z_n h\varphi(x)k\varphi(\tau(y))k^{-1}h^{-1})d\omega_K(h)}{f(z_n)}d\omega_K(k) - |\Phi|f(x)\frac{\sum_{\psi\in\Phi}\int_K f(z_n k\psi(y)k^{-1})d\omega_K(k)}{f(z_n)}\right| \leq \frac{\delta}{|f(z_n)|}.$$

Since the convergence is uniform, we have

$$\left||\Phi|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|^2 f(x)f(y)\right| \leq 0,$$

thus $(E\,(K))$ holds and the proof is complete. □

4. APPLICATIONS

If $K \subset Z(G)$, we obtain the following corollary.

Corollary 4.1. *Let $\delta > 0$ and let f be a complex-valued function on G satisfying the Kannappan condition (see* [10]*)*

$$f(zxy) = f(zyx), \quad x, y \in G, \tag{*}$$

and the functional inequality

$$\left|\sum_{\varphi\in\Phi} f(x\varphi(y)) - |\Phi|f(x)f(y)\right| \leq \delta, \quad x, y \in G. \tag{4.1}$$

Then either

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G, \tag{4.2}$$

or

$$\sum_{\varphi\in\Phi} f(x\varphi(y)) = |\Phi|f(x)f(y), \quad x, y \in G. \tag{4.3}$$

If G is abelian, then the condition (*) holds and we have the following:

If $\Phi = \{i\}$ (resp. $\Phi = \{i, \sigma\}$), where $i(x) = x$ and $\sigma(x) = -x$, we find the Baker's stability see [8] *(resp.* [9]*).*

If $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$ and $x \in G$, where χ is a character of K (see [11]), then we have the following corollary.

Corollary 4.2. *Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ such that $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$,*

$$\int_K\int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) \tag{*}$$

and

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k) - |\Phi|f(x)f(y)\right| \leq \delta, \quad x, y \in G. \tag{4.4}$$

Then either

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G, \tag{4.5}$$

or

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k) = |\Phi|f(x)f(y), \;\; x, y \in G. \tag{4.6}$$

Proposition 4.3. *If the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative then the condition (*) holds.*

Proof. Since $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$, then we have $\chi\omega_K \star f \star \chi\omega_K = f$. Suppose that the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative, then we get:

$$\begin{aligned}
\int_K \int_K f(xkyk^{-1}hzh^{-1})d\omega_K(k)d\omega_K(h) &= \int_K \int_K f(xkyhzh^{-1}k^{-1})d\omega_K(k)d\omega_K(h) \\
&= \langle \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x, f\rangle \\
&= \langle \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x, \chi\omega_K \star f \star \chi\omega_K\rangle \\
&= \langle \overline{\chi}\omega_K \star \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x \star \overline{\chi}\omega_K, f\rangle \\
&= \langle \overline{\chi}\omega_K \star \delta_z \star \overline{\chi}\omega_K \star \delta_x \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K, f\rangle \\
&= \int_K \int_K f(ykxk^{-1}hzh^{-1})d\omega_K(k)d\omega_K(h).
\end{aligned}$$

□

Let f be bi-K-invariant (i.e $f(hxk) = f(x), h, k \in K, x \in G$), then we have:

Corollary 4.4. *Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ be bi-K-invariant such that for all $x, y, z \in G$,*

$$\int_K \int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)d\omega_K(k)d\omega_K(h), \tag{*}$$

and

$$\left|\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))d\omega_K(k) - |\Phi|f(x)f(y)\right| \leq \delta, \;\; x, y \in G. \tag{4.7}$$

Then either

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G, \tag{4.8}$$

or

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))d\omega_K(k) = |\Phi|f(x)f(y), \;\; x, y \in G. \tag{4.9}$$

Proposition 4.5. *If the pair (G, K) is a Gelfand pair (i.e $\omega_K \star M(G) \star \omega_K$ is commutative), then the condition (*) holds.*

Proof. We take $\chi = 1$ (unit character of K) in Proposition 4.3 (see [3] and [6]). □

In the next corollary, we assume that $G = K$ is a compact group.

Lemma 4.6. *If f is central, then f satisfies the condition (*). Consequently, we have*

$$\int_G f(xtyt^{-1})dt = \int_G f(ytxt^{-1})dt, \quad x, y \in G. \tag{4.10}$$

Corollary 4.7. *Let $\delta > 0$ and let f be a complex measurable and essentially bounded function on G such that*

$$\left|\sum_{\varphi\in\Phi}\int_G f(xt\varphi(y)t^{-1})dt - |\Phi|f(x)f(y)\right| \leq \delta, \quad x,y \in G. \tag{4.11}$$

Then

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G. \tag{4.12}$$

Proof. Let $f \in L_\infty(G)$ be a solution of the inequality (4.11), then f is bounded, if not, then f satisfies the second case of Theorem 3.2 which implies that f is central (i.e the condition (*) holds) and f is a solution of the following functional equation

$$\sum_{\varphi\in\Phi}\int_G f(xt\varphi(y)t^{-1})dt = |\Phi|f(x)f(y), \quad x,y \in G. \tag{4.13}$$

In view of the proposition in [5], we have that f is continuous. Since G is compact, then the proof is accomplished. □

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