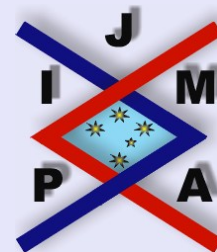


ESTIMATES FOR THE ∂ -NEUMANN OPERATOR ON STRONGLY PSEUDO-CONVEX DOMAIN WITH LIPSCHITZ BOUNDARY

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Abstract

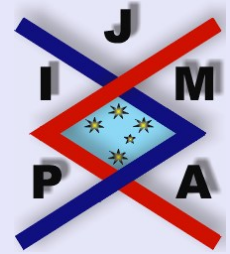
On a bounded strongly pseudo-convex domain X in \mathbb{C}^n with a Lipschitz boundary, we prove that the $\bar{\partial}$ -Neumann operator N can be extended as a bounded operator from Sobolev $(-1/2)$ -spaces to the Sobolev $(1/2)$ -spaces. In particular, N is compact operator on Sobolev $(-1/2)$ -spaces.

2000 Mathematics Subject Classification: Primary 35N15; Secondary 32W05.

Key words: Sobolev estimate, Neumann problem, Lipschitz domains.

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1. Introduction

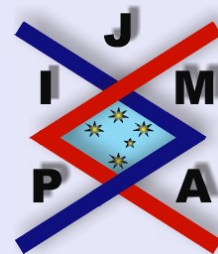
Let X be a bounded pseudo-convex domain in \mathbb{C}^n with the standard Hermitian metric. The $\bar{\partial}$ -Neumann operator N is the (bounded) inverse of the (unbounded) Laplace-Beltrami operator \square . The $\bar{\partial}$ -Neumann problem has been studied extensively when the domain X has smooth boundaries (see [12], [1], [3], [18], [19], [21], and [22]). Dahlberg [6] and Jerison and Kenig [17] established the work on the Dirichlet and classical Neumann problem on Lipschitz domains. The compactness of N on Lipschitz pseudo-convex domains is studied in Henkin and Iordan [14]. Let $W_{(p,q)}^s(X)$ be the Hilbert spaces of (p, q) -forms with $W^s(X)$ -coefficients. Henkin, Iordan, and Kohn in [15] and Michel and Shaw in [23] showed that N is bounded from $L_{(p,q)}^2(X)$ to $W_{(p,q)}^{1/2}(X)$ on domains with piecewise smooth strongly pseudo-convex boundary by two different methods. Also Michel and Shaw in [24] proved that N is bounded on $W_{(p,q)}^{1/2}(X)$ when the domain is only bounded pseudo-convex Lipschitz with a plurisubharmonic defining function. Other results in this direction belong to Bonami and Charpentier [4], Straube [26], Engliš [10], and Ehsani [7], [8], and [9]. In fact, the main aim of this work is to establish the following:

Theorem 1.1. *Let $X \subset\subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain with Lipschitz boundary. For each $0 \leq p \leq n$, $1 \leq q \leq n - 1$, the $\bar{\partial}$ -Neumann operator*

$$N : L_{(p,q)}^2(X) \longrightarrow L_{(p,q)}^2(X)$$

satisfies the following estimate: for any $\varphi \in L_{(p,q)}^2(X)$, there exists a constant $c > 0$ such that

$$(1.1) \quad \|N\varphi\|_{1/2(X)} \leq c\|\varphi\|_{-1/2(X)},$$



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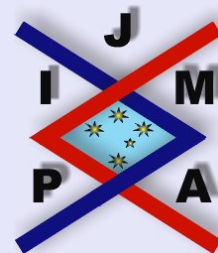
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where $c = c(X)$ is independent of φ ; i.e., N can be extended as a bounded operator from $W_{(p,q)}^{-1/2}(X)$ into $W_{(p,q)}^{1/2}(X)$. In particular, N is a compact operator on $L_{(p,q)}^2(X)$ and $W_{(p,q)}^{-1/2}(X)$.



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2. Notations and the $\bar{\partial}$ –Neumann Problem

We will use the standard notation of Hörmander [16] for differential forms. Let X be a bounded domain of \mathbb{C}^n . We express a (p, q) –form φ on X as follows:

$$\varphi = \sum_{I, J} \varphi_{IJ} dz^I \wedge d\bar{z}^J,$$

where I and J are strictly increasing multi-indices with lengths p and q , respectively. We denote by $\Lambda_{(p,q)}(X)$ the space of differential forms of class C^∞ and of type (p, q) on X . Let

$$\Lambda_{(p,q)}(\bar{X}) = \{\varphi|_{\bar{X}}; \varphi \in \Lambda_{(p,q)}(\mathbb{C}^n)\},$$

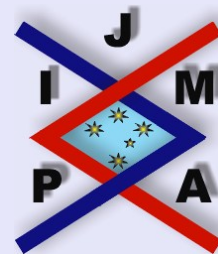
be the subspace of $\Lambda_{(p,q)}(X)$ whose elements can be extended smoothly up to the boundary ∂X of X . For $\varphi, \psi \in \Lambda_{(p,q)}(\bar{X})$, the inner product and norm are defined as usual by

$$\langle \varphi, \psi \rangle = \sum_{I, J} \int_X \varphi_{IJ} \bar{\psi}_{IJ} dv, \quad \text{and} \quad \|\varphi\|^2 = \int_X |\varphi|^2 dv,$$

where dv is the Lebesgue measure. Let $\Lambda_{0,(p,q)}(X)$ be the subspace of $\Lambda_{(p,q)}(\bar{X})$ whose elements have compact support disjoint from ∂X .

The operator $\bar{\partial} : \Lambda_{(p,q-1)}(X) \longrightarrow \Lambda_{(p,q)}(X)$ is defined by

$$\bar{\partial}\varphi = \sum_k \sum_{IJ} \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J.$$



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The formal adjoint operator δ of $\bar{\partial}$ is defined by :

$$\langle \delta\varphi, \psi \rangle = \langle \varphi, \bar{\partial}\psi \rangle$$

for any $\varphi \in \Lambda_{(p,q)}(X)$ and $\psi \in \Lambda_{0,(p,q-1)}(X)$. It is easily seen that $\bar{\partial}$ is a closed, linear, densely defined operator, and $\bar{\partial}$ forms a complex, i.e., $\bar{\partial}^2 = 0$. We denote by $L^2_{(p,q)}(X)$ the Hilbert space of all (p, q) forms with square integrable coefficients. We denote again by $\bar{\partial} : L^2_{(p,q-1)}(X) \longrightarrow L^2_{(p,q)}(X)$ the maximal extension of the original $\bar{\partial}$. Then $\bar{\partial}$ is a closed, linear, densely defined operator, and forms a complex, i.e., $\bar{\partial}^2 = 0$. Therefore, the adjoint operator $\bar{\partial}^* : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q-1)}(X)$ of $\bar{\partial}$ is also a closed, linear, defined operator. We denote the domain and the range of $\bar{\partial}$ in $L^2_{(p,q)}(X)$ by $\text{Dom}_{(p,q)}(\bar{\partial})$ and $\text{Range}_{(p,q)}(\bar{\partial})$ respectively.

We define the Laplace-Beltrami operator

$$\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q)}(X)$$

on

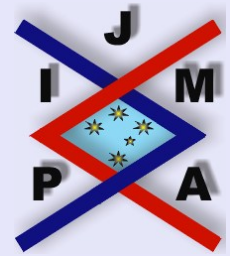
$$\text{Dom}_{(p,q)}(\square) = \{\varphi \in \text{Dom}_{(p,q)}(\bar{\partial}) \cap \text{Dom}_{(p,q)}(\bar{\partial}^*); \bar{\partial}\varphi \in \text{Dom}_{(p,q+1)}(\bar{\partial}^*)$$

and $\bar{\partial}^*\varphi \in \text{Dom}_{(p,q-1)}(\bar{\partial})\}$.

Let

$$\text{Ker}_{(p,q)}(\square) = \{\varphi \in \text{Dom}_{(p,q)}(\bar{\partial}) \cap \text{Dom}_{(p,q)}(\bar{\partial}^*); \bar{\partial}\varphi = 0 \quad \text{and} \quad \bar{\partial}^*\varphi = 0\}.$$

Definition 2.1. A domain $X \subset \subset \mathbb{C}^n$ is said to be strongly pseudo-convex with C^∞ -boundary if there exist an open neighborhood U of the boundary ∂X of X and a C^∞ function $\lambda : U \longrightarrow \mathfrak{R}$ having the following properties:



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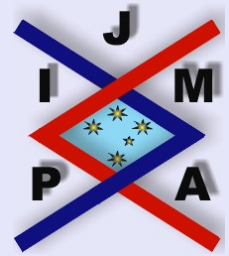


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(i) $X \cap U = \{z \in U; \lambda(z) < 0\}$.

(ii) $\sum_{\alpha, \beta=1}^n \frac{\partial^2 \lambda(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq L(z)|\eta|^2; z \in U, \eta = (\eta^1, \dots, \eta^n) \in \mathbb{C}^n$ and $L(z) > 0$.

(iii) The gradient $\nabla \lambda(z) = \left(\frac{\partial \lambda(z)}{\partial x^1}, \frac{\partial \lambda(z)}{\partial y^1}, \dots, \frac{\partial \lambda(z)}{\partial x^n}, \frac{\partial \lambda(z)}{\partial y^n} \right) \neq 0$

for $z = (z^1, \dots, z^n) \in U; z^\alpha = x^\alpha + iy^\alpha$.

Let $f : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}$ be a function that satisfies the Lipschitz condition

(2.1) $|f(x) - f(x')| \leq T|x - x'|$ for all $x, x' \in \mathbb{R}^{2n-1}$.

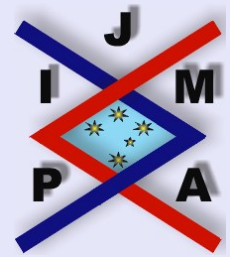
The smallest T in which (2.1) holds is called the bound of the Lipschitz constant. By choosing finitely many balls $\{V_j\}$ covering ∂X , the Lipschitz constant for a Lipschitz domain is the smallest T such that the Lipschitz constant is bounded in every ball $\{V_j\}$.

Definition 2.2. A bounded domain X in \mathbb{C}^n is called a strongly pseudo-convex domain with Lipschitz boundary ∂X if there exists a Lipschitz defining function ϱ in a neighborhood of \bar{X} such that the following condition holds:

(i) Locally near every point of the boundary ∂X , after a smooth change of coordinates, ∂X is the graph of a Lipschitz function.

(ii) There exists a constant $c_1 > 0$ such that,

(2.2) $\sum_{\alpha, \beta}^n \frac{\partial^2 \varrho}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq c_1|\eta|^2, \quad \eta = (\eta^1, \dots, \eta^n) \in \mathbb{C}^n,$



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where (2.2) is defined in the distribution sense.

Let $W^s(X)$, $s \geq 0$, be defined as the space of all $u|_X$ such that $u \in W^s(\mathbb{C}^n)$. We define the norm of $W^s(X)$ by

$$\|u\|_{s(X)} = \inf\{\|v\|_{s(\mathbb{C}^n)}, v \in W^s(\mathbb{C}^n), v|_X = u\}.$$

We use $W_{(p,q)}^s(X)$ to denote Hilbert spaces of (p, q) -forms with $W^s(X)$ coefficients and their norms are denoted by $\|\cdot\|_{s(X)}$. Let $W_0^s(X)$ be the completion of $C_0^\infty(X)$ -functions under the $W^s(X)$ -norm. Restricting to a small neighborhood U near a boundary point, we shall choose special boundary coordinates $t_1, \dots, t_{2n-1}, \lambda$ such that t_1, \dots, t_{2n-1} restricted to ∂X are coordinates for ∂X . Let $D_{t_j} = \partial/\partial t_j$, $j = 1, \dots, 2n-1$, and $D_\lambda = \partial/\partial \lambda$. Thus D_{t_j} 's are the tangential derivatives on ∂X , and D_λ is the normal derivative. For a multi-index $\beta = (\beta_1, \dots, \beta_{2n-1})$, where each β_j is a nonnegative integer, D_t^β denotes the product of D_{t_j} 's with order $|\beta| = \beta_1 + \dots + \beta_{2n-1}$, i.e., $D_t^\beta = D_{t_1}^{\beta_1} \dots D_{t_{2n-1}}^{\beta_{2n-1}}$. For any $\phi \in C_0^\infty(\bar{X})$ with compact support in U , we define the tangential Fourier transform for ϕ in a special boundary chart by

$$\tilde{\phi}(\nu, \lambda) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t, \nu \rangle} \phi(t, \lambda) dt,$$

where $\nu = (\nu_1, \dots, \nu_{2n-1})$ and $\langle t, \nu \rangle = t_1\nu_1 + \dots + t_{2n-1}\nu_{2n-1}$. We define the tangential Sobolev norms $\|\cdot\|_s$ by

$$\|\phi\|_s = \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 (1 + |\nu|^2)^s |\tilde{\phi}(\nu, \lambda)| d\lambda d\nu.$$

We recall the L^2 existence theorem for the $\bar{\partial}$ -Neumann operator on any bounded pseudo-convex domain $X \subset \mathbb{C}^n$. Following Hörmander L^2 - estimates for $\bar{\partial}$ on any bounded pseudoconvex domains, one can prove that \square has closed range and $\text{Ker}_{(p,q)}(\square) = \{0\}$. The $\bar{\partial}$ -Neumann operator N is the inverse of \square . In fact, one can prove

Proposition 2.1 (Hörmander [16]). *Let X be a bounded pseudo-convex domain in \mathbb{C}^n , $n \geq 2$. For each $0 \leq p \leq n$ and $1 \leq q \leq n$, there exists a bounded linear operator*

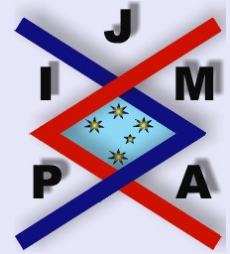
$$N : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q)}(X)$$

such that we have the following:

- (i) $\text{Range}_{(p,q)}(N) \subset \text{Dom}_{(p,q)}(\square)$ and $\square N = N \square = I$ on $\text{Dom}_{(p,q)}(\square)$.
- (ii) For any $\varphi \in L^2_{(p,q)}(X)$, $\varphi = \bar{\partial} \bar{\partial}^* N \varphi + \bar{\partial}^* \bar{\partial} N \varphi$.
- (iii) If δ is the diameter of X , we have the following estimates:

$$\begin{aligned} \|N\varphi\| &\leq \frac{e\delta^2}{q} \|\varphi\| \\ \|\bar{\partial} N\varphi\| &\leq \sqrt{\frac{e\delta^2}{q}} \|\varphi\| \\ \|\bar{\partial}^* N\varphi\| &\leq \sqrt{\frac{e\delta^2}{q}} \|\varphi\| \end{aligned}$$

for any $\varphi \in L^2_{(p,q)}(X)$.



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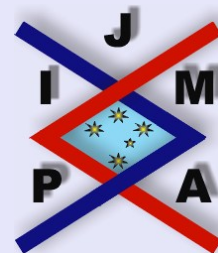
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For a detailed proof of this proposition see Shaw [25], Proposition 2.3, and Chen and Shaw [5], Theorem 4.4.1.

Theorem 2.2 (Rellich Lemma). *Let X be a bounded domain in \mathbb{C}^n with Lipschitz boundary. If $s > t \geq 0$, the inclusion $W^s(X) \hookrightarrow W^t(X)$ is compact.*



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3. Proof of the Main Theorem

To prove the main theorem we first obtain the following estimates on each smooth subdomain. As Lemma 2.1 in Michel and Shaw [23], we prove the following lemma:

Lemma 3.1. *Let $X \subset\subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain with Lipschitz boundary. Then, there exists an exhaustion $\{X_\mu\}$ of X with the following conditions:*

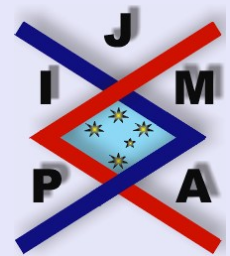
- (i) $\{X_\mu\}$ is an increasing sequence of relatively compact subsets of X and $\cup_\mu X_\mu = X$.
- (ii) Each $\{X_\mu\}$ has a C^∞ plurisubharmonic defining Lipschitz function λ_μ such that

$$\sum_{\alpha, \beta=1}^n \frac{\partial^2 \lambda_\mu(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq c_1 |\eta|^2$$

for $z \in \partial X_\mu$ and $\eta \in \mathbb{C}^n$, where $c_1 > 0$ is a constant independent of μ .

- (iii) There exist positive constants c_2, c_3 such that $c_2 \leq |\nabla \lambda_\mu| \leq c_3$ on ∂X_μ , where c_2, c_3 are independent of μ .

Proof. Let $\aleph = \{z \in X \mid -\delta_0 < \varrho(z) < 0\}$, where $\delta_0 > 0$ is sufficiently small. Thus, there exists a constant $c_1 > 0$ such that the function $\sigma_0(z) = \varrho(z) - c_1 |z|^2$ is a plurisubharmonic on \aleph . Let δ_μ be a decreasing sequence such that $\delta_\mu \searrow 0$, and we define $X_{\delta_\mu} = \{z \in X \mid \varrho(z) < -\delta_\mu\}$. Then $\{X_{\delta_\mu}\}$ is a sequence of relatively compact subsets of X with union equal to X . Let $\Psi \in C_0^\infty(\mathbb{C}^n)$ be a function depending only on $|z_1|, \dots, |z_n|$ and such that



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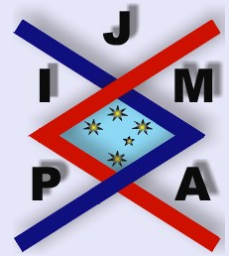


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(i) $\Psi \geq 0$.

(ii) $\Psi = 0$ when $|z| > 1$.

(iii) $\int \Psi d\lambda = 1$, where $d\lambda$ is the Lebesgue measure.

We define $\Psi_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} \Psi\left(\frac{z}{\varepsilon}\right)$ for $\varepsilon > 0$.

For each $z \in X_{\delta_\mu}$, $0 < \varepsilon < \delta_\mu$, we define

$$\varrho_\varepsilon(z) = \varrho \star \Psi_\varepsilon(z) = \int \varrho(z - \varepsilon\zeta) \Psi(\zeta) d\lambda(\zeta).$$

Then $\varrho_\varepsilon \in C^\infty(X_{\delta_\mu})$ and $\varrho_\varepsilon \searrow \varrho$ on X_{δ_μ} when $\varepsilon \searrow 0$. Since

$$\begin{aligned} \frac{\partial^2 \varrho_\varepsilon(z)}{\partial z^\alpha \partial \bar{z}^\beta} &= \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \{(\sigma_0(z) + c_1|z|^2) \star \Psi_\varepsilon(z)\} \\ &= \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \{\sigma_0(z) \star \Psi_\varepsilon(z) + c_1|z|^2 \star \Psi_\varepsilon(z)\} \\ &= \frac{\partial^2 \sigma_0(z)}{\partial z^\alpha \partial \bar{z}^\beta} \star \Psi_\varepsilon(z) + c_1 \frac{\partial^2 |z|^2}{\partial z^\alpha \partial \bar{z}^\beta} \star \Psi_\varepsilon(z) \end{aligned}$$

for $z \in X_{\delta_\mu} \cap \mathbb{N}$, and $\eta \in \mathbb{C}^n$ it follows that

$$\begin{aligned} &\sum_{\alpha, \beta=1}^n \frac{\partial^2 \varrho_\varepsilon(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \\ &= \sum_{\alpha, \beta=1}^n \frac{\partial^2 \sigma_0(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \star \Psi_\varepsilon(z) + c_1 \sum_{\alpha, \beta=1}^n \frac{\partial^2 |z|^2}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \star \Psi_\varepsilon(z) \\ &\geq c_1 |\eta|^2. \end{aligned}$$

Each $\varrho_{\varepsilon_\mu}$ is well defined if $0 < \varepsilon_\mu < \delta_{\mu+1}$ for $z \in X_{\delta_{\mu+1}}$. Let $c_3 = \sup_{\bar{X}} |\nabla \varrho|$, then for ε_μ sufficiently small, we have $\varrho(z) < \varrho_{\varepsilon_\mu}(z) < \varrho(z) + c_3\varepsilon_\mu$ on $X_{\delta_{\mu+1}}$. For each μ , we choose $\varepsilon_\mu = \frac{1}{2c_3}(\delta_{\mu-1} - \delta_\mu)$ and $\zeta_\mu \in (\delta_{\mu+1}, \delta_\mu)$. We define $X_\mu = \{z \in \mathbb{C}^n \mid \varrho_{\varepsilon_\mu} < -\zeta_\mu\}$. Since $\varrho(z) < \varrho_{\varepsilon_\mu}(z) < -\zeta_\mu < -\delta_{\mu+1}$, we have that $X_\mu \subset X_{\delta_{\mu+1}}$. Also, if $z \in X_{\delta_{\mu-1}}$, then $\varrho_{\varepsilon_\mu}(z) < \varrho(z) + c_3\varepsilon_\mu < -\delta_\mu < -\zeta_\mu$. Thus we have

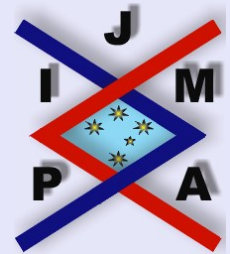
$$X_{\delta_{\mu+1}} \supset X_\mu \supset X_{\delta_{\mu-1}}$$

and (i) is satisfied. Then the function $\lambda_\mu = \varrho_{\varepsilon_\mu} + \zeta_\mu$ satisfies (ii). Now, we prove (iii). First, since a Lipschitz function is almost everywhere differentiable (see Evans and Gariepy [11] for a proof of this fact), the gradient of a Lipschitz function exists almost everywhere and we have $|\nabla \varrho| \leq c_3$ a.e. in \bar{X} and $|\nabla \lambda_\mu| \leq c_3$ on ∂X_μ . Secondly, we show that $|\nabla \lambda_\mu|$ is uniformly bounded from below. To do that, since ∂X is Lipschitz from our assumption, then there exists a finite covering $\{V_j\}_{1 \leq j \leq m}$ of ∂X such that $\bar{V}_j \subset \bar{U}_j$ for $1 \leq j \leq m$, a finite set of unit vectors $\{\chi_j\}_{1 \leq j \leq m}$ and $c_2 > 0$ such that the inner product $(\nabla \varrho, \chi_j) \geq c_2 > 0$ a.e. for $z \in V_j$, $1 \leq j \leq m$. Since this is preserved by convolution, (iii) is proved. Moreover, we have $\nabla \lambda_\mu \neq 0$ in a small neighborhood of ∂X_μ . Thus ∂X_μ is smooth. Then, the proof is complete. \square

We use a subscript μ to indicate operators on X_μ .

Proposition 3.2. *Let $\{X_\mu\}$ be the same as in Lemma 3.1. There exists a constant $c_4 > 0$, such that for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\bar{\partial}_\mu^*)$, $0 \leq p \leq n$, $1 \leq q \leq n - 1$,*

$$(3.1) \quad \|\varphi\|_{1/2(X_\mu)}^2 \leq c_4 \left(\|\bar{\partial}\varphi\|_{X_\mu}^2 + \|\bar{\partial}_\mu^*\varphi\|_{X_\mu}^2 \right),$$



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where c_4 is independent of φ and μ . If $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\square_\mu)$, then

$$(3.2) \quad \|\varphi\|_{1/2(X_\mu)}^2 \leq c_4 \|\square_\mu \varphi\|_{X_\mu}^2,$$

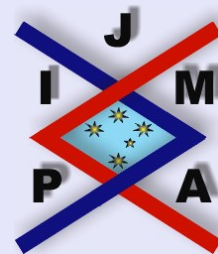
where c_4 is independent of φ and μ .

Proof. Since $|\nabla \lambda_\mu| \neq 0$ on a neighborhood W of ∂X_μ , then the function $\eta_\mu = \lambda_\mu / |\nabla \lambda_\mu|$ is defined on W . We extend η_μ to be negative smoothly inside X_μ . Then η_μ is a defining function in a neighborhood of \bar{X}_μ such that $\eta_\mu < 0$ on X_μ , $\eta_\mu = 0$ on ∂X_μ and $|\nabla \eta_\mu| = 1$ on W . Then, by simple calculation as in Lemma 2.2 in Michel and Shaw [23] and by using the identity of Morrey-Kohn-Hörmander which was proved in Chen and Shaw [5], Proposition 4.3.1, and from (ii) and (iii) in Lemma 3.1, it follows that there exists a constant $c_5 > 0$ such that for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\bar{\partial}^*)$,

$$(3.3) \quad \sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \leq c_5 \left(\|\bar{\partial} \varphi\|_{X_\mu}^2 + \|\bar{\partial}_\mu^* \varphi\|_{X_\mu}^2 \right).$$

Let $z \in \partial X_\mu$ and U be a special boundary chart containing z . From Kohn [20], Proposition 3.10 and Chen and Shaw [5], Lemma 5.2.2, the tangential Sobolev norm $\sum_{j=1}^n \|D^j \varphi\|_{\epsilon-1}$, and the ordinary Sobolev norm $\|\varphi\|_\epsilon$ are equivalent for $\varphi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$ where the support of φ lies in $U \cap \bar{X}_\mu$, $D^j \varphi = \partial \varphi / \partial x_j$, ($j = 1, 2, \dots, 2n$), and $\epsilon > 0$. Then, from Folland and Kohn [12], Theorems 2.4.4 and 2.4.5, it follows that there exists a neighborhood $V \subset U$ of z and a positive constant c_6 such that

$$(3.4) \quad \|\varphi\|_{1/2(X_\mu)}^2 \leq c_6 \left(\sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \|\varphi\|_{X_\mu}^2 + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \right),$$



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for $\varphi \in \Lambda_{0,(p,q)}(V \cap \bar{X}_\mu)$. Since X_μ is a Lipschitz domain, then c_6 depends only on the Lipschitz constant. Also from Lemma 3.1, if $\{X_\mu\}_{\mu=1}^\infty$ is uniformly Lipschitz, then the constant c_6 can be chosen to depend only on the Lipschitz constant of ∂X_μ , which is independent of μ . Now cover ∂X_μ by finitely many charts $\{V_i\}_{i=1}^m$ such that this conclusion holds on each chart, and choose V_0 so that $X_\mu - \cup_1^m V_i \subset V_0 \subset \bar{V}_0 \subset X_\mu$. Then, the estimate (3.4) holds for all $\varphi \in \Lambda_{0,(p,q)}(V_0)$. Using a partition of unity subordinate to $\{V_i\}_0^m$, the estimate (3.4) now reads

$$(3.5) \quad \|\varphi\|_{1/2(X_\mu)}^2 \leq c_6 \left(\sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \|\varphi\|_{X_\mu}^2 + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \right),$$

for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\bar{\partial}_\mu^*)$. It follows from Proposition 2.1 that

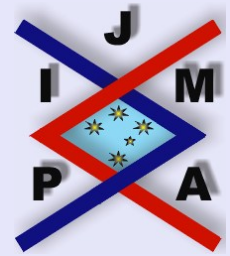
$$\|\varphi\|_{X_\mu}^2 \leq \frac{e\delta^2}{q} \left(\|\bar{\partial}\varphi\|_{X_\mu}^2 + \|\bar{\partial}^*\varphi\|_{X_\mu}^2 \right).$$

Therefore, by taking $c_4 = c_6 \left(\frac{e\delta^2}{q} + c_5 \right)$, and by using (3.3) and (3.5) inequality (3.1) is proved. Also, since

$$\|\bar{\partial}\varphi\|_{X_\mu}^2 + \|\bar{\partial}^*\varphi\|_{X_\mu}^2 \leq \|\square_\mu\varphi\|_{X_\mu} \|\varphi\|_{X_\mu},$$

when $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\square_\mu)$. Then, (3.2) is proved also. \square

Proof of Theorem 1.1. We shall apply the Michel and Shaw technique in [23] with the suitable modifications required. Let $\{X_\mu\}$ be the same as in Lemma 3.1 and N_μ denote the $\bar{\partial}$ -Neumann operator on $L^2_{(p,q)}(X_\mu)$. Since X is a strongly



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pseudo-convex domain with Lipschitz boundary, then by using Lemma 3.1, it can be approximated by domains with smooth boundary which are uniformly Lipschitz. Then, X_μ is a Lipschitz domain, and so $C^\infty(\bar{X}_\mu)$ is dense in $W^s(X_\mu)$ in the $W^s(X_\mu)$ -norm. Then, to prove this theorem, it suffices to prove (1.1) for any $\varphi \in \Lambda_{(p,q)}(\bar{X})$. By using the boundary regularity for N_μ which was established by Kohn [19], we have $N_\mu\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\square_\mu)$. The $\bar{\partial}$ -Neumann operator N is the inverse of the operator \square . By using (iii) in Proposition 2.1, we have

$$\|N_\mu\varphi\|_{X_\mu} \leq \frac{e\delta^2}{q} \|\varphi\|_{X_\mu} \leq \frac{e\delta^2}{q} \|\varphi\|_X,$$

and

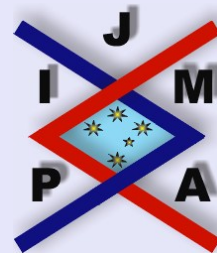
$$\|\bar{\partial}N_\mu\varphi\|_{X_\mu} + \|\bar{\partial}_\mu^*N_\mu\varphi\|_{X_\mu} \leq 2\sqrt{\frac{e\delta^2}{q}} \|\varphi\|_{X_\mu} \leq 2\sqrt{\frac{e\delta^2}{q}} \|\varphi\|_X.$$

Then, there is no loss of generality if we assume that

$$N_\mu\varphi = 0 \quad \text{in} \quad X \setminus X_\mu.$$

Then there is a subsequence of $N_\mu\varphi$, still denoted by $N_\mu\varphi$, converging weakly to some element $\psi \in L^2_{(p,q)}(X)$ and $\bar{\partial}\psi \in L^2_{(p,q+1)}(X)$. This implies that $\psi \in \text{Dom}_{(p,q)}(\bar{\partial})$. Now, we show that $\psi \in \text{Dom}_{(p,q)}(\bar{\partial}_\mu^*)$ as follows: for any $u \in \text{Dom}_{(p,q-1)}(\bar{\partial}) \cap L^2_{(p,q-1)}(X)$,

$$\begin{aligned} |\langle \psi, \bar{\partial}u \rangle_X| &= \lim_{\mu \rightarrow \infty} |\langle N_\mu\varphi, \bar{\partial}u \rangle_{X_\mu}| \\ &= \lim_{\mu \rightarrow \infty} |\langle \bar{\partial}_\mu^*N_\mu\varphi, u \rangle_{X_\mu}| \\ &\leq \|\varphi\|_X \|u\|_X. \end{aligned}$$



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Thus $\psi \in \text{Dom}_{(p,q)}(\bar{\partial}_\mu^*)$. Also, we show that $\bar{\partial}\psi \in \text{Dom}_{(p,q+1)}(\bar{\partial}_\mu^*)$ and $\bar{\partial}_\mu^*\psi \in \text{Dom}_{(p,q-1)}(\bar{\partial})$ as follows: by using (ii) in Proposition 2.1, we have

$$(3.6) \quad \|\bar{\partial}\bar{\partial}_\mu^*N_\mu\varphi\|_{X_\mu}^2 + \|\bar{\partial}_\mu^*\bar{\partial}N_\mu\varphi\|_{X_\mu}^2 = \|\varphi\|_{X_\mu}^2 \leq \|\varphi\|_X^2.$$

Thus $\bar{\partial}\bar{\partial}_\mu^*\psi$ is the L^2 weak limit of some subsequence of $\bar{\partial}\bar{\partial}_\mu^*N_\mu\varphi$ and $\bar{\partial}_\mu^*\psi \in \text{Dom}_{(p,q-1)}(\bar{\partial})$. By using (3.6), we have, for any $v \in \text{Dom}_{(p,q)}(\bar{\partial}) \cap L^2_{(p,q)}(X)$,

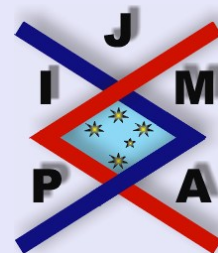
$$\begin{aligned} |\langle \bar{\partial}\psi, \bar{\partial}v \rangle_X| &= \lim_{\mu \rightarrow \infty} \left| \langle \bar{\partial}N_\mu\varphi, \bar{\partial}v \rangle_{X_\mu} \right| \\ &= \lim_{\mu \rightarrow \infty} \left| \langle \bar{\partial}_\mu^*\bar{\partial}N_\mu\varphi, v \rangle_{X_\mu} \right| \\ &\leq \|\varphi\|_X \|v\|_X. \end{aligned}$$

Thus $\bar{\partial}\psi \in \text{Dom}_{(p,q+1)}(\bar{\partial}_\mu^*)$ and $\bar{\partial}_\mu^*\bar{\partial}\psi$ is the weak limit of a subsequence of $\bar{\partial}_\mu^*\bar{\partial}N_\mu\varphi$. This implies that $\psi \in \text{Dom}_{(p,q)}(\square_\mu)$ and $\square_\mu\psi = \varphi$. Since N is one to one on $L^2_{(p,q)}(X)$, then we conclude that $\psi = N\varphi$. Since X_μ is a Lipschitz domain. Hence $\Lambda(\bar{X})$ are dense in $W^s(X)$ in $W^s(X)$ -norm. If $s \leq 1/2$, we can show that $\Lambda_0(\bar{X})$ are dense in $W^s(X)$ as in Theorem 1.4.2.4 in Grisvard [13]. Thus

$$W^{1/2}(X) = W_0^{1/2}(X).$$

It follows from the Generalized Schwartz inequality (see Proposition (A.1.1) in Folland and Kohn [12]) that

$$\left| \langle h, f \rangle_{X_\mu} \right| \leq \|h\|_{1/2(X_\mu)} \|f\|_{-1/2(X_\mu)},$$



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for any $h \in W_{(p,q)}^{1/2}(X_\mu)$ and $f \in W_{(p,q)}^{-1/2}(X_\mu)$. By using (3.1), there exists a constant $c_4 > 0$ such that for any $\varphi \in L_{(p,q)}^2(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\square_\mu)$, $0 \leq p \leq n$ and $1 \leq q \leq n$,

$$\begin{aligned} \|\varphi\|_{1/2(X_\mu)}^2 &\leq c_4(\|\bar{\partial}\varphi\|_{X_\mu}^2 + \|\bar{\partial}_\mu^*\varphi\|_{X_\mu}^2) \\ &= c_4 \langle \varphi, \square_\mu \varphi \rangle_{X_\mu} \\ (3.7) \quad &\leq c_4 \|\varphi\|_{1/2(X_\mu)} \|\square_\mu \varphi\|_{-1/2(X_\mu)}, \end{aligned}$$

where c_4 is independent of φ and μ . Substituting $N_\mu \varphi$ into (3.7), we have

$$(3.8) \quad \|N_\mu \varphi\|_{1/2(X_\mu)} \leq c_4 \|\square_\mu N_\mu \varphi\|_{-1/2(X_\mu)} = c_4 \|\varphi\|_{-1/2(X_\mu)},$$

where c_4 is independent of φ and μ . By using the extension operator on Euclidean space (see Theorem 1.4.3.1 in Grisvard [13]), it follows that for any Lipschitz domain $X_\mu \subset \mathbb{C}^n$,

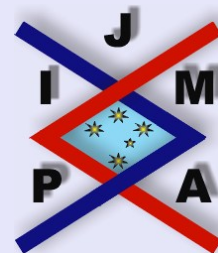
$$R_\mu : W^{1/2}(X_\mu) \longrightarrow W^{1/2}(\mathbb{C}^n)$$

such that for each $\varphi \in W^{1/2}(X_\mu)$, $R_\mu \varphi = \varphi$ on X_μ and

$$(3.9) \quad \|R_\mu \varphi\|_{1/2(\mathbb{C}^n)} \leq c_5 \|\varphi\|_{1/2(X_\mu)},$$

for some positive constant c_5 . The constant c_5 in (3.9) can be chosen independent of μ since extension exists for any Lipschitz domain (see Theorem 1.4.3.1 in Grisvard [13]). By applying R_μ to $N_\mu \varphi$ component-wise, we have, by using (3.8) and (3.9), that

$$\|R_\mu N_\mu \varphi\|_{1/2(X)} \leq \|R_\mu N_\mu \varphi\|_{1/2(\mathbb{C}^n)} \leq c_5 \|N_\mu \varphi\|_{1/2(X_\mu)} \leq c \|\varphi\|_{-1/2(X_\mu)},$$



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where $c > 0$ is independent of μ . Since $W_{(p,q)}^{1/2}(X)$ is a Hilbert space, then from the weak compactness for Hilbert spaces, there exists a subsequence of $R_\mu N_\mu \varphi$ which converges weakly in $W_{(p,q)}^{1/2}(X)$. Since $R_\mu N_\mu \varphi$ converges weakly to $N\varphi$ in $L_{(p,q)}^2(X)$, we conclude that $N\varphi \in W_{(p,q)}^{1/2}(X)$ and

$$\|N\varphi\|_{1/2(X)} \leq \lim_{\mu \rightarrow \infty} \|R_\mu N_\mu \varphi\|_{1/2(X_\mu)} \leq c \|\varphi\|_{-1/2(X)}.$$

Thus, N can be extended as a bounded operator from $W_{(p,q)}^{-1/2}(X)$ to $W_{(p,q)}^{1/2}(X)$.

To prove that N is compact, we note that for any bounded domain X with Lipschitz boundary there exists a continuous linear operator

$$R : W^{1/2}(X) \longrightarrow W^{1/2}(\mathbb{C}^n)$$

such that $R\phi|_X = \phi$. Also, we note that the inclusion map

$$W^{1/2}(X) \longrightarrow L^2(X) = W^0(X)$$

is compact. Thus, by using the Rellich Lemma for \mathbb{C}^n , we conclude that

$$W^{1/2}(X) \hookrightarrow W^{-1/2}(X)$$

is compact and this proves that N is compact on $W_{(p,q)}^{-1/2}(X)$ and $L_{(p,q)}^2(X)$. \square



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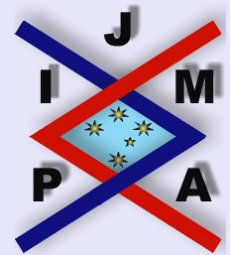
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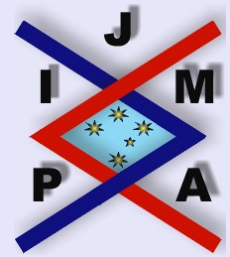
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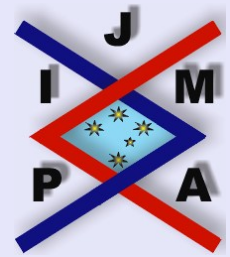
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