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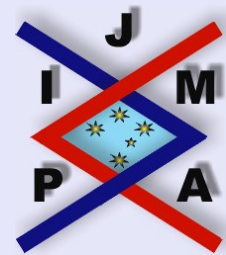
ON HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETERS

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Abstract

In this paper, by means of a sharpening of Hölder's inequality, Hardy-Hilbert's integral inequality with parameters is improved. Some new inequalities are established.

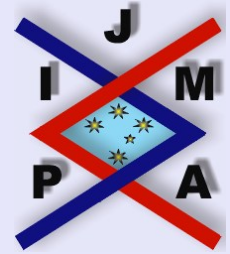
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Key words: Hardy-Hilbert integral inequality, Hölder's inequality, Weight function, Beta function.

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1. Introduction

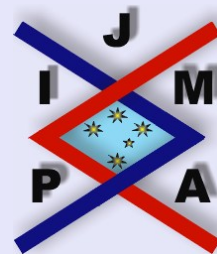
Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g > 0$. If $0 < \int_0^\infty f^p(t)dt < +\infty, 0 < \int_0^\infty g^q(t)dt < +\infty$, then

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t) dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t) dt \right)^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is best possible. The inequality (1.1) is well known as Hardy-Hilbert's integral inequality. In recent years, some improvements and extensions of Hilbert's inequality and Hardy-Hilbert's inequality have been given in [2] – [6], Yang [2] gave a generalization of (1.1) as follows:

If $\lambda > 2 - \min\{p, q\}, \alpha < T \leq \infty$ then

$$(1.2) \quad \int_\alpha^T \int_\alpha^T \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dx dy < \left\{ \int_\alpha^T \left[k_\lambda(p) - \theta_\lambda(p) \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{p+\lambda-2}{p}} \right] (t-\alpha)^{1-\lambda} f^p(t) dt \right\}^{\frac{1}{p}} \\ \times \left\{ \int_\alpha^T \left[k_\lambda(q) - \theta_\lambda(q) \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{q+\lambda-2}{q}} \right] (t-\alpha)^{1-\lambda} g^q(t) dt \right\}^{\frac{1}{q}} \\ (T < \infty)$$



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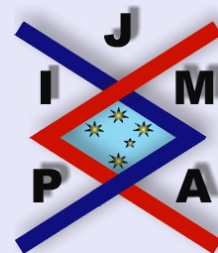
and

$$(1.3) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy \\ < k_{\lambda}(p) \left\{ \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} f^p(t) dt \right\}^{\frac{1}{p}} \left\{ \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} g^q(t) dt \right\}^{\frac{1}{q}},$$

where

$$k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right), \\ \theta_{\lambda}(r) = \int_0^1 \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u}\right)^{(2-\lambda)/r} du \quad (r = p, q).$$

The main purpose of this paper is to build a few new inequalities which include improvements of the inequalities (1.2) and (1.3), and extensions of corresponding results in [3] – [5].



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2. Lemmas and their Proofs

For convenience, we firstly introduce some notations:

$$(f^r, g^s) = \int_{\alpha}^T f^r(x)g^s(x) dx, \quad \|f\|_p = \left(\int_{\alpha}^T f^p(x) dx \right)^{\frac{1}{p}}, \quad \|f\|_2 = \|f\|.$$

We next introduce a function defined by

$$S_r(H, x) = (H^{r/2}, x) \|H\|_r^{-r/2},$$

where x is a parametric variable vector which is a variable unit vector. Under the general case, it is properly chosen such that the specific problems discussed are simplified.

Clearly, $S_r(H, x) = 0$ when the vector x selected is orthogonal to $H^{p/2}$. Throughout this paper, the exponent m indicates $m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$, $\alpha < T \leq \infty$.

In order to verify our assertions, we need to build the following lemmas.

Lemma 2.1. *Let $f(x), g(x) > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then*

$$(2.1) \quad (f, g) < \|f\|_p \|g\|_q (1 - R)^m,$$

where $R = (S_p(f, h) - S_q(g, h))^2$, $\|h\| = 1$, $f^{p/2}(x)$, $g^{q/2}(x)$ and $h(x)$ are linearly independent.



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Proof. First of all, we discuss the case of $p \neq q$. Without loss of generality, suppose that $p > q > 1$, since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$. Then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain,

$$\begin{aligned}
 (2.2) \quad (f, g) &= \int_a^T f(x)g(x) dx \\
 &= \int_a^T (f \cdot g^{q/p})g^{1-(q/p)} dx \\
 &\leq \left(\int_a^T (f \cdot g^{q/p})^R dx \right)^{\frac{1}{R}} \left(\int_a^T (g^{1-(q/p)})^Q dx \right)^{\frac{1}{Q}} \\
 &= (f^{p/2}, g^{q/2})^{\frac{2}{p}} \|g\|_q^{q(1-\frac{2}{p})}.
 \end{aligned}$$

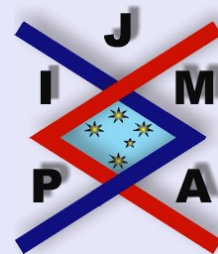
And the equality in (2.2) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, the equality in (2.2) holds if and only if, there exists a c_1 such that $(f \cdot g^{q/p})^R = c_1 (g^{1-(q/p)})^Q$. It is easy to deduce that $f^{p/2} = c_1 g^{q/2}$.

In our previous paper [3], with the help of the positive definiteness of the Gram matrix, we established an important inequality of the form

$$(2.3) \quad (\alpha, \beta)^2 \leq \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\| x - \|\beta\| y)^2 = \|\alpha\|^2 \|\beta\|^2 (1 - \bar{\gamma})$$

where $\bar{\gamma} = \left(\frac{y}{\|\alpha\|} - \frac{x}{\|\beta\|} \right)^2$, $x = (\beta, \gamma)$, $y = (\alpha, \gamma)$ with $\|\gamma\| = 1$ and $xy \geq 0$. The equality in (2.3) holds if and only if α and β are linearly dependent; or the vector γ is a linear combination of α and β , and $xy = 0$ but $x \neq y$. If α , β and γ in (2.3) are replaced by $f^{p/2}$, $g^{q/2}$ and h respectively, then we get

$$(2.4) \quad (f^{p/2}, g^{q/2})^2 \leq \|f\|_p^p \|g\|_q^q (1 - R),$$



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where $R = (S_p(f, h) - S_q(g, h))^2$ with $\|h\| = 1$. The equality in (2.4) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent, or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{p/2}, h)(g^{q/2}, h) = 0$, but $(f^{p/2}, h) \neq (g^{q/2}, h)$. Since $f^{p/2}$ and $g^{q/2}$ are linearly independent, it is impossible to have equality in (2.4). Substituting (2.4) into (2.2), we obtain after simplifications

$$(2.5) \quad (f, g) < \|f\|_p \|g\|_q (1 - R)^{\frac{1}{p}}.$$

Provided that $h(x)$ is properly chosen, then $R \neq 0$ is achieved. (The choice of $h(x)$ is quite flexible, as long as condition $\|h\| = 1$ is satisfied, on which we can refer to [3, 4], etc.). Noticing the symmetry of p and q , the inequality (2.1) follows from (2.5).

Next, we discuss the case of $p = q$. According to the hypothesis: when f, g and h are linearly independent, we immediately obtain from (2.3) the following result:

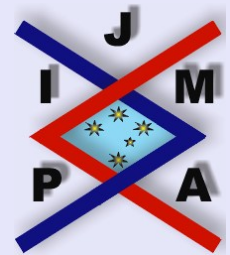
$$(f, g) < \|f\| \|g\| (1 - \bar{r})^{\frac{1}{2}},$$

where $\bar{r} = \left(\frac{(f, h)}{\|f\|} - \frac{(g, h)}{\|g\|} \right)^2$, and $\|h\| = 1$. Thus the lemma is proved. \square

Lemma 2.2. Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, \alpha < T < \infty$. Define the weight function ω_λ as follows:

$$(2.6) \quad \omega_\lambda(\alpha, T, r, x) = \int_\alpha^T \frac{1}{(x + y - 2\alpha)^\lambda} \left(\frac{x - \alpha}{y - \alpha} \right)^{\frac{2-\lambda}{r}} dy \quad x \in (\alpha, T].$$

Setting $\omega_\lambda(\alpha, \infty, r, x) = \lim_{T \rightarrow \infty} \omega_\lambda(\alpha, T, r, x)$ and $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$,



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$$(2.7) \quad \bar{\theta}_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{(2-\lambda)(1-1/r)} du, \quad (r = p, q),$$

then we have

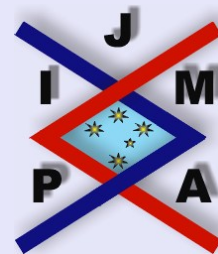
$$(2.8) \quad \omega_\lambda(\alpha, \infty, r, x) = k_\lambda(p)(x - \alpha)^{1-\lambda}, \quad x \in (\alpha, \infty)$$

and

$$(2.9) \quad \omega_\lambda(\alpha, T, r, x) < \left(k_\lambda(p) - \bar{\theta}(r) \left(\frac{x - \alpha}{T - \alpha} \right)^{1+(\lambda-2)(1-1/r)} \right) (x - \alpha)^{1-\lambda}, \quad x \in (\alpha, T),$$

where $B(m, n)$ is the beta function.

The proof of this lemma is given in the paper [2]; it is omitted here.



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3. Main Results

In order to state it conveniently, we need again to define the functions and introduce some notations

$$F = \frac{f(x)}{(x+y-2\alpha)^{\lambda/p}} \left(\frac{x-\alpha}{y-\alpha} \right)^{\frac{2-\lambda}{pq}}, \quad G = \frac{g(y)}{(x+y-2\alpha)^{\lambda/q}} \left(\frac{y-\alpha}{x-\alpha} \right)^{\frac{2-\lambda}{pq}},$$

$$S_p(F, h_T) = \left\{ \int_{\alpha}^T \int_{\alpha}^T F^{p/2} h_T dx dy \right\} \left\{ \int_{\alpha}^T \int_{\alpha}^T F^p dx dy \right\}^{-\frac{1}{2}},$$

$$S_q(G, h_T) = \left\{ \int_{\alpha}^T \int_{\alpha}^T G^{q/2} h_T dx dy \right\} \left\{ \int_{\alpha}^T \int_{\alpha}^T G^q dx dy \right\}^{-\frac{1}{2}},$$

where $h_T = h_T(x, y)$ is a unit vector with two variants, namely

$$\|h_T\| = \left\{ \int_{\alpha}^T \int_{\alpha}^T h_T^2 dx dy \right\}^{\frac{1}{2}} = 1, \quad \alpha < T \leq \infty,$$

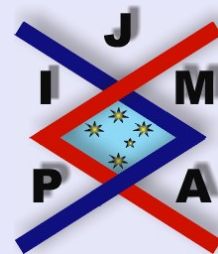
and $F^{p/2}$, $G^{q/2}$, h_T are linearly independent.

Theorem 3.1. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $\alpha < T \leq \infty$, $f(t), g(t) > 0$. If

$$0 < \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} f^p(t) dt < +\infty \quad \text{and}$$

$$0 < \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} g^q(t) dt < +\infty,$$

then



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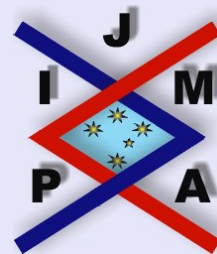


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(i) For $T < \infty$, we have

$$(3.1) \quad \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy$$

$$< \left\{ \int_{\alpha}^T \left(k_{\lambda}(p) - \theta_{\lambda}(p) \left(\frac{t-\alpha}{T-\alpha} \right)^{(p+\lambda-2)/p} \right) (t-\alpha)^{1-\lambda} f^p(t) dt \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_{\alpha}^T \left(k_{\lambda}(q) - \theta_{\lambda}(q) \left(\frac{t-\alpha}{T-\alpha} \right)^{(q+\lambda+2)/q} \right) (t-\alpha)^{1-\lambda} g^q(t) dt \right\}^{\frac{1}{q}} (1-R_T)^m,$$

where

$$k_{\lambda}(p) = B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right),$$

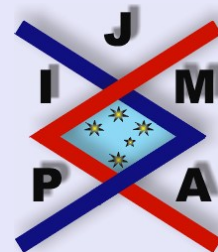
$$\theta_{\lambda}(r) = \int_0^1 \frac{1}{(1+u)^{\lambda}} \left(\frac{1}{u} \right)^{\frac{2-\lambda}{r}} du \quad (r = p, q).$$

(ii) For $T = \infty$, we have

$$(3.2) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy$$

$$< k_{\lambda}(p) \left(\int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} f^p(t) dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} g^q(t) dt \right)^{\frac{1}{q}} (1-R_{\infty})^m,$$



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where $R_T = (S_p(F, h_T) - S_q(G, h_T))^2$,

$$(3.3) \quad h_T(x, y) = \begin{cases} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{a-x}}{(x+y-2a)^{\frac{1}{2}}} \left(\frac{x-a}{y-a}\right)^{\frac{1}{4}}, & T = \infty; \\ \frac{T-\alpha}{(x-\alpha)(y-\alpha)} e^{\left(1-\frac{T-\alpha}{2(x-\alpha)}-\frac{T-\alpha}{2(y-\alpha)}\right)}, & T < \infty. \end{cases}$$

Proof. By Lemma 2.1, we get

$$(3.4) \quad \begin{aligned} & \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy \\ &= \int_{\alpha}^T \int_{\alpha}^T FG dx dy \\ &\leq \left\{ \int_{\alpha}^T \int_{\alpha}^T F^p dx dy \right\}^{\frac{1}{p}} \left\{ \int_{\alpha}^T \int_{\alpha}^T G^q dx dy \right\}^{\frac{1}{q}} (1 - R_T)^m \\ &= \left(\int_{\alpha}^T \omega_{\lambda}(\alpha, \beta, q, t) f^p(t) dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\alpha}^T \omega_{\lambda}(\alpha, \beta, p, t) g^q(t) dt \right)^{\frac{1}{q}} (1 - R_T)^m, \end{aligned}$$

where $\omega_{\lambda}(\alpha, T, r, t)$ ($r = p, q$) is the function defined by (2.6).

Now notice that $\theta_{\lambda}(p) = \bar{\theta}_{\lambda}(q)$, $\theta_{\lambda}(q) = \bar{\theta}_{\lambda}(p)$ and substituting (2.9) and (2.8) into (3.4) respectively, the inequalities (3.1) and (3.2) follow.

It remains to discuss the expression of R_T . We may choose the function h_T indicated by (3.3).

When $T = \infty$, setting $s = x - \alpha$, $t = y - \alpha$, then

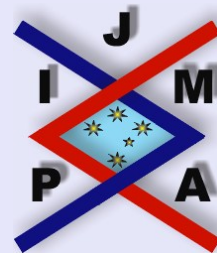
$$\begin{aligned} \|h_\infty\| &= \left(\int_\alpha^\infty \int_\alpha^\infty h_\infty^2(x, y) dx dy \right)^{\frac{1}{2}} \\ &= \left\{ \frac{2}{\pi} \int_0^\infty e^{-2s} ds \int_0^\infty \frac{1}{s+t} \left(\frac{s}{t}\right)^{\frac{1}{2}} dt \right\}^{\frac{1}{2}} = 1. \end{aligned}$$

When $T < \infty$, setting $\xi = \frac{T-\alpha}{x-\alpha}$, $\eta = \frac{T-\alpha}{y-\alpha}$, then we have

$$\begin{aligned} \|h_T\| &= \left(\int_\alpha^T \int_\alpha^T h_T^2 dx dy \right)^{\frac{1}{2}} \\ &= \left\{ \int_\alpha^T \frac{T-\alpha}{(x-\alpha)^2} e^{(1-\frac{T-\alpha}{x-\alpha})} dx \cdot \int_\alpha^T \frac{T-\alpha}{(y-\alpha)^2} e^{(1-\frac{T-\alpha}{y-\alpha})} dy \right\}^{\frac{1}{2}} \\ &= \left\{ \int_1^\infty e^{1-\xi} d\xi \cdot \int_1^\infty e^{1-\eta} d\eta \right\}^{\frac{1}{2}} = 1. \end{aligned}$$

According to Lemma 2.1 and the given h_T , we have $R_T = (S_p(F, h_T) - S_q(G, h_T))^2$. It is obvious that $F^{p/2}$, $G^{q/2}$ and h_T are linearly independent, so it is impossible for equality to hold in (3.4). Thus the proof of theorem is completed. \square

Remark 3.1. Clearly, the inequalities (3.1) and (3.2) are the improvements of (1.2) and (1.3) respectively.



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Owing to $p, q > 1$, when $\lambda = 1, 2, 3$, the condition $\lambda > 2 - \min(p, q)$ is satisfied, then we have

$$\theta_1(r) = \int_0^1 \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{r}} du > \int_0^1 \frac{1}{1+u} du = \ln 2,$$

$$k_1(p) = B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2},$$

$$k_2(p) = B\left(\frac{p+2-2}{p}, \frac{q+2-2}{q}\right) = B(1, 1) = 1,$$

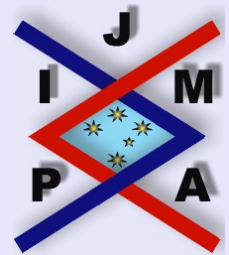
$$\theta_3(r) = \int_0^1 \frac{1}{(1+u)^3} \left(\frac{1}{u}\right)^{-\frac{1}{r}} du > \int_0^1 \frac{u}{(1+u)^3} du = \frac{1}{8},$$

$$k_3(p) = \frac{1}{2pq} B\left(\frac{1}{q}, \frac{1}{p}\right) = \frac{(p-1)\pi}{2p^2 \sin(\pi/p)}.$$

By Theorem 3.1, some corollaries are established as follows:

Corollary 3.2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = 1$, $\alpha < T \leq \infty$ and $f(t), g(t) > 0$, $0 < \int_{\alpha}^T f^p(t) dt < +\infty$ and $0 < \int_{\alpha}^T g^q(t) dt < +\infty$, then we have*

$$(3.5) \quad \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{x+y-2\alpha} dx dy < \left\{ \int_{\alpha}^T \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{1}{q}} \cdot \ln 2 \right) f^p(t) dt \right\}^{\frac{1}{p}}$$



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$$\times \left\{ \int_{\alpha}^T \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{t-\alpha}{T-\alpha} \right)^{\frac{1}{p}} \cdot \ln 2 \right) \cdot g^q(t) dt \right\}^{\frac{1}{q}} (1-r_1)^m,$$

for $T < \infty$,

and

$$(3.6) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{x+y-2\alpha} dx dy$$

$$< \frac{\pi}{\sin(\pi/p)} \left(\int_{\alpha}^{\infty} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{\alpha}^{\infty} g^q(t) dt \right)^{\frac{1}{q}} (1-\bar{r}_1)^m.$$

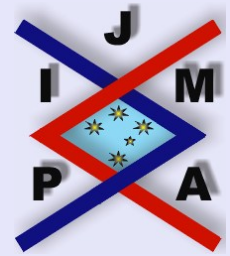
Remark 3.2. When $\alpha = 0$ and $p = q = 2$, the inequality (3.6) is reduced to a result which is equivalent to inequality (3.1) in [3] after simple computations. As a result, the inequalities (3.1), (3.2) and (3.5) – (3.6) are all extensions of (3.1) in [3].

Corollary 3.3. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < T \leq \infty$ and $f(t), g(t) > 0$. If

$$0 < \int_{\alpha}^T \frac{1}{t-\alpha} f^p(t) dt < +\infty$$

and

$$0 < \int_{\alpha}^T \frac{1}{t-\alpha} g^q(t) dt < +\infty,$$



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then we obtain

$$(3.7) \quad \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{(x+y-2\alpha)^2} dx dy$$

$$< \left\{ \int_{\alpha}^T \left(1 - \frac{t-\alpha}{2(T-\alpha)} \right) \frac{1}{t-\alpha} \cdot f^p(t) dt \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_{\alpha}^T \left(1 - \frac{t-\alpha}{2(T-\alpha)} \right) \frac{1}{t-\alpha} \cdot g^q(t) dt \right\}^{\frac{1}{q}} (1-r_2)^m,$$

for $T < \infty$,

and

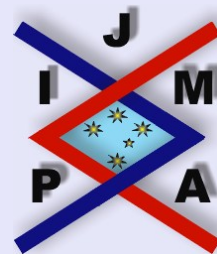
$$(3.8) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^2} dx dy$$

$$< \left(\int_{\alpha}^{\infty} \frac{1}{t-\alpha} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{\alpha}^{\infty} \frac{1}{t-\alpha} g^q(t) dt \right)^{\frac{1}{q}} (1-\bar{r}_2)^m.$$

Corollary 3.4. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = 3$, $\alpha < T \leq \infty$ and $f(t), g(t) > 0$,

$$0 < \int_{\alpha}^T \frac{1}{(t-\alpha)^2} f^p(t) dt < +\infty,$$

$$0 < \int_{\alpha}^T \frac{1}{(t-\alpha)^2} g^q(t) dt < +\infty,$$



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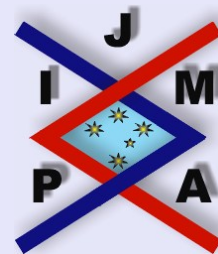
then we get

$$\begin{aligned}
 (3.9) \quad & \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{(x+y-2\alpha)^3} dx dy \\
 & < \left\{ \int_{\alpha}^T \left(\frac{(p-1)\pi}{2p^2 \sin(\pi/p)} - \frac{1}{8} \left(\frac{t-\alpha}{T-\alpha} \right)^{1+\frac{1}{p}} \right) \frac{1}{(t-\alpha)^2} f^p(t) dt \right\}^{\frac{1}{p}} \\
 & \times \left\{ \int_{\alpha}^T \left(\frac{(p-1)\pi}{2p^2 \sin(\pi/p)} - \frac{1}{8} \left(\frac{t-\alpha}{T-\alpha} \right)^{1+\frac{1}{q}} \right) \frac{1}{(t-\alpha)^2} g^q(t) dt \right\}^{\frac{1}{q}} (1-r_3)^m \\
 & \qquad \qquad \qquad T < \infty,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.10) \quad & \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^3} dx dy \\
 & < \frac{(p-1)\pi}{2p^2 \sin(\pi/p)} \left(\int_{\alpha}^{\infty} \frac{1}{(t-\alpha)^2} f^p(t) dt \right)^{\frac{1}{p}} \\
 & \qquad \qquad \qquad \times \left(\int_{\alpha}^{\infty} \frac{1}{(t-\alpha)^2} g^q(t) dt \right)^{\frac{1}{q}} (1-\bar{r}_3)^m.
 \end{aligned}$$

Since $k_{\lambda}(2) = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $\theta_{\lambda}(2) = \frac{1}{2}B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, and $\lambda > 2 - \min(2, 2) = 0$, we also have



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Corollary 3.5. If $p = q = 2$, $\lambda > 0$, $\alpha < T \leq \infty$ and $f(t), g(t) > 0$,

$$0 < \int_{\alpha}^T (t - \alpha)^{1-\lambda} f^2(t) dt < +\infty,$$

$$0 < \int_{\alpha}^T (t - \alpha)^{1-\lambda} g^2(t) dt < +\infty,$$

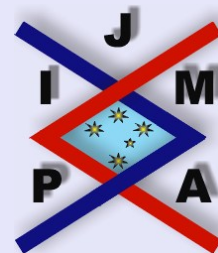
then we have

$$\begin{aligned} (3.11) \quad & \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy \\ & < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_{\alpha}^T \left[1 - \frac{1}{2} \left(\frac{t-\alpha}{T-\alpha} \right)^{\lambda/2} \right] (t-\alpha)^{1-\lambda} f^2(t) dt \right\}^{\frac{1}{2}} \\ & \times \left\{ \int_{\alpha}^T \left[1 - \frac{1}{2} \left(\frac{t-\alpha}{T-\alpha} \right)^{\lambda/2} \right] (t-\alpha)^{1-\lambda} g^2(t) dt \right\}^{\frac{1}{2}} (1 - \bar{R})^m, \end{aligned}$$

for $T < \infty$

and

$$\begin{aligned} (3.12) \quad & \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y-2\alpha)^{\lambda}} dx dy \\ & < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} f^2(t) dt \right\}^{\frac{1}{2}} \\ & \times \left\{ \int_{\alpha}^{\infty} (t-\alpha)^{1-\lambda} g^2(t) dt \right\}^{\frac{1}{2}} (1 - \tilde{R})^m. \end{aligned}$$



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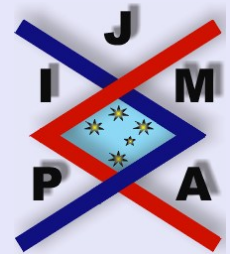
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Remark 3.3. The inequalities (3.11), (3.12) are new generalizations of (20) in [4] and improvements of the inequalities (4) and (12) in [6] respectively.



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- [1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1952.
- [2] BICHENG YANG, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, **261** (2001), 295–306.
- [3] MINGZHE GAO, LI TAN AND L. DEBNATH, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, **229** (1999), 682–689.
- [4] MINGZHE GAO, On the Hilbert inequality, *Zeitschrift für Analysis und ihre Anwendungen*, **18**(4) (1999), 1117–1122.
- [5] MINGZHE GAO, SHANG RONG WEI AND LEPING HE, On the Hilbert inequality with weights, *Zeitschrift für Analysis und ihre Anwendungen*, **21**(1) (2002), 257–263.
- [6] BICHENG YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1998), 778–785.



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