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# A GENERAL $L_2$ INEQUALITY OF GRÜSS TYPE

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## Abstract

Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general $L_2$ inequality of Grüss type, which includes some existing results as special cases.



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## Contents

# 1. Introduction

Inequalities of the Grüss type have been the subject renewed research interest in the past few years. The monograph [3] has had much impact on the stream of current research in this area.

Inequalities of the Grüss type can be found in e.g. [4, 5, 6, 7, 8, 9, 10] and references therein.

Recently, N. Ujević in [10] proved the following two theorems among others.

**Theorem 1.1.** *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function, whose derivatives $f' \in L_2[0,1]$. Then,*

$$\left| \frac{1}{6}\left[ f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] - \int_0^1 f(t)dt \right| \leq \frac{1}{6}\sqrt{\sigma(f')}, \tag{1.1}$$

*where $\sigma(\cdot)$ is defined by*

$$\sigma(f) = \|f\|_2^2 - \left( \int_0^1 f(t)dt \right)^2. \tag{1.2}$$

*Inequality (1.1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.*

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, for any $x \in [0,1]$, we have*

$$\left| f(x) - \left(x - \frac{1}{2}\right)[f(1) - f(0)] - \int_0^1 f(t)dt \right| \leq \frac{1}{2\sqrt{3}}\sqrt{\sigma(f')}. \tag{1.3}$$

*Inequality (1.3) is sharp in the sense that the constant $1/(2\sqrt{3})$ cannot be replaced by a smaller one.*

Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general $L_2$ inequality of the Grüss type under very natural assumptions. Our results improve and generalize some existing observations.

## 2. A $L_2$ version of Grüss Type Inequality

In what follows, let $f$ be defined on $[0,1]$,

$$||f||_2 = \left(\int_0^1 f(t)dt\right)^2$$

and $L_2[0,1] = \{f| \; ||f||_2 < \infty\}$.

Some more notations and the following lemmas are needed before we proceed. In the rest of the paper, a standing assumption is that $x \in [0,1]$, $n$ is a positive integer and $0 = t_0 < t_1 < \cdots < t_n = 1$ is an equidistant subdivision of the interval $[0,1]$ such that $t_{i+1} - t_i = h = 1/n$, $i = 0, 1, \ldots, n-1$.

We start with the following lemma.

**Lemma 2.1 ([1], cf. [11]).** *Let $f : [0,1] \to \mathbb{R}$ be such that its $(k-1)$th derivative $f^{(k-1)}$ is absolutely continuous for some positive integer $k$. Then for any $x \in [0,1]$, we have the Euler-Maclaurin formula*

$$\int_0^1 f(t)dt = Q_k(f,x) + E_k(Q_k; f, x), \tag{2.1}$$

*where*

$$\begin{aligned} Q_k(f,x) &= h\sum_{i=0}^{n-1} f(t_i + xh) - \sum_{\nu=1}^{k} \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!} B_\nu(x) h^\nu, \\ E_k(Q_k; f, x) &= \frac{h^k}{k!}\int_0^1 \widetilde{B}_k(x - nt) f^{(k)}(t)dt, \end{aligned} \tag{2.2}$$

*and $\widetilde{B}_k(t) := B_k(t - \lfloor t \rfloor)$ where $B_k(t)$ is the $k$th Bernoulli polynomial.*

**Lemma 2.2.** *For any $x, y \in [0,1]$, we have*

$$\int_0^1 \widetilde{B}_k(x-t)\widetilde{B}_k(y-t)dt = \frac{(-1)^{k-1}(k!)^2}{(2k)!}\widetilde{B}_{2k}(x-y). \tag{2.3}$$

*Proof.* We use a technique of [2]. Setting $n = 1$, $t_0 = 0$ and

$$f(t) = \frac{(-1)^k k!}{(2k)!}\widetilde{B}_{2k}(x-t),$$

in (2.1), then we have

$$f^{(k)}(t) = \widetilde{B}_k(x-t).$$

By the periodicity of $\widetilde{B}_{2k}(t)$ and the property of $B_{2k}(t)$ (see e.g. [12]), we can easily get

$$\int_0^1 \widetilde{B}_{2k}(x-t)dt = \int_0^1 B_{2k}(t)dt = 0. \tag{2.4}$$

Then we have

$$\int_0^1 f(t)dt = 0. \tag{2.5}$$

From (2.2) and the periodicity of this special function $f$, we have for any $y \in [0,1]$

$$\begin{aligned} Q_k(f,y) &= f(y) - \sum_{\nu=1}^{k} \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!}B_\nu(y) = f(y), \\ E_k(Q_k;f,y) &= \frac{1}{k!}\int_0^1 \widetilde{B}_k(y-t)\widetilde{B}_k(x-t)dt. \end{aligned} \tag{2.6}$$

Now from (2.1), (2.5) and (2.6), (2.3) follows. $\square$

By Lemmas 2.1 and 2.2, we have

**Corollary 2.3.** *Suppose the conditions in Lemma 2.1 hold, then we have*

$$E_k(Q_k; f, x) \leq h^k c_k(2) \|f^{(k)}\|_2, \tag{2.7}$$

*where*

$$c_k(2) = \sqrt{\frac{(-1)^{k-1}}{(2k)!} B_{2k}}.$$

**Remark 1.** *From Corollary 2.3, the right side of (2.7) is independent of $x$.*

**Lemma 2.4 ([1], cf. [11]).** *Suppose that the following quadrature rule*

$$\int_0^1 f(t)dt = \sum_{j=0}^{m-1} p_j f(x_j) \tag{2.8}$$

*is exact for any polynomial of degree $\leq k-1$ for some positive integer $k$. Let $f : [0,1] \to \mathbb{R}$ be such that its $(k-1)$th derivative $f^{(k-1)}$ is absolutely continuous. Then we have*

$$\int_0^1 f(t)dt = \overline{Q}(f) + E_k(\overline{Q}; f), \tag{2.9}$$

*where*

$$\overline{Q}(f) = h\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} p_j f(t_i + x_j h),$$

(2.10)

$$E_k(\overline{Q}; f) = \frac{h^k}{k!}\int_0^1 g_k(nt) f^{(k)}(t)dt,$$

*and*

(2.11)
$$g_k(t) = \sum_{j=0}^{m-1} p_j(\widetilde{B}_k(x_j - t) - B_k(x_j)).$$

By the Hölder inequality, we have

(2.12)
$$|E_k(\overline{Q}; f)| \le \overline{c}_k(2)||f^{(k)}||_2,$$

where

(2.13)
$$\overline{c}_k(2) = \frac{h^k}{k!}||g_k||_2.$$

**Remark 2.** *It is easy to see that* (2.12) *is sharp in the sense that the constant* $\overline{c}_k(2)$ *cannot be replaced by a smaller one.*

We are now able to find an explicit expression for $\overline{c}_k(2)$.

**Theorem 2.5.** *Suppose that the quadrature rule (2.3) is exact for any polynomial of degree $\leq k-1$ for some positive integer $k$. Then the following equality is valid.*

(2.14)
$$\overline{c}_k(2) = \frac{h^k}{k!}\left\{\sum_{i,j=0}^{m-1} p_i p_j \left(\frac{(-1)^{k-1}(k!)^2}{(2k)!}\widetilde{B}_{2k}(x_i-x_j)+B_k(x_i)B_k(x_j)\right)\right\}^{\frac{1}{2}}.$$

*Proof.* A straightforward computation on using (2.4) and Lemma 2.2 gives

$$\begin{aligned}
\|g_k\|_2^2 &= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\widetilde{B}_k(x_i-t)-B_k(x_i)\right)\left(\widetilde{B}_k(x_j-t)-B_k(x_j)\right)dt \\
&= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\widetilde{B}_k(x_i-t)\widetilde{B}_k(x_j-t)+B_k(x_i)B_k(x_j)\right)dt \\
&= \frac{(-1)^{k-1}(k!)^2}{(2k)!}\sum_{i,j=0}^{m-1} p_i p_j \widetilde{B}_{2k}(x_i-x_j)+\frac{1}{k!}\sum_{i,j=0}^{m-1} p_i p_j B_k(x_i)B_k(x_j),
\end{aligned}$$

which in combination with (2.13) proves the conclusion as desired. □

# 3. Examples

**Example 3.1.** *For the Trapezoid rule,* $m = 2, x_0 = 0, x_1 = 1, p_0 = p_1 = 1/2$*. It is well known that the Trapezoid rule has degree of precision 1 (*$k = 2$*). A direct calculation using (2.14) yields*

$$\bar{c}_1(2) = \frac{h}{2\sqrt{3}}, \qquad \bar{c}_2(2) = \frac{h^2}{2\sqrt{30}}.$$

*If* $f$ *is absolutely continuous, then we can obtain*

$$\left|\frac{1}{2}\left[f(0) + f(1)\right] - \int_0^1 f(t)dt\right| \le \frac{1}{2\sqrt{3}}||f'||_2. \tag{3.1}$$

*Replacing* $f(t)$ *by* $f(t) - t\int_0^1 f(t)dt$ *in (3.1), we get*

$$\left|\frac{1}{2}\left[f(0) + f(1)\right] - \int_0^1 f(t)dt\right| \le \frac{1}{2\sqrt{3}}\sqrt{\sigma(f')}, \tag{3.2}$$

*since the Trapezoid rule has degree of precision* $1$*.*

**Example 3.2.** *Consider the following quadrature rule*

$$\int_0^1 f(t)dt = \left(x - \frac{1}{2}\right) f(0) + f(x) - \left(x - \frac{1}{2}\right) f(1), \qquad x \in [0, 1], \tag{3.3}$$

*which has degree of precision* $1$ $(k = 2)$*. A direct calculation using Corollary 2.3 gives*

$$c_1(2) = \frac{1}{2\sqrt{3}},$$

*from which and the similar argument of Example 3.1, follows (1.3).*

**Example 3.3.** *For Simpson's rule,* $m = 3, x_0 = 0, x_1 = 1/2, x_2 = 1, p_0 = p_2 = 1/6, p_1 = 2/3$. *It is well known that Simpson's rule has degree of precision* $3$ $(k = 4)$. *A direct calculation leads to the following.*

$$\bar{c}_1(2) = \frac{h}{6}; \qquad \bar{c}_2(2) = \frac{h^2}{12\sqrt{30}};$$
$$\bar{c}_3(2) = \frac{h^3}{48\sqrt{105}}; \qquad \bar{c}_4(2) = \frac{h^4}{576\sqrt{14}}.$$

*The inequality* (2.12) *in combination with* $\bar{c}_1(2) = h/6$ *yields*

$$\left|\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)dt\right| \leq \frac{1}{6}||f'||_2.$$

*Again replacing* $f(t)$ *by* $f(t) - t\int_0^1 f(t)dt$ *in the above inequality, we easily get Theorem* 1.1.

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