ON GENERALIZATION OF ČEBYŠEV TYPE INEQUALITIES

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Abstract: A generalization of Pečarić's extension of Montgomery's identity is established and used to derive new Čebyšev type inequalities.

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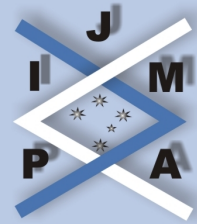
1. Introduction

In the present work we establish a generalization of Pečarić's extension of 'Montgomery's' identity and use it to derive new Čebyšev type inequalities.

We recall the Čebyšev inequality [1], given by the following:

$$(1.1) \qquad |T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty,$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and g' are bounded,

$$(1.2) \qquad T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx$$
$$- \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = \underset{t \in [a,b]}{ess\sup}|p(t)|$.

Pachpatte in [6] established new inequalities of the Čebyšev type by using Pečarič's extension of the Montgomery identity [7].
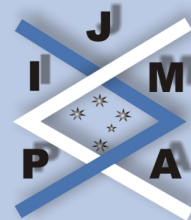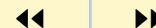
2.   Statement of Results

From [3], if $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative $f'(t)$ integrable on $[a, b]$, then the Montgomery identity holds:

$$(2.1) \qquad f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P(x, t) f'(t)\, dt,$$

where $P(x, t)$ is the Peano kernel defined by

$$P(x, t) = \begin{cases} \dfrac{t-a}{b-a}, & a \le t \le x \\[2ex] \dfrac{t-b}{b-a}, & x < t \le b. \end{cases}$$

We assume that $w : [a, b] \to [0, +\infty[$ is some probability density function, i.e. $\int_a^b w(t)\, dt = 1$, and set $W(t) = \int_a^t w(x)\, dx$ for $a \le t \le b$, $W(t) = 0$ for $t < a$ and for $t > b$. We then have the following identity given by Pečarić in [7], that is the weighted generalization of the Montgomery identity:

$$(2.2) \qquad f(x) = \int_a^b w(t) f(t)\, dt + \int_a^b P_w(x, t) f'(t)\, dt,$$

where the weighted Peano kernel P_w is:

$$(2.3) \qquad P_w(x, t) = \begin{cases} W(t), & a \le t \le x \\[1ex] W(t) - 1, & x < t \le b \end{cases}$$

Let $\varphi : [0, 1] \to \mathbb{R}$ be a differentiable function on $[0, 1]$, with $\varphi(0) = 0$, $\varphi(1) \ne 0$ and φ' integrable on $[0, 1]$. To simplify the notation, for some given functions w, f,
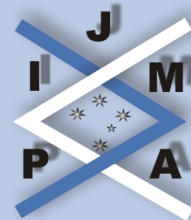
$g : [a, b] \to \mathbb{R}$, we set

$$(2.4) \quad T\left(w, f, g, \varphi'\right) = \int_a^b w(x) \varphi'\left(\int_a^x w(t)\, dt\right) f(x) g(x)\, dx$$
$$- \frac{1}{\varphi(1)} \left[\int_a^b w(x) \varphi'\left(\int_a^x w(t)\, dt\right) f(x)\, dx \right]$$
$$\times \left[\int_a^b w(x) \varphi'\left(\int_a^x w(t)\, dt\right) g(x)\, dx \right].$$

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable and $f'(t)$ integrable on $[a, b]$, then,*

$$(2.5) \quad f(x) = \frac{1}{\varphi(1)} \int_a^b w(t) \varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt$$
$$+ \frac{1}{\varphi(1)} \int_a^b P_{w,\varphi}(x, t)\, f'(t)\, dt,$$

where $P_{w,\varphi}$ is a generalization of the weighted Peano kernel defined by:

$$(2.6) \quad P_{w,\varphi}(x, t) = \begin{cases} \varphi(W(t)), & a \le t \le x; \\ \varphi(W(t)) - \varphi(1), & x < t \le b. \end{cases}$$

Proof. Using the hypothesis on φ,

$$(2.7) \quad \int_a^b P_{w,\varphi}(x, t)\, f'(t)\, dt$$
$$= \int_a^x \varphi(W(t))\, f'(t)\, dt + \int_x^b \left(\varphi(W(t)) - \varphi(1)\right) f'(t)\, dt$$

$$= \int_a^b \varphi\left(W\left(t\right)\right) f'(t)dt - \varphi\left(1\right) \int_x^b f'(t)dt$$

$$= \left[\varphi\left(W\left(t\right)\right) f\left(t\right)\right]_a^b - \int_a^b w\left(t\right) \varphi'\left(W\left(t\right)\right) f\left(t\right) dt - \varphi\left(1\right)\left[f\left(b\right) - f\left(x\right)\right]$$

$$= \varphi\left(1\right) f\left(x\right) - \int_a^b w\left(t\right) \varphi'\left(\int_a^t w\left(s\right) ds\right) f\left(t\right) dt.$$

Multiplying both sides by $1/\varphi\left(1\right)$, we obtain,

$$f\left(x\right) = \frac{1}{\varphi\left(1\right)} \int_a^b w\left(t\right) \varphi'\left(\int_a^t w\left(s\right) ds\right) f\left(t\right) dt + \frac{1}{\varphi\left(1\right)} \int_a^b P_{w,\varphi}\left(x,t\right) f'(t)dt$$
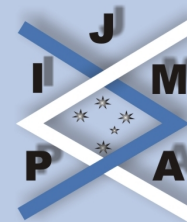
and this completes the proof. $\qquad \square$

Theorem 2.2. *Let f, g $[a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and f', g' be integrable on $[a,b]$ and let w, φ be as in Theorem 2.1, then,*

$$\left|T\left(w,f,g,\varphi'\right)\right| \leq \frac{1}{\varphi^2(1)} \left\|f'\right\|_\infty \left\|g'\right\|_\infty \left\|\varphi'\right\|_\infty \int_a^b w\left(x\right) H^2\left(x\right) dx,$$

where $H\left(x\right) = \int_a^b \left|P_{w,\varphi}\left(x,t\right)\right| dt$ and $\left\|\varphi'\right\|_\infty = \underset{t\in[0,1]}{ess\sup} \left|\varphi'\left(t\right)\right|.$

Since the functions f and g satisfy the hypothesis of Theorem 2.1, the following identities hold:

$$(2.8) \qquad f\left(x\right) = \frac{1}{\varphi\left(1\right)} \int_a^b w\left(t\right) \varphi'\left(\int_a^t w\left(s\right) ds\right) f\left(t\right) dt$$

$$+ \frac{1}{\varphi\left(1\right)} \int_a^b P_{w,\varphi}\left(x,t\right) f'(t)dt$$
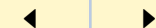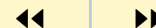
and

$$(2.9) \quad g(x) = \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt$$

$$+ \frac{1}{\varphi(1)} \int_a^b P_{w,\varphi}(x,t)\, g'(t)\, dt \ .$$

Using (2.8) and (2.9) we obtain,

$$\left[f(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt \right]$$

$$\times \left[g(x) - \frac{1}{\varphi(1)} \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt \right]$$

$$= \frac{1}{\varphi^2(1)} \left[\int_a^b P_{w,\varphi}(x,t)\, f'(t) dt \right] \left[\int_a^b P_{w,\varphi}(x,t)\, g'(t)\, dt \right].$$

Consequently,

$$(2.10) \quad f(x)\, g(x) - \frac{1}{\varphi(1)} f(x) \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt$$

$$- \frac{1}{\varphi(1)} g(x) \int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt$$

$$+ \frac{1}{\varphi^2(1)} \left(\int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) f(t)\, dt \right)$$

$$\times \left(\int_a^b w(t)\, \varphi'\left(\int_a^t w(s)\, ds\right) g(t)\, dt \right)$$

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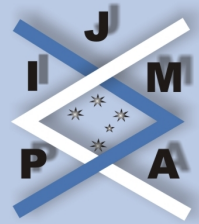
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$$= \frac{1}{\varphi^2(1)} \left[\int_a^b P_{w,\varphi}(x,t) f'(t) dt \right] \left[\int_a^b P_{w,\varphi}(x,t) g'(t) dt \right].$$

Multiplying both sides of (2.10) by $w(x) \varphi' \left(\int_a^x w(s) ds \right)$ and then integrating the resultant identity with respect to x from a to b, we get,

$$(2.11) \quad T(w,f,g,\varphi') = \frac{1}{\varphi^2(1)} \int_a^b w(x) \varphi' \left(\int_a^x w(t) dt \right)$$
$$\times \left[\int_a^b P_{w,\varphi}(x,t) f'(t) dt \right] \left[\int_a^b P_{w,\varphi}(x,t) g'(t) dt \right] dx.$$

Finally,

$$|T(w,f,g,\varphi')| \leq \frac{1}{\varphi^2(1)} \|f'\|_\infty \|g'\|_\infty \|\varphi'\|_\infty \int_a^b w(x) H^2(x) dx.$$

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