ON A GEOMETRIC INEQUALITY BY J. SÁNDOR

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ler's inequality.

Abstract: In this short note, we sharpen and generalize a geometric inequality by J. Sándor.

As applications of our results, we give an alternative proof of Sándor's inequality

and solve two conjectures posed by Liu.

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1. Introduction and Main Results

Let P be an arbitrary point P in the plane of triangle ABC. Let a, b, c be the lengths of these sides, \triangle the area, s the semi-perimeter, R the circumradius and r the inradius, respectively. Denote by R_1 , R_2 , R_3 the distances from P to the vertices A, B, C, respectively.

The following interesting geometric inequality from 1986 is due to J. Sándor [8], a proof of this inequality can be found in the monograph [9].

Theorem 1.1. For triangle ABC and an arbitrary point P, we have

$$(1.1) (R_1 R_2)^2 + (R_2 R_3)^2 + (R_3 R_1)^2 \ge \frac{16}{9} \triangle^2.$$

Recently, J. Liu [6] also independently proved inequality (1.1).

In this short note, we sharpen and generalize inequality (1.1) and obtain the following results.

Theorem 1.2. We have

$$(1.2) (R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \ge \frac{a^2b^2c^2}{a^2 + b^2 + c^2}.$$

Theorem 1.3. If

$$k \ge k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462,$$

then

(1.3)
$$(R_1 R_2)^k + (R_2 R_3)^k + (R_3 R_1)^k \ge 3 \left(\frac{4}{9} \sqrt{3} \triangle\right)^k.$$

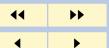


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2. Preliminary Results

Lemma 2.1 (Hayashi's inequality, see [7, pp. 297, 311]). For any $\triangle ABC$ and an arbitrary point P, we have

$$(2.1) aR_2R_3 + bR_3R_1 + cR_1R_2 \ge abc,$$

with equality holding if and only if P is the orthocenter of the acute triangle ABC or one of the vertices of the triangle ABC.

Lemma 2.2 (see [2] and [4]). For $\triangle ABC$, if

$$0 \le t \le t_0 = \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3},$$

then we have

$$(2.2) a^t + b^t + c^t \le 3\left(\sqrt{3}R\right)^t.$$

Lemma 2.3. Let

$$k \ge k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462.$$

Then

(2.3)
$$\frac{(abc)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \ge 3\left(\frac{4}{9}\sqrt{3}\triangle\right)^k.$$

Proof. From the well known identities abc = 4Rrs and $\triangle = rs$, inequality (2.3) is equivalent to

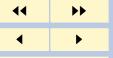
$$\frac{(4Rrs)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \ge 3\left(\frac{4}{9}\sqrt{3}rs\right)^k,$$



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or

(2.4)
$$a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}} \le 3\left(\sqrt{3}R\right)^{\frac{k}{k-1}}.$$

It is easy to see that the function

$$f(x) = \frac{x}{x-1}$$

is strictly monotone decreasing on $(1, +\infty)$. If we let

$$t = \frac{k}{k-1} = f(k)$$
 $\left(k \ge k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2}\right),$

then

$$0 < f(k) = t \le \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3} = f(k_0),$$

and inequality (2.4) is equivalent to (2.2).

The proof of Lemma 2.3 is thus complete from Lemma 2.2.

Lemma 2.4 ([3]). For any $\lambda \geq 1$, we have

$$(2.5) \ [R - \lambda(\lambda+1)r]s^2 + r[4(\lambda^2-4)R^2 + (5\lambda^2+12\lambda+4)Rr + (\lambda^2+3\lambda+2)r^2] \ge 0.$$

Lemma 2.5. In triangle ABC, we have

$$a^{9} + b^{9} + c^{9} = 2s[s^{8} - 18r(R + 2r)s^{6} + 18r^{2}(21Rr + 7r^{2} + 12R^{2})s^{4} - 6r^{3}(105r^{2}R + 240rR^{2} + 14r^{3} + 160R^{3})s^{2} + 9r^{4}(r + 2R)(r + 4R)^{3}].$$

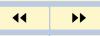


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Proof. The identity directly follows from the known identities a + b + c = 2s, $ab + bc + ca = s^2 + 4Rr + r^2$, abc = 4Rrs and the following identity:

$$a^{9} + b^{9} + c^{9}$$

$$= 3a^{3}b^{3}c^{3} - 45abc(ab + bc + ca)(a + b + c)^{4} + 54abc(ab + bc + ca)^{2}(a + b + c)^{2}$$

$$- 27a^{2}b^{2}c^{2}(ab + bc + ca)(a + b + c) + (a + b + c)^{9}$$

$$- 9(ab + bc + ca)(a + b + c)^{7} + 9(ab + bc + ca)^{4}(a + b + c)$$

$$- 30(ab + bc + ca)^{3}(a + b + c)^{3} + 18a^{2}b^{2}c^{2}(a + b + c)^{3}$$

$$+ 27(ab + bc + ca)^{2}(a + b + c)^{5} + 9abc(a + b + c)^{6} - 9abc(ab + bc + ca)^{3}.$$

Lemma 2.6 ([5]). If $x, y, z \ge 0$, then

$$x + y + z + 3\sqrt[3]{xyz} \ge 2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right).$$



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3. Proof of the Main Result

The proof of Theorem 1.2 is easy to find from the following inequality (3.1) for k = 2 of the proof of Theorem 1.3. Now, we prove Theorem 1.3.

The proof of Theorem 1.3. Hölder's inequality and Lemma 2.1 imply for k > 1 that

$$\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}} \left[(R_1 R_2)^k + (R_2 R_3)^k + (R_3 R_1)^k \right]^{\frac{1}{k}}$$

$$\geq aR_2 R_3 + bR_3 R_1 + cR_1 R_2 \geq abc,$$

or

$$(3.1) (R_1 R_2)^k + (R_2 R_3)^k + (R_3 R_1)^k \ge \frac{(abc)^k}{\left\lceil a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}} \right\rceil^{k-1}}.$$

Combining inequality (3.1) and Lemma 2.3, we immediately see that Theorem 1.3 is true. \Box

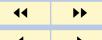


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4. Applications

4.1. Alternative Proof of Theorem 1.1

From Theorem 1.2, in order to prove inequality (1.1), we only need to prove the following inequality:

(4.1)
$$\frac{a^2b^2c^2}{a^2+b^2+c^2} \ge \frac{16}{9} \triangle^2.$$

With the known identities abc = 4Rrs and $\triangle = rs$, inequality (4.1) is equivalent to

$$a^2 + b^2 + c^2 \le 9R^2.$$

This is simply inequality (2.2) for $t = 2 < t_0$ in Lemma 2.2. This completes the proof of inequality (1.1).

Remark 1. The above proof of inequality (1.1) is simpler than Liu's proof [6].

4.2. Solution of Two Conjectures

In 2008, J. Liu [6] posed the following two geometric inequality conjectures, (4.2) and (4.3), involving R_1 , R_2 , R_3 , R and r.

Conjecture 4.1. For $\triangle ABC$ and an arbitrary point P, we have

$$(4.2) (R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \ge 8(R^2 + 2r^2)r^2,$$

and

$$(4.3) (R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \ge 24r^3.$$

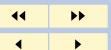


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Proof. First of all, from *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$s^2 \le 4R^2 + 4Rr + 3r^2$$

and Euler's inequality [1, pp. 48, Theorem 5.1]

$$R \geq 2r$$
,

we have

$$2r^{2}(4R^{2} + 4Rr + 3r^{2} - s^{2}) + (R - 2r)(4R^{2} + Rr + 2r^{2})r \ge 0$$

$$\iff \frac{16R^{2}r^{2}s^{2}}{2(s^{2} - 4Rr - r^{2})} \ge 8(R^{2} + 2r^{2})r^{2}.$$

Using Theorem 1.2 and the known identities [7, pp.52]

$$abc = 4Rrs$$
 and $a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$

we see that inequality (4.2) holds true.

Secondly, from (3.1), in order to prove inequality (4.3), we only need to prove

(4.4)
$$\frac{(abc)^{\frac{3}{2}}}{[a^3 + b^3 + c^3]^{\frac{1}{2}}} \ge 24r^3.$$

With the known identities [7, pp. 52]

$$abc = 4Rrs$$
 and $a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$

inequality (4.4) is equivalent to

$$(4.5) \frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^2 - 6Rr - 3r^2)]^{\frac{1}{2}}} \ge 24r^3$$

$$\iff 18r^3(4R^2 + 4Rr + 3r^2 - s^2) + R^3(s^2 - 16Rr + 5r^2)$$

$$+ Rr(R - 2r)(16R^2 + 27Rr - 18r^2) \ge 0.$$



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From Gerretsen's inequality [1, pp. 50, Theorem 5.8]

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2$$

and Euler's inequality [1, pp. 48, Theorem 5.1]

we can conclude that inequality (4.5) holds, further, inequality (4.4) is true. This completes the proof of Conjecture 4.1.

Corollary 4.2. For $\triangle ABC$ and an arbitrary point P, we have

$$(4.6) R_1^3 + R_2^3 + R_3^3 + 3R_1R_2R_3 \ge 48r^3.$$

Proof. Inequality (4.6) can directly be obtained from Lemma 2.6 and inequality (4.3).

4.3. Sharpened Form of Above Conjectures

The inequalities (4.2) and (4.3) of Conjecture 4.1 can be sharpened as follows.

Theorem 4.3. For $\triangle ABC$ and an arbitrary point P, we have

$$(4.7) (R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \ge 8(R+r)Rr^2,$$

and

$$(4.8) (R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \ge 12Rr^2.$$

Proof. The proof of inequality (4.7) is left to the readers. Now, we prove inequality (4.8).



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From inequality (2.5) for $\lambda=2$ in Lemma 2.4, the well-known *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2,$$

Euler's inequality [1, pp. 48, Theorem 5.1]

$$R \ge 2r$$

and the known identities [7, pp. 52]

$$abc = 4Rrs$$
 and $a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$,

we obtain that

$$(4.9) [(R-6r)s^{2} + 12r^{2}(4R+r)] + 3r(4R^{2} + 4Rr + 3r^{2} - s^{2}) + R(s^{2} - 16Rr + 5r^{2}) + r(R - 2r)(4R - 3r) \ge 0 \Leftrightarrow \frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^{2} - 6Rr - 3r^{2})]^{\frac{1}{2}}} \ge 12Rr^{2} \Leftrightarrow \frac{(abc)^{\frac{3}{2}}}{[a^{3} + b^{3} + c^{3}]^{\frac{1}{2}}} \ge 12Rr^{2}.$$

Inequality (4.8) follows by Lemma 2.4.

Theorem 4.3 is thus proved.

4.4. Generalization of Inequality (4.3)

Theorem 4.4. If $k \geq \frac{9}{8}$, then

$$(4.10) (R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \ge 3(4r^2)^k.$$



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Proof. From the monotonicity of the power mean, we only need to prove that inequality (4.10) holds for $k = \frac{9}{8}$. By using inequality (3.1), we only need to prove the following inequality

$$\frac{(abc)^{\frac{9}{8}}}{(a^9 + b^9 + c^9)^{\frac{1}{8}}} \ge 3(4r^2)^{\frac{9}{8}}.$$

From Gerretsen's inequality [1, pp. 50, Theorem 5.8]

$$s^2 \ge 16Rr - 5r^2$$

and Euler's inequality [1, pp. 48, Theorem 5.1]

$$R \geq 2r$$
,

it is obvious that

$$P = (R - 2r)[4096R^{10} + 12544R^9r + 34992R^8r^2 + 89667R^7r^3 + 218700R^6r^4 + 516132R^5r^5 + 1189728R^4r^6 + 2493180R^3r^7 + 6018624(R - 2r)Rr^8 + 6753456r^{10} + 201204(R^2 - 4r^2)Rr^7] + 2799360r^{11} > 0,$$

and

$$\begin{split} Q &= (s^2 - 16Rr + 5r^2)\{R^9(s^2 - 16Rr + 5r^2) \\ &+ 3R^4r(R - 2r)(16R^5 + 27R^4r + 54R^3r^2 \\ &+ 108R^2r^3 + 216Rr^4 + 432r^5) + 324r^7[8(R^2 - 12r^2)^2 + 30r^2(R - 2r)^2 \\ &+ 39Rr^3 + 267r^4]\} + 17496r^7(R^2 - 3Rr + 6r^2)(R^2 - 12Rr + 24r^2)^2 \\ &+ 3r^2(R - 2r)\{(R - 2r)[256R^9 + 864R^8r + 2457R^2r^2(R^5 - 32r^5) \\ &+ 6372R^2r^3(R^4 - 16r^4) + 15660R^2r^4(R^3 - 8r^3) + 31320R^2r^5(R^2 - 4r^2) \\ &+ 220104R^2r^6(R - 2r) + 2618784(R - 2r)r^8 + 51840R^2r^7 + 501120Rr^8] \\ &+ 687312r^{10}\} > 0. \end{split}$$

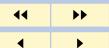


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Therefore, with the fundamental inequality [7, pp.1–3]

$$-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \ge 0,$$

we have

$$\begin{split} W &= (R^9 - 13122r^9)s^8 + 236196r^{10}(2r+R)s^6 - 236196r^{11}(7r^2 + 12R^2 + 21Rr)s^4 \\ &\quad + 78732r^{12}(105Rr^2 + 160R^3 + 240R^2r + 14r^3)s^2 \\ &\quad - 118098r^{13}(2R+r)(4R+r)^3 \\ &= 13122r^9[s^4 + 9r^3(2R+r)][-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R+r)^3] \\ &\quad + r^3s^2(R-2r)P + s^2(s^2 - 16Rr + 5r^2)Q \\ &> 0. \end{split}$$

Hence, from Lemma 2.4, we get that

(4.12)
$$3\left(\frac{Rs}{3r}\right)^9 - (a^9 + b^9 + c^9) = \frac{s}{6561r^9}W \ge 0,$$

or

$$3\left(\frac{Rs}{3r}\right)^9 \ge a^9 + b^9 + c^9.$$

Inequality (4.13) is simply (4.11). Thus, we complete the proof of Theorem 4.4. \Box



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5. Two Open Problems

Finally, we pose two open problems as follows.

Open Problem 1. For a triangle ABC and an arbitrary point P, prove or disprove

(5.1)
$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \ge 72r^3.$$

Open Problem 2. For a triangle ABC and an arbitrary point P, determine the best constant k such that the following inequality holds:

$$(5.2) (R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \ge 12[R + k(R - 2r)]r^2.$$



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References

- [1] O. BOTTEMA, R.Ž. DJORDEVIĆ, R.R. JANIĆ, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
- [2] J. BERKES, Einige Dreiecksungleichungen, *Elem. Math.*, **18** (1963), 31–32. (in German)
- [3] X.-G. CHU, Two triangle inequalities containing parameter, *J. Binzhou Teachers College*, **16**(1) (2000), 27–30. (in Chinese)
- [4] J.-C. KUANG, *Chángyòng Bùděngshi (Applied Inequalities)*, 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, 2004, 194.
- [5] S.-H. LI, AM–GM Inequality and Cauchy Inequality, East China Normal University Press, Shanghai City, China, 2005, 35. (in Chinese)
- [6] J. LIU, Nine sine inequality, manuscript, 2008, 33. (in Chinese)
- [7] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Acad. Publ., Dordrecht, Boston, London, 1989.
- [8] J. SÁNDOR, Problem 20942*, Mat. Lapok, 11-12 (1986), 486.
- [9] J. SÁNDOR, Geometric Theorems, Diophantine Equations and Arithmetic Functions, American Research Press, Rehoboth, NM, USA, 2002, 31. [ON-LINE: www.gallup.unm.edu/~smarandache/JozsefSandor2.pdf]



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