

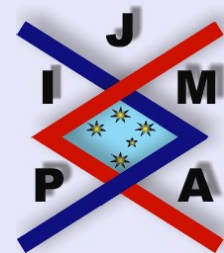
## NEIGHBOURHOODS AND PARTIAL SUMS OF STARLIKE FUNCTIONS BASED ON RUSCHEWEYH DERIVATIVES

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Abstract

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## Abstract

In this paper a new class  $S_p^\lambda(\alpha, \beta)$  of starlike functions is introduced. A subclass  $TS_p^\lambda(\alpha, \beta)$  of  $S_p^\lambda(\alpha, \beta)$  with negative coefficients is also considered. These classes are based on Ruscheweyh derivatives. Certain neighbourhood results are obtained. Partial sums  $f_n(z)$  of functions  $f(z)$  in these classes are considered and sharp lower bounds for the ratios of real part of  $f(z)$  to  $f_n(z)$  and  $f'(z)$  to  $f'_n(z)$  are determined.

*2000 Mathematics Subject Classification:* 30C45.

*Key words:* Univalent , Starlike , Convex.

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# 1. Introduction

Let  $S$  denote the family of functions of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk  $U = \{z : |z| < 1\}$ . Also denote by  $T$ , the subclass of  $S$  consisting of functions of the form

$$(1.2) \quad f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$$

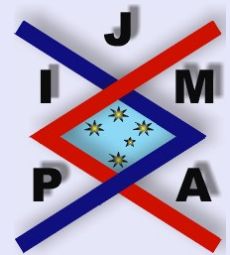
which are univalent and normalized in  $U$ .

For  $f \in S$ , and of the form (1.1) and  $g(z) \in S$  given by  $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ , we define the Hadamard product (or convolution)  $f * g$  of  $f$  and  $g$  by

$$(1.3) \quad (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For  $-1 \leq \alpha < 1$  and  $\beta \geq 0$ , we let  $S_p^\lambda(\alpha, \beta)$  be the subclass of  $S$  consisting of functions of the form (1.1) and satisfying the analytic criterion

$$(1.4) \quad \operatorname{Re} \left\{ \frac{z (D^\lambda f(z)')}{D^\lambda f(z)} - \alpha \right\} > \beta \left| \frac{z (D^\lambda f(z)')}{D^\lambda f(z)} - 1 \right|,$$



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where  $D^\lambda$  is the Ruscheweyh derivative [6] defined by

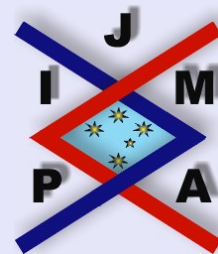
$$D^\lambda f(z) = f(z) * \frac{1}{(1-z)^{\lambda+1}} = z + \sum_{k=2}^{\infty} B_k(\lambda) a_k z^k$$

and

$$(1.5) \quad B_k(\lambda) = \frac{(\lambda+1)_{k-1}}{(k-1)!} = \frac{(\lambda+1)(\lambda+1)\cdots(\lambda+k-1)}{(k-1)!}, \quad \lambda \geq 0.$$

We also let  $TS_p^\lambda(\alpha, \beta) = S_p^\lambda(\alpha, \beta) \cap T$ . It can be seen that, by specializing on the parameters  $\alpha, \beta, \lambda$  the class  $TS_p^\lambda(\alpha, \beta)$  reduces to the classes introduced and studied by various authors [1, 9, 11, 12].

The main aim of this work is to study coefficient bounds and extreme points of the general class  $TS_p^\lambda(\alpha, \beta)$ . Furthermore, we obtain certain neighbourhoods results for functions in  $TS_p^\lambda(\alpha, \beta)$ . Partial sums  $f_n(z)$  of functions  $f(z)$  in the class  $S_p^\lambda(\alpha, \beta)$  are considered.




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## 2. The Classes $S_p^\lambda(\alpha, \beta)$ and $TS_p^\lambda(\alpha, \beta)$

In this section we obtain a necessary and sufficient condition and extreme points for functions  $f(z)$  in the class  $TS_p^\lambda(\alpha, \beta)$ .

**Theorem 2.1.** A sufficient condition for a function  $f(z)$  of the form (1.1) to be in  $S_p^\lambda(\alpha, \beta)$  is that

$$(2.1) \quad \sum_{k=2}^{\infty} \frac{[(1 + \beta)k - (\alpha + \beta)]}{1 - \alpha} B_k(\lambda) |a_k| \leq 1,$$

$-1 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0$  and  $B_k(\lambda)$  is as defined in (1.5).

*Proof.* It suffices to show that

$$\beta \left| \frac{z (D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z (D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned} & \beta \left| \frac{z (D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z (D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right\} \\ & \leq (1 + \beta) \left| \frac{z (D^\lambda f(z))'}{D^\lambda f(z)} - 1 \right| \\ & \leq \frac{(1 + \beta) \sum_{k=2}^{\infty} (k - 1) B_k(\lambda) |a_k| |z|^{k-1}}{1 - \sum_{k=2}^{\infty} B_k(\lambda) |a_k| |z|^{k-1}} \end{aligned}$$



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$$\leq \frac{(1 + \beta) \sum_{k=2}^{\infty} (k - 1) B_k(\lambda) |a_k|}{1 - \sum_{k=2}^{\infty} B_k(\lambda) |a_k|}.$$

This last expression is bounded above by  $1 - \alpha$  if

$$\sum_{k=2}^{\infty} [(1 + \beta) k - (\alpha + \beta)] B_k(\lambda) |a_k| \leq 1 - \alpha,$$

and the proof is complete.  $\square$

Now we prove that the above condition is also necessary for  $f \in T$ .

**Theorem 2.2.** *A necessary and sufficient condition for  $f$  of the form (1.2) namely  $f(z) = z - \sum_{k=2}^{\infty} b_k z^k$ ,  $a_k \geq 0$ ,  $z \in U$  to be in  $TS_p^\lambda(\alpha, \beta)$ ,  $-1 \leq \alpha < 1$ ,  $\beta \geq 0$ ,  $\lambda \geq 0$  is that*

$$(2.2) \quad \sum_{k=2}^{\infty} [(1 + \beta) k - (\alpha + \beta)] B_k(\lambda) a_k \leq 1 - \alpha.$$

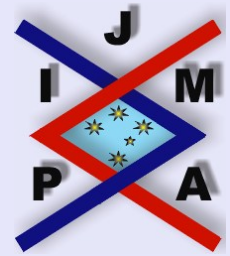
*Proof.* In view of Theorem 2.1, we need only to prove the necessity. If  $f \in TS_p^\lambda(\alpha, \beta)$  and  $z$  is real then

$$\frac{1 - \sum_{k=2}^{\infty} k a_k B_k(\lambda) z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k B_k(\lambda) z^{k-1}} - \alpha \geq \frac{1 - \sum_{k=2}^{\infty} (k - 1) a_k B_k(\lambda) z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k B_k(\lambda) z^{k-1}}.$$

Letting  $z \rightarrow 1$  along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty} [(1 + \beta) k - (\alpha + \beta)] B_k(\lambda) a_k \leq 1 - \alpha.$$

$\square$



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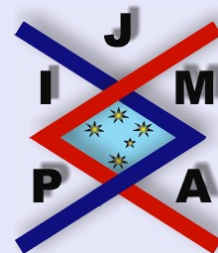
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**Theorem 2.3.** *The extreme points of  $TS_p^\lambda(\alpha, \beta)$ ,  $-1 \leq \alpha < 1$ ,  $\beta \geq 0$  are the functions given by*

$$(2.3) \quad f_1(z) = 1 \text{ and } f_k(z) = z - \frac{1 - \alpha}{[(1 + \beta)k - (\alpha + \beta)] B_k(\lambda)} z^k,$$

$k = 2, 3, \dots$  where  $\lambda > -1$  and  $B_k(\lambda)$  is as defined in (1.5).

**Corollary 2.4.** *A function  $f \in TS_p^\lambda(\alpha, \beta)$  if and only if  $f$  may be expressed as  $\sum_{k=1}^{\infty} \mu_k f_k(z)$  where  $\mu_k \geq 0$ ,  $\sum_{k=1}^{\infty} \mu_k = 1$  and  $f_1, f_2, \dots$  are as defined in (2.3).*




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### 3. Neighbourhood Results

The concept of neighbourhoods of analytic functions was first introduced by Goodman [4] and then generalized by Ruscheweyh [5]. In this section we study neighbourhoods of functions in the family  $TS_p^\lambda(\alpha, \beta)$ .

**Definition 3.1.** For  $f \in S$  of the form (1.1) and  $\delta \geq 0$ , we define  $\eta - \delta$ -neighbourhood of  $f$  by

$$M_\delta^\eta(f) = \left\{ g \in S : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k^{\eta+1} |a_k - b_k| \leq \delta \right\},$$

where  $\eta$  is a fixed positive integer.

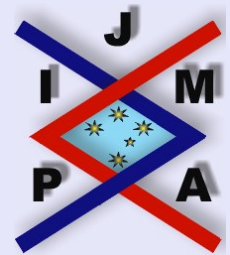
We may write  $M_\delta^\eta(f) = N_\delta(f)$  and  $M_\delta^1(f) = M_\delta(f)$  [5]. We also notice that  $M_\delta(f)$  was defined and studied by Silverman [7] and also by others [2, 3].

We need the following two lemmas to study the  $\eta - \delta$ -neighbourhood of functions in  $TS_p^\lambda(\alpha, \beta)$ .

**Lemma 3.1.** Let  $m \geq 0$  and  $-1 \leq \gamma < 1$ . If  $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$  satisfies  $\sum_{k=2}^{\infty} k^{\mu+1} |b_k| \leq \frac{1-\gamma}{1+\beta}$  then  $g \in S_p^\mu(\gamma, \beta)$ . The result is sharp.

*Proof.* In view of the first part of Theorem 2.1, it is sufficient to show that

$$\frac{k(1+\beta) - (\gamma + \beta)}{1 - \gamma} B_k(\mu) = \frac{k^{\mu+1}}{(1 - \gamma)} (1 + \beta).$$




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But

$$\frac{k(1+\beta) - (\gamma + \beta)}{1 - \gamma} B_k(\mu) = \frac{(k(1+\beta) - (\gamma + \beta))(\mu + 1) \cdots (\mu + k - 1)}{(1 - \gamma)(k - 1)!}$$

$$\leq \frac{k(1+\beta)(\mu + 1)(\mu + 2) \cdots (\mu + k - 1)}{(1 - \gamma)(k - 1)!}.$$

Therefore we need to prove that

$$H(k, \mu) = \frac{(\mu + 1)(\mu + 2) \cdots (\mu + k - 1)}{k^\mu (k - 1)!} \leq 1.$$

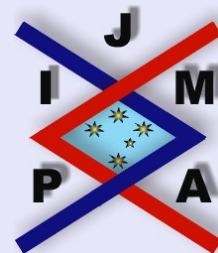
Since  $H(k, \mu) = [(\mu + 1)/2^\mu] \leq 1$ , we need only to show that  $H(k, \mu)$  is a decreasing function of  $k$ . But  $H(k + 1, \mu) \leq H(k, \mu)$  is equivalent to  $(1 + \mu/k) \leq (1 + 1/k)^\mu$ . The result follows because the last inequality holds for all  $k \geq 2$ .  $\square$

**Lemma 3.2.** Let  $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T$ ,  $-1 \leq \alpha < 1$ ,  $\beta \geq 0$  and  $\varepsilon \geq 0$ . If  $\frac{f(z) + \varepsilon z}{1 + \varepsilon} \in TS_p^\lambda(\alpha, \beta)$  then

$$\sum_{k=2}^{\infty} k^{\mu+1} a_k \leq \frac{2^{\eta+1} (1 - \alpha) (1 + \varepsilon)}{(2 - \alpha + \beta) (1 + \lambda)},$$

where either  $\eta = 0$  and  $\lambda \geq 0$  or  $\eta = 1$  and  $1 \leq \lambda \leq 2$ . The result is sharp with the extremal function

$$f(z) = z - \frac{(1 - \alpha)(1 + \varepsilon)}{(2 - \alpha + \beta)(1 + \lambda)} z^2, \quad z \in U.$$



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*Proof.* Letting  $g(z) = \frac{f(z)+\varepsilon z}{1+\varepsilon}$  we have  $g(z) = z - \sum_{k=2}^{\infty} \frac{a_k}{1+\varepsilon} z^k, z \in U$ .

In view of Corollary 2.4  $g(z)$ , may be written as  $g(z) = \sum_{k=1}^{\infty} \mu_k g_k(z)$ , where  $\mu_k \geq 0, \sum_{k=1}^{\infty} \mu_k = 1$ ,

$$g_1(z) = z \text{ and } g_k(z) = z - \frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)} z^k, \quad k = 2, 3, \dots$$

Therefore we obtain

$$\begin{aligned} g(z) &= \mu_1 z + \sum_{k=2}^{\infty} \mu_k \left( z - \frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)} z^k \right) \\ &= z - \sum_{k=2}^{\infty} \mu_k \left( \frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)} \right) z^k. \end{aligned}$$

Since  $\mu_k \geq 0$  and  $\sum_{k=1}^{\infty} \mu_k \leq 1$ , it follows that

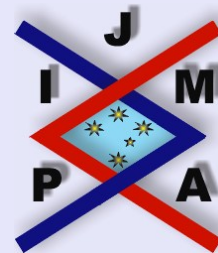
$$\sum_{k=2}^{\infty} k^{\eta+1} a_k \leq \sup_{k \geq 2} k^{\eta+1} \left( \frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)} \right).$$

The result will follow if we can show that  $A(k, \eta, \alpha, \varepsilon, \lambda) = \frac{k^{\eta+1}(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}$  is a decreasing function of  $k$ . In view of  $B_{k+1}(\lambda) = \frac{\lambda+k}{k} B_k(\lambda)$  the inequality

$$A(k+1, \eta, \alpha, \varepsilon, \lambda) \leq A(k, \eta, \alpha, \varepsilon, \lambda)$$

is equivalent to

$$(k+1)^{\eta+1} (k-\alpha+\beta) \leq k^{\eta} (k+1-\alpha+\beta) (\lambda+k).$$



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This yields

$$(3.1) \quad \lambda(k - \alpha + \beta) + \lambda + \alpha - \beta \geq 0$$

whenever  $\eta = 0$  and  $\lambda \geq 0$  and

$$(3.2) \quad k[(k + 1)(\lambda - 1) + (2 - \lambda)(\alpha - \beta)] + \alpha - \beta \geq 0,$$

whenever  $\eta = 1$  and  $1 \leq \lambda \leq 2$ . Since (3.1) and (3.2) holds for all  $k \geq 2$ , the proof is complete.  $\square$

**Theorem 3.3.** *Suppose either  $\eta = 0$  and  $\lambda \geq 0$  or  $\eta = 1$  and  $1 \leq \lambda \leq 2$ .*

*Let  $-1 \leq \alpha < 1$ , and*

$$-1 \leq \gamma < \frac{(2 - \alpha + \beta)(1 + \lambda) - 2^{\eta+1}(1 - \alpha)(1 + \varepsilon)(1 + \beta)}{(2 - \alpha + \beta)(1 + \lambda)(1 + \beta)}.$$

*Let  $f \in T$  and for all real numbers  $0 \leq \varepsilon < \delta$ , assume  $\frac{f(z) + \varepsilon z}{1 + \varepsilon} \in TS_p^\lambda(\alpha, \beta)$ .*

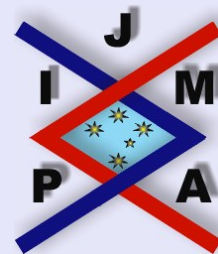
*Then the  $\eta$ - $\delta$  - neighbourhood of  $f$ , namely  $M_\delta^\eta(f) \subset S_p^\eta(\gamma, \beta)$  where*

$$\delta = \frac{(1 - \gamma)(2 - \alpha + \beta)(1 + \lambda) - 2^{\eta+1}(1 - \alpha)(1 + \varepsilon)(1 + \beta)}{(2 - \alpha + \beta)(1 + \lambda)(1 + \beta)}.$$

*The result is sharp, with the extremal function  $f(z) = \frac{(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)}z^2$ .*

*Proof.* For a function  $f$  of the form (1.2), let  $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$  be in  $M_\delta^\eta(f)$ . In view of Lemma 3.2, we have

$$\sum_{k=2}^{\infty} k^{\eta+1} |b_k| = \sum_{k=2}^{\infty} k^{\eta+1} |a_k - b_k - a_k|$$



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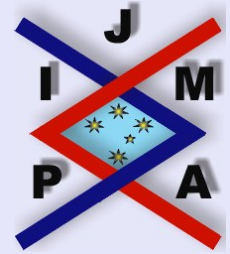
$$\leq \delta + \frac{2^{\eta+1} (1 - \alpha) (1 + \varepsilon)}{(2 - \alpha + \beta) (1 + \lambda)}.$$

Applying Lemma 3.1, it follows that  $g \in S_p^\eta(\gamma, \beta)$  if  $\delta + \frac{2^{\eta+1}(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)} \leq \frac{1-\gamma}{1+\beta}$ .  
That is, if

$$\delta \leq \frac{(1 - \gamma) (2 - \alpha + \beta) (1 + \lambda) - 2^{\eta+1} (1 - \alpha) (1 + \varepsilon) (1 + \beta)}{(2 - \alpha + \beta) (1 + \lambda) (1 + \beta)}.$$

This completes the proof. □

**Remark 3.1.** By taking  $\beta = 0$  and letting  $\lambda = 0$ ,  $\lambda = 1$  and  $\eta = 0 = \varepsilon$ , we note that Theorems 1,2,4 in [8] follow immediately from Theorem 3.3.




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## 4. Partial Sums

Following the earlier works by Silverman [8] and Silvia [10] on partial sums of analytic functions. We consider in this section partial sums of functions in the class  $S_p^\lambda(\alpha, \beta)$  and obtain sharp lower bounds for the ratios of real part of  $f(z)$  to  $f_n(z)$  and  $f'(z)$  to  $f'_n(z)$ .

**Theorem 4.1.** *Let  $f(z) \in S_p^\lambda(\alpha, \beta)$  be given by (1.1) and define the partial sums  $f_1(z)$  and  $f_n(z)$ , by*

$$(4.1) \quad f_1(z) = z; \text{ and } f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (n \in \mathbb{N} / \{1\})$$

Suppose also that

$$(4.2) \quad \sum_{k=2}^{\infty} c_k |a_k| \leq 1,$$

where  $\left( c_k := \frac{[(1+\beta)k - (\alpha+\beta)]B_k(\lambda)}{1-\alpha} \right)$ . Then  $f \in S_p^\lambda(\alpha, \beta)$ . Furthermore,

$$(4.3) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{c_{n+1}} z \in U, \quad n \in \mathbb{N}$$

and

$$(4.4) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} > \frac{c_{n+1}}{1 + c_{n+1}}.$$



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*Proof.* It is easily seen that  $z \in S_p^\lambda(\alpha, \beta)$ . Thus from Theorem 3.3 and by hypothesis (4.2), we have

$$(4.5) \quad N_1(z) \subset S_p^\lambda(\alpha, \beta),$$

which shows that  $f \in S_p^\lambda(\alpha, \beta)$  as asserted by Theorem 4.1.

Next, for the coefficients  $c_k$  given by (4.2) it is not difficult to verify that

$$(4.6) \quad c_{k+1} > c_k > 1.$$

Therefore we have

$$(4.7) \quad \sum_{k=2}^n |a_k| + c_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} c_k |a_k| \leq 1$$

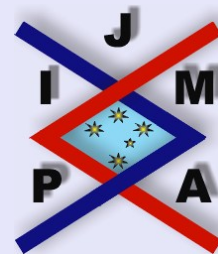
by using the hypothesis (4.2).

By setting

$$(4.8) \quad \begin{aligned} g_1(z) &= c_{n+1} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{c_{n+1}} \right) \right\} \\ &= 1 + \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^n a_k z^{k-1}} \end{aligned}$$

and applying (4.7), we find that

$$(4.9) \quad \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{c_{n+1} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^n |a_k| - c_{n+1} \sum_{k=n+1}^{\infty} |a_k|} \leq 1, \quad z \in U,$$




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which readily yields the assertion (4.3) of Theorem 4.1. In order to see that

$$(4.10) \quad f(z) = z + \frac{z^{n+1}}{c_{n+1}}$$

gives sharp result, we observe that for  $z = re^{i\pi/n}$  that  $\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \rightarrow 1 - \frac{1}{c_{n+1}}$  as  $z \rightarrow 1^-$ .

Similarly, if we take

$$(4.11) \quad g_2(z) = (1 + c_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}} \right\} \\ = 1 - \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}$$

and making use of (4.7), we can deduce that

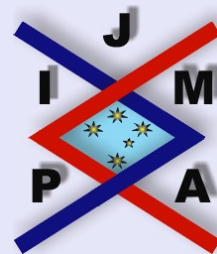
$$(4.12) \quad \left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^n |a_k| - (1 + c_{n+1}) \sum_{k=n+1}^{\infty} |a_k|} \\ \leq 1, \quad z \in U,$$

which leads us immediately to the assertion (4.4) of Theorem 4.1.

The bound in (4.4) is sharp for each  $n \in \mathbb{N}$  with the extremal function  $f(z)$  given by (4.10). The proof of Theorem 4.1. is thus complete.  $\square$

**Theorem 4.2.** *If  $f(z)$  of the form (1.1) satisfies the condition (2.1). Then*

$$(4.13) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq 1 - \frac{n+1}{c_{n+1}}.$$



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*Proof.* By setting

$$\begin{aligned}
 (4.14) \quad g(z) &= c_{n+1} \left\{ \frac{f'_n(z)}{f'_n(z)} - \left( 1 - \frac{n+1}{c_{n+1}} \right) \right\} \\
 &= \frac{1 + \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^n k a_k z^{k-1}} \\
 &= 1 + \frac{\frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^n k a_k z^{k-1}}, \\
 \left| \frac{g(z) - 1}{g(z) + 1} \right| &\leq \frac{\frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k|}{2 - 2 \sum_{k=2}^n k |a_k| - \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k|}.
 \end{aligned}$$

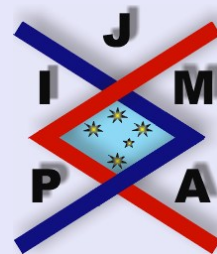
Now  $\left| \frac{g(z)-1}{g(z)+1} \right| \leq 1$  if

$$(4.15) \quad \sum_{k=2}^n k |a_k| + \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k| \leq 1$$

since the left hand side of (4.15) is bounded above by  $\sum_{k=2}^n c_k |a_k|$  if

$$(4.16) \quad \sum_{k=2}^n (c_k - k) |a_k| + \sum_{k=n+1}^{\infty} c_k - \frac{c_{n+1}}{n+1} k |a_k| \geq 0,$$

and the proof is complete. The result is sharp for the extremal function  $f(z) = z + \frac{z^{n+1}}{c_{n+1}}$ .  $\square$



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**Theorem 4.3.** If  $f(z)$  of the form (1.1) satisfies the condition (2.1) then

$$\operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{c_{n+1}}{n+1+c_{n+1}}.$$

*Proof.* By setting

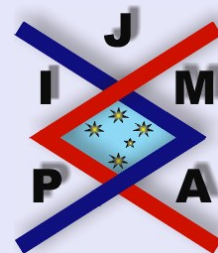
$$\begin{aligned} g(z) &= [(n+1) + c_{n+1}] \left\{ \frac{f'_n(z)}{f'(z)} - \frac{c_{n+1}}{n+1+c_{n+1}} \right\} \\ &= 1 - \frac{\left(1 + \frac{c_{n+1}}{n+1}\right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^n k a_k z^{k-1}} \end{aligned}$$

and making use of (4.16), we can deduce that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\left(1 + \frac{c_{n+1}}{n+1}\right) \sum_{k=n+1}^{\infty} k |a_k|}{2 - 2 \sum_{k=2}^n k |a_k| - \left(1 + \frac{c_{n+1}}{n+1}\right) \sum_{k=n+1}^{\infty} k |a_k|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 4.3. □

**Remark 4.1.** We note that  $\beta = 1$ , and choosing  $\lambda = 0$ ,  $\lambda = 1$  these results coincide with the results obtained in [13].




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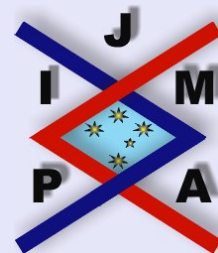
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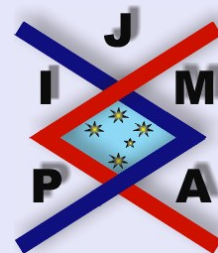
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