

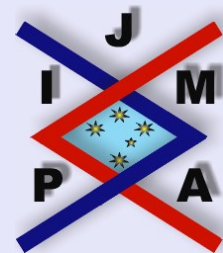
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ON A Q -ANALOGUE OF SÁNDOR'S FUNCTION

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Abstract

In this paper we obtain a q -analogue of J. Sándor's theorems [6], on employing the q -analogue of Stirling's formula established by D. S. Moak [5].

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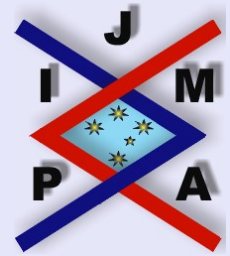
Key words: q -gamma function, q -Stirling's formula, Asymptotic formula.

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Dedicated to Professor Katsumi Shiratani on the occasion of his 71st birthday

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1. Introduction

F. H. Jackson defined a q -analogue of the gamma function which extends the q -factorial

$$(n!)_q = 1(1+q)(1+q+q^2)\cdots(1+q+\dots+q^{n-1}), \text{ cf. [3, 4],}$$

which becomes the ordinary factorial as $q \rightarrow 1$. He defined the q -analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (q-1)^{1-x} q^{\binom{x}{2}}, \quad q > 1,$$

where

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

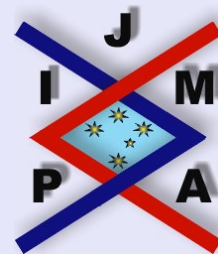
It is well-known that $\Gamma_q(x) \rightarrow \Gamma(x)$ as $q \rightarrow 1$, where $\Gamma(x)$ is the ordinary gamma function. In [2], R. Askey obtained a q -analogue of many of the classical facts about the gamma function.

In his interesting paper [6], J. Sándor defined the functions S and S_* by

$$S(x) = \min\{m \in N : x \leq m!\}, \quad x \in (1, \infty),$$

and

$$S_*(x) = \max\{m \in N : m! \leq x\}, \quad x \in [1, \infty).$$



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He has studied many important properties of S_* and proved the following theorems:

Theorem 1.1.

$$S_*(x) \sim \frac{\log x}{\log \log x} \quad (x \rightarrow \infty).$$

Theorem 1.2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

In [1], C. Adiga and T. Kim have obtained a generalization of Theorems 1.1 and 1.2.

We now define the q -analogues of S and S_* as follows:

$$S_q(x) = \min\{m \in N : x \leq \Gamma_q(m+1)\}, \quad x \in (1, \infty),$$

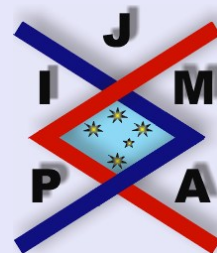
and

$$S_q^*(x) = \max\{m \in N : \Gamma_q(m+1) \leq x\}, \quad x \in [1, \infty),$$

where $0 < q < 1$.

Clearly $S_q(x) \rightarrow S(x)$ and $S_q^*(x) \rightarrow S_*(x)$ as $q \rightarrow 1^-$.

In Section 2 of this paper we study some properties of S_q and S_q^* , which are similar to those of S and S_* studied by Sándor [6]. In Section 3 we prove two theorems which are the q -analogues of Theorems 1.1 and 1.2 of Sándor [6].



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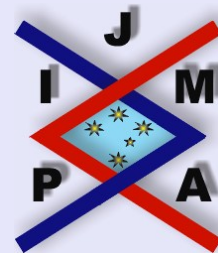
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To prove our main theorems we make use of the following q -analogue of Stirling's formula established by D.S. Moak [5]:

$$(1.1) \quad \log \Gamma_q(z) \sim \left(z - \frac{1}{2}\right) \log \left(\frac{q^z - 1}{q - 1}\right) + \frac{1}{\log q} \int_{-\log q}^{-z \log^n q} \frac{udu}{e^u - 1} \\ + C_q + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log q}{q^z - 1}\right)^{2k-1} q^z P_{2k-1}(q^z),$$

where C_q is a constant depending upon q , and $P_n(z)$ is a polynomial of degree n satisfying,

$$P_n(z) = (z - z^2)P'_{n-1}(z) + (nz + 1)P_{n-1}(z), \quad P_0 = 1, \quad n \geq 1.$$



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2. Some Properties of S_q and S_q^*

From the definitions of S_q and S_q^* , it is clear that

$$(2.1) \quad S_q(x) = m \quad \text{if } x \in (\Gamma_q(m), \Gamma_q(m+1)], \quad \text{for } m \geq 2,$$

and

$$(2.2) \quad S_q^*(x) = m \quad \text{if } x \in [\Gamma_q(m+1), \Gamma_q(m+2)), \quad \text{for } m \geq 1.$$

(2.1) and (2.2) imply

$$S_q(x) = \begin{cases} S_q^*(x) + 1, & \text{if } x \in (\Gamma_q(k+1), \Gamma_q(k+2)), \\ S_q^*(x), & \text{if } x = \Gamma_q(k+2). \end{cases}$$

Thus

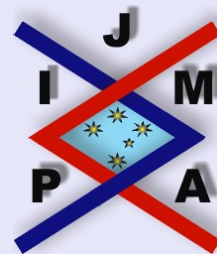
$$S_q^*(x) \leq S_q(x) \leq S_q^*(x) + 1.$$

Hence it suffices to study the function S_q^* . The following are the simple properties of S_q^* .

- (1) S_q^* is surjective and monotonically increasing.
- (2) S_q^* is continuous for all $x \in [1, \infty) \setminus A$, where $A = \{\Gamma_q(k+1) : k \geq 2\}$.
Since

$$\lim_{x \rightarrow \Gamma_q(k+1)^+} S_q^*(x) = k \quad \text{and} \quad \lim_{x \rightarrow \Gamma_q(k+1)^-} S_q^*(x) = (k-1), \quad (k \geq 2),$$

S_q^* is continuous from the right at $x = \Gamma_q(k+1)$, $k \geq 2$, but it is not continuous from the left.



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(3) S_q^* is differentiable on $(1, \infty) \setminus A$ and since

$$\lim_{x \rightarrow \Gamma_q(k+1)^+} \frac{S_q^*(x) - S_q^*(\Gamma_q(k+1))}{x - \Gamma_q(k+1)} = 0$$

for $k \geq 1$, it has a right derivative in $A \cup \{1\}$.

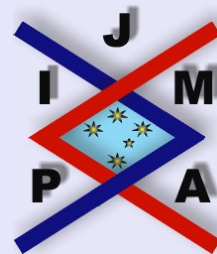
(4) S_q^* is Riemann integrable on $[a, b]$, where $\Gamma_q(k+1) \leq a < b, k \geq 1$.

(i) If $[a, b] \subset [\Gamma_q(k+1), \Gamma_q(k+2)], k \geq 1$, then

$$\int_a^b S_q^*(x) dx = \int_a^b k dx = k(b-a).$$

(ii) For $n > k$, we have

$$\begin{aligned} & \int_{\Gamma_q(k+1)}^{\Gamma_q(n+1)} S_q^*(x) dx \\ &= \sum_{m=1}^{(n-k)} \int_{\Gamma_q(k+m)}^{\Gamma_q(k+m+1)} S_q^*(x) dx \\ &= \sum_{m=1}^{(n-k)} (k+m-1) [\Gamma_q(k+m+1) - \Gamma_q(k+m)] \\ &= \sum_{m=1}^{(n-k)} (k+m-1) \Gamma_q(k+m) [q + q^2 + \dots + q^{k+m-1}]. \end{aligned}$$



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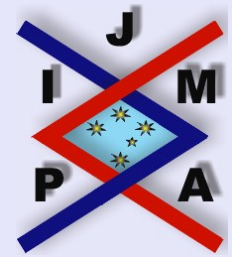


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(iii) If $a \in [\Gamma_q(k + 1), \Gamma_q(k + 2))$ and $b \in [\Gamma_q(n), \Gamma_q(n + 1))$ then

$$\begin{aligned} & \int_a^b S_q^*(x) dx \\ &= \int_a^{\Gamma_q(k+2)} S_q^*(x) dx + \int_{\Gamma_q(k+2)}^{\Gamma_q(n)} S_q^*(x) dx + \int_{\Gamma_q(n)}^b S_q^*(x) dx \\ &= k[\Gamma_q(k + 2) - a] + \sum_{m=1}^{n-k-2} (k + m)\Gamma_q(k + m + 1) \\ & \quad \times (q + q^2 + \dots + q^{k+m}) + (n - 1)[b - \Gamma_q(n)], \end{aligned}$$

by (ii).

3. Main Theorems

We now prove our main theorems.

Theorem 3.1. *If $0 < q < 1$, then*

$$S_q^*(x) \sim \frac{\log x}{\log\left(\frac{1}{1-q}\right)}.$$

Proof. If $\Gamma_q(n+1) \leq x < \Gamma_q(n+2)$, then

$$(3.1) \quad \log \Gamma_q(n+1) \leq \log x < \log \Gamma_q(n+2).$$

By (1.1) we have

$$(3.2) \quad \begin{aligned} \log \Gamma_q(n+1) &\sim \left(n + \frac{1}{2}\right) \\ \log\left(\frac{q^{n+1} - 1}{q - 1}\right) &\sim n \log\left(\frac{1}{1-q}\right). \end{aligned}$$

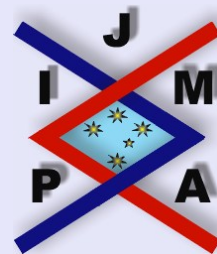
Dividing (3.1) throughout by $n \log\left(\frac{1}{1-q}\right)$, we obtain

$$(3.3) \quad \frac{\log \Gamma_q(n+1)}{n \log\left(\frac{1}{1-q}\right)} \leq \frac{\log x}{S_q^*(x) \log\left(\frac{1}{1-q}\right)} < \frac{\log \Gamma_q(n+2)}{n \log\left(\frac{1}{1-q}\right)}.$$

Using (3.2) in (3.3) we deduce

$$\lim_{n \rightarrow \infty} \frac{\log x}{S_q^*(x) \log\left(\frac{1}{1-q}\right)} = 1.$$

This completes the proof. □



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Theorem 3.2. *The series*

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{1}{n(S_q^*(n))^\alpha}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$.

Proof. Since

$$S_q^*(x) \sim \frac{\log x}{\log\left(\frac{1}{1-q}\right)},$$

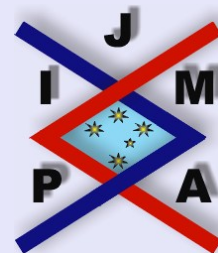
we have

$$A \frac{\log n}{\log\left(\frac{1}{1-q}\right)} < S_q^*(n) < B \frac{\log n}{\log\left(\frac{1}{1-q}\right)},$$

for all $n \geq N > 1, A, B > 0$. Therefore to examine the convergence or divergence of the series (3.4) it suffices to study the series

$$\log\left(\frac{1}{1-q}\right) \sum_{n=1}^{\infty} \frac{1}{n(\log n)^\alpha}.$$

By the integral test, $\sum \frac{1}{n(\log n)^\alpha}$ converges for $\alpha > 1$ and diverges for $0 \leq \alpha \leq 1$. If $\alpha < 0$, then $\frac{1}{n(\log n)^\alpha} > \frac{1}{n}$ for $n \geq 3$. Hence $\sum \frac{1}{n(\log n)^\alpha}$ diverges by the comparison test. \square



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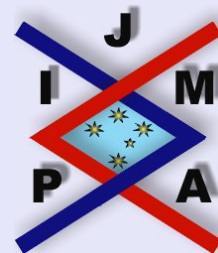
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