AN INEQUALITY ON TERNARY QUADRATIC FORMS IN TRIANGLES

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Abstract: In this short note, we give a proof of a conjecture about ternary quadratic forms involving two triangles and several interesting applications.

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1. Introduction

In [3], Liu proved the following theorem.

Theorem 1.1. *For any* $\triangle ABC$ *and real numbers* x, y, z*, the following inequality holds.*

$$x^2 \cos^2 \frac{A}{2} + y^2 \cos^2 \frac{B}{2} + z^2 \cos^2 \frac{C}{2} \geq yz \sin^2 A + zx \sin^2 B + xy \sin^2 C. \tag{1.1}$$

In [6], Tao proved the following theorem.

Theorem 1.2. *For any* $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$*, the following inequality holds.*

$$\cos \frac{A_1}{2} \cos \frac{A_2}{2} + \cos \frac{B_1}{2} \cos \frac{B_2}{2} + \cos \frac{C_1}{2} \cos \frac{C_2}{2} \geq \sin A_1 \sin A_2 + \sin B_1 \sin B_2 + \sin C_1 \sin C_2. \tag{1.2}$$

Then, in [4], Liu proposed the following conjecture.

Conjecture 1.3. *For any* $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$ *and real numbers* x, y, z*, the following inequality holds.*

$$x^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + y^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + z^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \geq yz \sin A_1 \sin A_2 + zx \sin B_1 \sin B_2 + xy \sin C_1 \sin C_2. \tag{1.3}$$

In this paper, we give a proof of this conjecture and some interesting applications.

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2. Preliminaries

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, s the semi-perimeter, S the area, R the circumradius and r the inradius, respectively. In addition we will customarily use the symbols $\sum$(cyclic sum) and $\prod$ (cyclic product):

$$\sum f(a) = f(a) + f(b) + f(c), \quad \prod f(a) = f(a)f(b)f(c).$$

To prove the inequality (1.1), we need the following well-known proposition about positive semidefinite quadratic forms.

Proposition 2.1 (see [2]). *Let p_i, q_i $(i = 1, 2, 3)$ be real numbers such that $p_i \geq 0$ $(i = 1, 2, 3)$, $4p_2p_3 \geq q_1^2$, $4p_3p_1 \geq q_2^2$, $4p_1p_2 \geq q_3^2$ and*

$$4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3. \tag{2.1}$$

Then the following inequality holds for any real numbers x, y, z,

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy. \tag{2.2}$$

Lemma 2.2. *For $\triangle ABC$, the following inequalities hold.*

$$2\cos\frac{B}{2}\cos\frac{C}{2} \geq \frac{3\sqrt{3}}{4}\sin^2 A > \sin^2 A, \tag{2.3}$$

$$2\cos\frac{C}{2}\cos\frac{A}{2} \geq \frac{3\sqrt{3}}{4}\sin^2 B > \sin^2 B, \tag{2.4}$$

$$2\cos\frac{A}{2}\cos\frac{B}{2} \geq \frac{3\sqrt{3}}{4}\sin^2 C > \sin^2 C. \tag{2.5}$$

Proof. We will only prove (2.3) because (2.4) and (2.5) can be done similarly. Since

$$S = \frac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}$$

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and

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}},$$

then it follows that

$$\begin{aligned}
&2\cos\frac{B}{2}\cos\frac{C}{2} \geq \frac{3\sqrt{3}}{4}\sin^2 A \\
&\qquad\qquad \Longleftrightarrow\ 2\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}} \geq \frac{3\sqrt{3}S^2}{b^2c^2} \\
&\qquad\qquad \Longleftrightarrow\ \frac{4s^2(s-b)(s-c)}{a^2bc} \geq \frac{27s^2(s-a)^2(s-b)^2(s-c)^2}{b^4c^4} \\
&\qquad\qquad \Longleftrightarrow\ \frac{4}{a^2} \geq \frac{27(s-a)^2(s-b)(s-c)}{b^3c^3} \\
&\qquad\qquad \Longleftrightarrow\ 4b^3c^3 \geq 27a^2(s-a)^2(s-b)(s-c). \qquad (2.6)
\end{aligned}$$

On the other hand, by the arithmetic-mean geometric-mean inequality, we have the following inequality.

$$\begin{aligned}
&27a^2(s-a)^2(s-b)(s-c) \\
&\qquad = 108\cdot\frac{1}{2}a(s-a)\cdot\frac{1}{2}a(s-a)\cdot(s-b)(s-c) \\
&\qquad \leq 108\left[\frac{\frac{1}{2}a(s-a)+\frac{1}{2}a(s-a)+(s-b)(s-c)}{3}\right]^3 \\
&\qquad = 4\left[bc-\frac{(b+c-a)^2}{4}\right]^3 < 4b^3c^3.
\end{aligned}$$

Therefore the inequality (2.6) holds, and hence (2.3) holds. □

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Lemma 2.3. *For $\triangle ABC$, the following equality holds.*

$$\sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} = \frac{(2R+5r)s^4 - 2(R+r)(16R+5r)rs^2 + (4R+r)^3 r^2}{2R^3 s^2}. \tag{2.7}$$

Proof. By the familiar identity: $a+b+c = 2s$, $ab+bc+ca = s^2 + 4Rr + r^2$, $abc = 4Rrs$ (see [5]) and the following identity

$$\begin{aligned}\sum a^5(b+c-a) = &-(a+b+c)^6 + 7(ab+bc+ca)(a+b+c)^4 \\ &- 13(a+b+c)^2(ab+bc+ca)^2 - 7abc(a+b+c)^3 \\ &+ 4(ab+bc+ca)^3 + 19abc(ab+bc+ca)(a+b+c) - 6a^2b^2c^2,\end{aligned}$$

it follows that

$$\sum a^5(b+c-a) = 4(2R+5r)rs^4 - 8(R+r)(16R+5r)r^2s^2 + 4(4R+r)^3r^3,$$

and hence

$$\begin{aligned}\sum \sin^4 A(1+\cos A) &= \sum \left(\frac{a}{2R}\right)^4 \frac{(b+c)^2 - a^2}{2bc} \\ &= \frac{(a+b+c)\sum a^5(b+c-a)}{32R^4 abc} \\ &= \frac{(2R+5r)s^4 - 2(R+r)(16R+5r)rs^2 + (4R+r)^3r^2}{16R^5}.\end{aligned}$$

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Thus, together with the familiar identity $\prod \cos \frac{A}{2} = \frac{s}{4R}$, it follows that

$$\sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} = \frac{\sum \sin^4 A \cos^2 \frac{A}{2}}{\prod \cos^2 \frac{A}{2}} = \frac{\sum \sin^4 A(1+\cos A)}{2\prod \cos^2 \frac{A}{2}} = \frac{(2R+5r)s^4 - 2(R+r)(16R+5r)rs^2 + (4R+r)^3 r^2}{2R^3 s^2}.$$

Therefore the equality (2.7) is proved. □

Lemma 2.4. *For $\triangle ABC$, the following inequality holds.*

$$-(2R+5r)s^4 + 2(2R+5r)(2R+r)(R+r)s^2 - (4R+r)^3 r^2 \geq 0. \tag{2.8}$$

Proof. First it is easy to verify that the inequality (2.8) is just the following inequality.

$$\begin{aligned}(2R+5r)&[-s^4 + (4R^2+20Rr-2r^2)s^2 - r(4R+r)^3] \\ &+ 2r(14R^2+31Rr-10r^2)(4R^2+4Rr+3r^2-s^2) \\ &+ 4(R-2r)(4R^3+6R^2r+3Rr^2-8r^3) \geq 0.\end{aligned} \tag{2.9}$$

Thus, together with the fundamental inequality

$$-s^4 + (4R^2+20Rr-2r^2)s^2 - r(4R+r)^3 \geq 0$$

(see [5, page 2]), Euler's inequality $R \geq 2r$ and Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$ (see [1, page 45]), it follows that the inequality (2.9) holds, and hence (2.8) holds. □

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Lemma 2.5. *For $\triangle ABC$, the following inequality holds.*

$$\sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} + 64 \prod \sin^2 \frac{A}{2} \leq 4. \tag{2.10}$$

Proof. By Lemma 2.3 and the familiar identity $\prod \sin \frac{A}{2} = \frac{r}{4R}$, it follows that

$$\begin{aligned} &\sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} + 64 \prod \sin^2 \frac{A}{2} \leq 4 \\ &\iff \frac{(2R+5r)s^4 - 2(R+r)(16R+5r)rs^2 + (4R+r)^3 r^2}{2R^3 s^2} + \frac{4r^2}{R^2} \leq 4 \\ &\iff \frac{-(2R+5r)s^4 + 2(2R+5r)(2R+r)(R+r)s^2 - (4R+r)^3 r^2}{2R^3 s^2} \geq 0. \end{aligned} \tag{2.11}$$

Thus, by Lemma 2.4, it follows that the inequality (2.11) holds, and hence (2.10) holds. □

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3. Proof of the Main Theorem

Now we give the proof of inequality (1.1).

Proof. First, it is easy to verify that

$$\cos\frac{A_1}{2}\cos\frac{A_2}{2}\geq 0, \tag{3.1}$$

$$\cos\frac{B_1}{2}\cos\frac{B_2}{2}\geq 0, \tag{3.2}$$

$$\cos\frac{C_1}{2}\cos\frac{C_2}{2}\geq 0. \tag{3.3}$$

Next, by Lemma 2.2, we have the following inequalities:

$$4\cos\frac{B_1}{2}\cos\frac{B_2}{2}\cdot\cos\frac{C_1}{2}\cos\frac{C_2}{2}\geq \sin^2 A_1\sin^2 A_2, \tag{3.4}$$

$$4\cos\frac{C_1}{2}\cos\frac{C_2}{2}\cdot\cos\frac{A_1}{2}\cos\frac{A_2}{2}\geq \sin^2 B_1\sin^2 B_2, \tag{3.5}$$

$$4\cos\frac{A_1}{2}\cos\frac{A_2}{2}\cdot\cos\frac{B_1}{2}\cos\frac{B_2}{2}\geq \sin^2 C_1\sin^2 C_2. \tag{3.6}$$

Thus, in order that Proposition 2.1 is applicable, we have to show the following inequality.

$$\begin{aligned}4\prod\cos\frac{A_1}{2}\prod\cos\frac{A_2}{2} &\geq \cos\frac{A_1}{2}\sin^2 A_1\cos\frac{A_2}{2}\sin^2 A_2+\cos\frac{B_1}{2}\sin^2 B_1\cos\frac{B_2}{2}\sin^2 B_2\\ &\quad+\cos\frac{C_1}{2}\sin^2 C_1\cos\frac{C_2}{2}\sin^2 C_2+\prod\sin A_1\prod\sin A_2.\end{aligned} \tag{3.7}$$

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However, in order to prove the inequality (3.7), we only need the following inequality.

$$\frac{\sin^2 A_1}{\cos\frac{B_1}{2}\cos\frac{C_1}{2}}\cdot\frac{\sin^2 A_2}{\cos\frac{B_2}{2}\cos\frac{C_2}{2}}+\frac{\sin^2 B_1}{\cos\frac{C_1}{2}\cos\frac{A_1}{2}}\cdot\frac{\sin^2 B_2}{\cos\frac{C_2}{2}\cos\frac{A_2}{2}} +\frac{\sin^2 C_1}{\cos\frac{A_1}{2}\cos\frac{B_1}{2}}\cdot\frac{\sin^2 C_2}{\cos\frac{A_2}{2}\cos\frac{B_2}{2}}+8\prod\sin\frac{A_1}{2}\cdot 8\prod\sin\frac{A_2}{2}\le 4. \tag{3.8}$$

In fact, by the Cauchy inequality and Lemma 2.5, we have that

$$\begin{aligned}&\left[\frac{\sin^2 A_1}{\cos\frac{B_1}{2}\cos\frac{C_1}{2}}\cdot\frac{\sin^2 A_2}{\cos\frac{B_2}{2}\cos\frac{C_2}{2}}+\frac{\sin^2 B_1}{\cos\frac{C_1}{2}\cos\frac{A_1}{2}}\cdot\frac{\sin^2 B_2}{\cos\frac{C_2}{2}\cos\frac{A_2}{2}}\right.\\ &\qquad\left.+\frac{\sin^2 C_1}{\cos\frac{A_1}{2}\cos\frac{B_1}{2}}\cdot\frac{\sin^2 C_2}{\cos\frac{A_2}{2}\cos\frac{B_2}{2}}+8\prod\sin\frac{A_1}{2}\cdot 8\prod\sin\frac{A_2}{2}\right]^2\\ &\le\left[\sum\frac{\sin^4 A_1}{\cos^2\frac{B_1}{2}\cos^2\frac{C_1}{2}}+64\prod\sin^2\frac{A_1}{2}\right]\\ &\qquad\times\left[\sum\frac{\sin^4 A_2}{\cos^2\frac{B_2}{2}\cos^2\frac{C_2}{2}}+64\prod\sin^2\frac{A_2}{2}\right]\\ &\le 16\end{aligned}$$

Therefore the inequality (3.8) holds, and hence (3.7) holds. Thus, together with inequality (3.4)–(3.7), Proposition 2.1 is applicable to complete the proof of (1.1). □

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4. Applications

Let P be a point in the $\triangle ABC$. Recall that A, B, C denote the angles, a, b, c the lengths of sides, w_a, w_b, w_c the lengths of interior angular bisectors, m_a, m_b, m_c the lengths of medians, h_a, h_b, h_c the lengths of altitudes, R_1, R_2, R_3 the distances of P to vertices A, B, C, r_1, r_2, r_3 the distances of P to the sidelines BC, CA, AB.

Corollary 4.1. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$a^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + b^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + c^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq bc\sin A_1 \sin A_2 + ca\sin B_1 \sin B_2 + ab\sin C_1 \sin C_2.$$

Corollary 4.2. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$w_a^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + w_b^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + w_c^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq w_bw_c\sin A_1 \sin A_2 + w_cw_a\sin B_1 \sin B_2 + w_aw_b\sin C_1 \sin C_2.$$

Corollary 4.3. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$m_a^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + m_b^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + m_c^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq m_bm_c\sin A_1 \sin A_2 + m_cm_a\sin B_1 \sin B_2 + m_am_b\sin C_1 \sin C_2.$$

Corollary 4.4. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality*

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holds.

$$h_a^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + h_b^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + h_c^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq h_b h_c \sin A_1 \sin A_2 + h_c h_a \sin B_1 \sin B_2 + h_a h_b \sin C_1 \sin C_2.$$

Corollary 4.5. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$R_1^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + R_2^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + R_3^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq R_2 R_3 \sin A_1 \sin A_2 + R_3 R_1 \sin B_1 \sin B_2 + R_1 R_2 \sin C_1 \sin C_2.$$

Corollary 4.6. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$r_1^2 \cos\frac{A_1}{2}\cos\frac{A_2}{2} + r_2^2 \cos\frac{B_1}{2}\cos\frac{B_2}{2} + r_3^2 \cos\frac{C_1}{2}\cos\frac{C_2}{2} \geq r_2 r_3 \sin A_1 \sin A_2 + r_3 r_1 \sin B_1 \sin B_2 + r_1 r_2 \sin C_1 \sin C_2.$$

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