

AN INEQUALITY ON TERNARY QUADRATIC FORMS IN TRIANGLES

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Abstract: In this short note, we give a proof of a conjecture about ternary quadratic forms involving two triangles and several interesting applications.



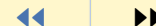
**Ternary Quadratic Forms in
Triangles**

Nu-Chun Hu

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1. Introduction

In [3], Liu proved the following theorem.

Theorem 1.1. *For any $\triangle ABC$ and real numbers x, y, z , the following inequality holds.*

$$(1.1) \quad x^2 \cos^2 \frac{A}{2} + y^2 \cos^2 \frac{B}{2} + z^2 \cos^2 \frac{C}{2} \geq yz \sin^2 A + zx \sin^2 B + xy \sin^2 C.$$

In [6], Tao proved the following theorem.

Theorem 1.2. *For any $\triangle A_1 B_1 C_1, \triangle A_2 B_2 C_2$, the following inequality holds.*

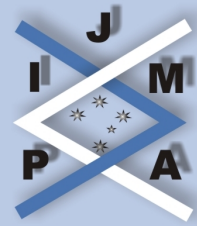
$$(1.2) \quad \cos \frac{A_1}{2} \cos \frac{A_2}{2} + \cos \frac{B_1}{2} \cos \frac{B_2}{2} + \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq \sin A_1 \sin A_2 + \sin B_1 \sin B_2 + \sin C_1 \sin C_2.$$

Then, in [4], Liu proposed the following conjecture.

Conjecture 1.3. *For any $\triangle A_1 B_1 C_1, \triangle A_2 B_2 C_2$ and real numbers x, y, z , the following inequality holds.*

$$(1.3) \quad x^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + y^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + z^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq yz \sin A_1 \sin A_2 + zx \sin B_1 \sin B_2 + xy \sin C_1 \sin C_2.$$

In this paper, we give a proof of this conjecture and some interesting applications.



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2. Preliminaries

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, s the semi-perimeter, S the area, R the circumradius and r the inradius, respectively. In addition we will customarily use the symbols \sum (cyclic sum) and \prod (cyclic product):

$$\sum f(a) = f(a) + f(b) + f(c), \quad \prod f(a) = f(a)f(b)f(c).$$

To prove the inequality (1.1), we need the following well-known proposition about positive semidefinite quadratic forms.

Proposition 2.1 (see [2]). *Let p_i, q_i ($i = 1, 2, 3$) be real numbers such that $p_i \geq 0$ ($i = 1, 2, 3$), $4p_2p_3 \geq q_1^2$, $4p_3p_1 \geq q_2^2$, $4p_1p_2 \geq q_3^2$ and*

$$(2.1) \quad 4p_1p_2p_3 \geq p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3.$$

Then the following inequality holds for any real numbers x, y, z ,

$$(2.2) \quad p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy.$$

Lemma 2.2. *For $\triangle ABC$, the following inequalities hold.*

$$(2.3) \quad 2 \cos \frac{B}{2} \cos \frac{C}{2} \geq \frac{3\sqrt{3}}{4} \sin^2 A > \sin^2 A,$$

$$(2.4) \quad 2 \cos \frac{C}{2} \cos \frac{A}{2} \geq \frac{3\sqrt{3}}{4} \sin^2 B > \sin^2 B,$$

$$(2.5) \quad 2 \cos \frac{A}{2} \cos \frac{B}{2} \geq \frac{3\sqrt{3}}{4} \sin^2 C > \sin^2 C.$$

Proof. We will only prove (2.3) because (2.4) and (2.5) can be done similarly. Since

$$S = \frac{1}{2}bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$$

and

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}},$$

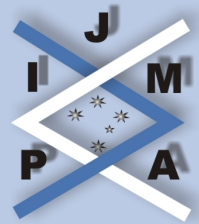
then it follows that

$$\begin{aligned} 2 \cos \frac{B}{2} \cos \frac{C}{2} &\geq \frac{3\sqrt{3}}{4} \sin^2 A \\ \Leftrightarrow 2 \sqrt{\frac{s(s-b)}{ca}} \sqrt{\frac{s(s-c)}{ab}} &\geq \frac{3\sqrt{3}S^2}{b^2c^2} \\ \Leftrightarrow \frac{4s^2(s-b)(s-c)}{a^2bc} &\geq \frac{27s^2(s-a)^2(s-b)^2(s-c)^2}{b^4c^4} \\ \Leftrightarrow \frac{4}{a^2} &\geq \frac{27(s-a)^2(s-b)(s-c)}{b^3c^3} \\ (2.6) \quad \Leftrightarrow 4b^3c^3 &\geq 27a^2(s-a)^2(s-b)(s-c). \end{aligned}$$

On the other hand, by the arithmetic-mean geometric-mean inequality, we have the following inequality.

$$\begin{aligned} &27a^2(s-a)^2(s-b)(s-c) \\ &= 108 \cdot \frac{1}{2}a(s-a) \cdot \frac{1}{2}a(s-a) \cdot (s-b)(s-c) \\ &\leq 108 \left[\frac{\frac{1}{2}a(s-a) + \frac{1}{2}a(s-a) + (s-b)(s-c)}{3} \right]^3 \\ &= 4 \left[bc - \frac{(b+c-a)^2}{4} \right]^3 < 4b^3c^3. \end{aligned}$$

Therefore the inequality (2.6) holds, and hence (2.3) holds. \square



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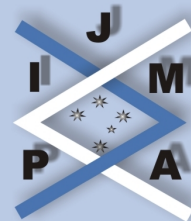


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Lemma 2.3. For $\triangle ABC$, the following equality holds.

$$(2.7) \quad \sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} = \frac{(2R + 5r)s^4 - 2(R + r)(16R + 5r)rs^2 + (4R + r)^3r^2}{2R^3s^2}.$$

Proof. By the familiar identity: $a + b + c = 2s$, $ab + bc + ca = s^2 + 4Rr + r^2$, $abc = 4Rrs$ (see [5]) and the following identity

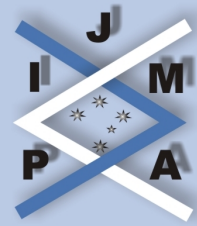
$$\begin{aligned} \sum a^5(b + c - a) &= -(a + b + c)^6 + 7(ab + bc + ca)(a + b + c)^4 \\ &\quad - 13(a + b + c)^2(ab + bc + ca)^2 - 7abc(a + b + c)^3 \\ &\quad + 4(ab + bc + ca)^3 + 19abc(ab + bc + ca)(a + b + c) - 6a^2b^2c^2, \end{aligned}$$

it follows that

$$\sum a^5(b + c - a) = 4(2R + 5r)rs^4 - 8(R + r)(16R + 5r)r^2s^2 + 4(4R + r)^3r^3,$$

and hence

$$\begin{aligned} \sum \sin^4 A(1 + \cos A) &= \sum \left(\frac{a}{2R}\right)^4 \frac{(b + c)^2 - a^2}{2bc} \\ &= \frac{(a + b + c) \sum a^5(b + c - a)}{32R^4abc} \\ &= \frac{(2R + 5r)s^4 - 2(R + r)(16R + 5r)rs^2 + (4R + r)^3r^2}{16R^5}. \end{aligned}$$



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Thus, together with the familiar identity $\prod \cos \frac{A}{2} = \frac{s}{4R}$, it follows that

$$\begin{aligned} \sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} &= \frac{\sum \sin^4 A \cos^2 \frac{A}{2}}{\prod \cos^2 \frac{A}{2}} \\ &= \frac{\sum \sin^4 A (1 + \cos A)}{2 \prod \cos^2 \frac{A}{2}} \\ &= \frac{(2R + 5r)s^4 - 2(R + r)(16R + 5r)rs^2 + (4R + r)^3 r^2}{2R^3 s^2}. \end{aligned}$$

Therefore the equality (2.7) is proved. □

Lemma 2.4. For $\triangle ABC$, the following inequality holds.

$$(2.8) \quad -(2R + 5r)s^4 + 2(2R + 5r)(2R + r)(R + r)s^2 - (4R + r)^3 r^2 \geq 0.$$

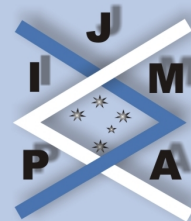
Proof. First it is easy to verify that the inequality (2.8) is just the following inequality.

$$(2.9) \quad \begin{aligned} (2R + 5r)[-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3] \\ + 2r(14R^2 + 31Rr - 10r^2)(4R^2 + 4Rr + 3r^2 - s^2) \\ + 4(R - 2r)(4R^3 + 6R^2r + 3Rr^2 - 8r^3) \geq 0. \end{aligned}$$

Thus, together with the fundamental inequality

$$-s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0$$

(see [5, page 2]), Euler's inequality $R \geq 2r$ and Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$ (see [1, page 45]), it follows that the inequality (2.9) holds, and hence (2.8) holds. □



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Lemma 2.5. For $\triangle ABC$, the following inequality holds.

$$(2.10) \quad \sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} + 64 \prod \sin^2 \frac{A}{2} \leq 4.$$

Proof. By Lemma 2.3 and the familiar identity $\prod \sin \frac{A}{2} = \frac{r}{4R}$, it follows that

$$(2.11) \quad \begin{aligned} & \sum \frac{\sin^4 A}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} + 64 \prod \sin^2 \frac{A}{2} \leq 4 \\ \Leftrightarrow & \frac{(2R+5r)s^4 - 2(R+r)(16R+5r)rs^2 + (4R+r)^3r^2}{2R^3s^2} + \frac{4r^2}{R^2} \leq 4 \\ \Leftrightarrow & \frac{-(2R+5r)s^4 + 2(2R+5r)(2R+r)(R+r)s^2 - (4R+r)^3r^2}{2R^3s^2} \geq 0. \end{aligned}$$

Thus, by Lemma 2.4, it follows that the inequality (2.11) holds, and hence (2.10) holds. \square



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3. Proof of the Main Theorem

Now we give the proof of inequality (1.1).

Proof. First, it is easy to verify that

$$(3.1) \quad \cos \frac{A_1}{2} \cos \frac{A_2}{2} \geq 0,$$

$$(3.2) \quad \cos \frac{B_1}{2} \cos \frac{B_2}{2} \geq 0,$$

$$(3.3) \quad \cos \frac{C_1}{2} \cos \frac{C_2}{2} \geq 0.$$

Next, by Lemma 2.2, we have the following inequalities:

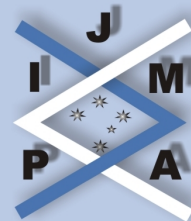
$$(3.4) \quad 4 \cos \frac{B_1}{2} \cos \frac{B_2}{2} \cdot \cos \frac{C_1}{2} \cos \frac{C_2}{2} \geq \sin^2 A_1 \sin^2 A_2,$$

$$(3.5) \quad 4 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \cdot \cos \frac{A_1}{2} \cos \frac{A_2}{2} \geq \sin^2 B_1 \sin^2 B_2,$$

$$(3.6) \quad 4 \cos \frac{A_1}{2} \cos \frac{A_2}{2} \cdot \cos \frac{B_1}{2} \cos \frac{B_2}{2} \geq \sin^2 C_1 \sin^2 C_2.$$

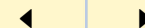
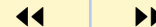
Thus, in order that Proposition 2.1 is applicable, we have to show the following inequality.

$$(3.7) \quad 4 \prod \cos \frac{A_1}{2} \prod \cos \frac{A_2}{2} \\ \geq \cos \frac{A_1}{2} \sin^2 A_1 \cos \frac{A_2}{2} \sin^2 A_2 + \cos \frac{B_1}{2} \sin^2 B_1 \cos \frac{B_2}{2} \sin^2 B_2 \\ + \cos \frac{C_1}{2} \sin^2 C_1 \cos \frac{C_2}{2} \sin^2 C_2 + \prod \sin A_1 \prod \sin A_2.$$



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However, in order to prove the inequality (3.7), we only need the following inequality.

$$(3.8) \quad \frac{\sin^2 A_1}{\cos \frac{B_1}{2} \cos \frac{C_1}{2}} \cdot \frac{\sin^2 A_2}{\cos \frac{B_2}{2} \cos \frac{C_2}{2}} + \frac{\sin^2 B_1}{\cos \frac{C_1}{2} \cos \frac{A_1}{2}} \cdot \frac{\sin^2 B_2}{\cos \frac{C_2}{2} \cos \frac{A_2}{2}} \\ + \frac{\sin^2 C_1}{\cos \frac{A_1}{2} \cos \frac{B_1}{2}} \cdot \frac{\sin^2 C_2}{\cos \frac{A_2}{2} \cos \frac{B_2}{2}} + 8 \prod \sin \frac{A_1}{2} \cdot 8 \prod \sin \frac{A_2}{2} \leq 4.$$

In fact, by the Cauchy inequality and Lemma 2.5, we have that

$$\left[\frac{\sin^2 A_1}{\cos \frac{B_1}{2} \cos \frac{C_1}{2}} \cdot \frac{\sin^2 A_2}{\cos \frac{B_2}{2} \cos \frac{C_2}{2}} + \frac{\sin^2 B_1}{\cos \frac{C_1}{2} \cos \frac{A_1}{2}} \cdot \frac{\sin^2 B_2}{\cos \frac{C_2}{2} \cos \frac{A_2}{2}} \right. \\ \left. + \frac{\sin^2 C_1}{\cos \frac{A_1}{2} \cos \frac{B_1}{2}} \cdot \frac{\sin^2 C_2}{\cos \frac{A_2}{2} \cos \frac{B_2}{2}} + 8 \prod \sin \frac{A_1}{2} \cdot 8 \prod \sin \frac{A_2}{2} \right]^2 \\ \leq \left[\sum \frac{\sin^4 A_1}{\cos^2 \frac{B_1}{2} \cos^2 \frac{C_1}{2}} + 64 \prod \sin^2 \frac{A_1}{2} \right] \\ \times \left[\sum \frac{\sin^4 A_2}{\cos^2 \frac{B_2}{2} \cos^2 \frac{C_2}{2}} + 64 \prod \sin^2 \frac{A_2}{2} \right] \\ \leq 16$$

Therefore the inequality (3.8) holds, and hence (3.7) holds. Thus, together with inequality (3.4)–(3.7), Proposition 2.1 is applicable to complete the proof of (1.1). \square



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4. Applications

Let P be a point in the $\triangle ABC$. Recall that A, B, C denote the angles, a, b, c the lengths of sides, w_a, w_b, w_c the lengths of interior angular bisectors, m_a, m_b, m_c the lengths of medians, h_a, h_b, h_c the lengths of altitudes, R_1, R_2, R_3 the distances of P to vertices A, B, C , r_1, r_2, r_3 the distances of P to the sidelines BC, CA, AB .

Corollary 4.1. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$a^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + b^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + c^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq bc \sin A_1 \sin A_2 + ca \sin B_1 \sin B_2 + ab \sin C_1 \sin C_2.$$

Corollary 4.2. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$w_a^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + w_b^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + w_c^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq w_b w_c \sin A_1 \sin A_2 + w_c w_a \sin B_1 \sin B_2 + w_a w_b \sin C_1 \sin C_2.$$

Corollary 4.3. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality holds.*

$$m_a^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + m_b^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + m_c^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq m_b m_c \sin A_1 \sin A_2 + m_c m_a \sin B_1 \sin B_2 + m_a m_b \sin C_1 \sin C_2.$$

Corollary 4.4. *For any $\triangle ABC$, $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, the following inequality*



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holds.

$$h_a^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + h_b^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + h_c^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq h_b h_c \sin A_1 \sin A_2 + h_c h_a \sin B_1 \sin B_2 + h_a h_b \sin C_1 \sin C_2.$$

Corollary 4.5. For any $\triangle ABC$, $\triangle A_1 B_1 C_1$, $\triangle A_2 B_2 C_2$, the following inequality holds.

$$R_1^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + R_2^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + R_3^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq R_2 R_3 \sin A_1 \sin A_2 + R_3 R_1 \sin B_1 \sin B_2 + R_1 R_2 \sin C_1 \sin C_2.$$

Corollary 4.6. For any $\triangle ABC$, $\triangle A_1 B_1 C_1$, $\triangle A_2 B_2 C_2$, the following inequality holds.

$$r_1^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + r_2^2 \cos \frac{B_1}{2} \cos \frac{B_2}{2} + r_3^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ \geq r_2 r_3 \sin A_1 \sin A_2 + r_3 r_1 \sin B_1 \sin B_2 + r_1 r_2 \sin C_1 \sin C_2.$$

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