

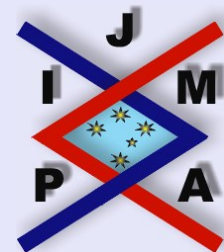
SOME RESULTS ON THE COMPLEX OSCILLATION THEORY OF DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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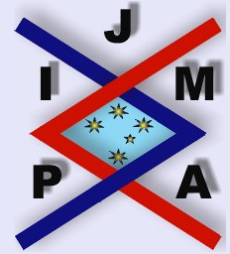
In this paper, we study the possible orders of transcendental solutions of the differential equation $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \dots + a_1(z)f' + a_0(z)f = 0$, where $a_0(z), \dots, a_{n-1}(z)$ are nonconstant polynomials. We also investigate the possible orders and exponents of convergence of distinct zeros of solutions of non-homogeneous differential equation $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \dots + a_1(z)f' + a_0(z)f = b(z)$, where $a_0(z), \dots, a_{n-1}(z)$ and $b(z)$ are nonconstant polynomials. Several examples are given.

2000 Mathematics Subject Classification: 34M10, 34M05, 30D35.

Key words: Differential equations, Order of growth, Exponent of convergence of distinct zeros, Wiman-Valiron theory.

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1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [3]). Let $\sigma(f)$ denote the order of an entire function f , that is,

$$(1.1) \quad \sigma(f) = \overline{\lim}_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r} = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \log M(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f (see [3]), and $M(r, f) = \max_{|z|=r} |f(z)|$.

We recall the following definition.

Definition 1.1. *Let f be an entire function. Then the exponent of convergence of distinct zeros of $f(z)$ is defined by*

$$(1.2) \quad \bar{\lambda}(f) = \overline{\lim}_{r \rightarrow +\infty} \frac{\log \bar{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

We define the logarithmic measure of a set $E \subset [1, +\infty[$ by

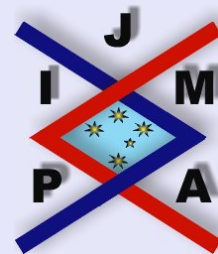
$$lm(E) = \int_1^{+\infty} \frac{\chi_E(t) dt}{t},$$

where χ_E is the characteristic function of set E .

In the study of the differential equations,

$$(1.3) \quad f'' + a_1(z) f' + a_0(z) f = 0, \quad f'' + a_1(z) f' + a_0(z) f = b(z),$$

where $a_0(z)$, $a_1(z)$ and $b(z)$ are nonconstant polynomials, Z.-X. Chen and C.-



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C. Yang proved the following results:

Theorem 1.1 ([1]). *Let a_0 and a_1 be nonconstant polynomials with degrees $\deg a_j = n_j$ ($j = 0, 1$). Let $f(z)$ be an entire solution of the differential equation*

$$(1.4) \quad f'' + a_1(z) f' + a_0(z) f = 0.$$

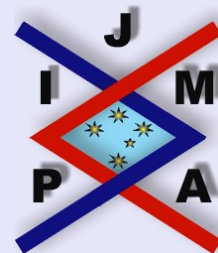
Then

- (i) *If $n_0 \geq 2n_1$, then any entire solution $f \not\equiv 0$ of the equation (1.4) satisfies $\sigma(f) = \frac{n_0+2}{2}$.*
- (ii) *If $n_0 < n_1 - 1$, then any entire solution $f \not\equiv 0$ of (1.4) satisfies $\sigma(f) = n_1 + 1$.*
- (iii) *If $n_1 - 1 \leq n_0 < 2n_1$, then any entire solution of (1.4) satisfies either $\sigma(f) = n_1 + 1$ or $\sigma(f) = n_0 - n_1 + 1$.*
- (iv) *In (iii), if $n_0 = n_1 - 1$, then the equation (1.4) possibly has polynomial solutions, and any two polynomial solutions of (1.4) are linearly dependent, all the polynomial solutions have the form $f_c(z) = cp(z)$, where p is some polynomial, c is an arbitrary constant.*

Theorem 1.2 ([1]). *Let a_0 , a_1 and b be nonconstant polynomials with degrees $\deg a_j = n_j$ ($j = 0, 1$). Let $f \not\equiv 0$ be an entire solution of the differential equation*

$$(1.5) \quad f'' + a_1(z) f' + a_0(z) f = b(z).$$

Then



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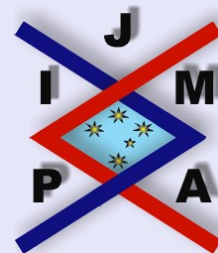


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(i) If $n_0 \geq 2n_1$, then $\bar{\lambda}(f) = \sigma(f) = \frac{n_0+2}{2}$.

(ii) If $n_0 < n_1 - 1$, then $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$.

(iii) If $n_1 - 1 < n_0 < 2n_1$, then $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$ or $\bar{\lambda}(f) = \sigma(f) = n_0 - n_1 + 1$, with at most one exceptional polynomial solution f_0 for three cases above.

(iv) If $n_0 = n_1 - 1$, then every transcendental entire solution f satisfies $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$ (or 0).

Remark 1. If the corresponding homogeneous equation of (1.5) has a polynomial solution $p(z)$, then (1.5) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (1.5), c is a constant). If the corresponding homogeneous equation of (1.5) has no polynomial solution, then (1.5) has at most one polynomial solution.

2. Statement and Proof of Results

For $n \geq 2$, we consider the linear differential equation

$$(2.1) \quad f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = 0,$$

where $a_0(z), \dots, a_{n-1}(z)$ are nonconstant polynomials with degrees $\deg a_j = d_j$ ($j = 0, \dots, n-1$). It is well-known that all solutions of equation (2.1) are entire functions of finite rational order see [7], [6, pp. 106-108], [8, pp. 65-67]. It is also known [5, p. 127], that for any solution f of (2.1), we have

$$(2.2) \quad \sigma(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{d_k}{n-k}.$$

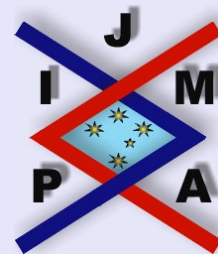
Recently G. Gundersen, M. Steinbart and S. Wang have investigated the possible orders of solutions of equation (2.1) in [2]. In the present paper, we prove two theorems which are analogous to Theorem 1.1 and Theorem 1.2 for higher order linear differential equations.

Theorem 2.1. *Let $a_0(z), \dots, a_{n-1}(z)$ be nonconstant polynomials with degrees $\deg a_j = d_j$ ($j = 0, 1, \dots, n-1$). Let $f(z)$ be an entire solution of the differential equation*

$$(2.3) \quad f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_1(z) f' + a_0(z) f = 0.$$

Then

- (i) *If $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ holds for all $j = 1, \dots, n-1$, then any entire solution $f \not\equiv 0$ of the equation (2.3) satisfies $\sigma(f) = \frac{d_0+n}{n}$.*



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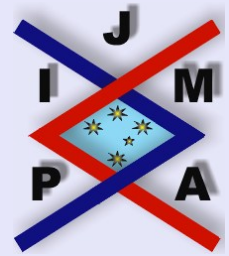


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(ii) If $d_j < d_{n-1} - (n - j - 1)$ holds for all $j = 0, \dots, n - 2$, then any entire solution $f \not\equiv 0$ of (2.3) satisfies $\sigma(f) = 1 + d_{n-1}$.

(iii) If $d_j - 1 \leq d_{j-1} < d_j + d_{n-1}$ holds for all $j = 1, \dots, n - 1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j - k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j - k}$ for all $0 \leq k < j - 1$, then the possible orders of any solution $f \not\equiv 0$ of (2.3) are:

$$1 + d_{n-1}, 1 + d_{n-2} - d_{n-1}, \dots, 1 + d_{j-1} - d_j, \dots, 1 + d_0 - d_1.$$

(iv) In (iii), if $d_{j-1} = d_j - 1$ for all $j = 1, \dots, n - 1$, then the equation (2.3) possibly has polynomial solutions, and any n polynomial solutions of (2.3) are linearly dependent, all the polynomial solutions have the form $f_c(z) = cp(z)$, where p is some polynomial, c is an arbitrary constant.

Theorem 2.2. Let $a_0(z), \dots, a_{n-1}(z)$ and $b(z)$ be nonconstant polynomials with degrees $\deg a_j = d_j$ ($j = 0, 1, \dots, n - 1$). Let $f \not\equiv 0$ be an entire solution of the differential equation

$$(2.4) \quad f^{(n)} + a_{n-1}(z) f^{(n-1)} + \dots + a_1(z) f' + a_0(z) f = b(z).$$

Then

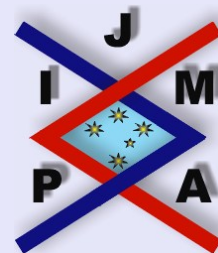
(i) If $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ holds for all $j = 1, \dots, n - 1$, then $\bar{\lambda}(f) = \sigma(f) = \frac{d_0 + n}{n}$.

(ii) If $d_j < d_{n-1} - (n - j - 1)$ holds for all $j = 0, \dots, n - 2$, then $\bar{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$.

(iii) If $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ holds for all $j = 1, \dots, n - 1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j - 1$, then $\bar{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$ or $\bar{\lambda}(f) = \sigma(f) = 1 + d_{n-2} - d_{n-1}$ or ... or $\bar{\lambda}(f) = \sigma(f) = 1 + d_{j-1} - d_j$ or ... or $\bar{\lambda}(f) = \sigma(f) = 1 + d_0 - d_1$, with at most one exceptional polynomial solution f_0 for three cases above.

(iv) If $d_{j-1} = d_j - 1$ for some $j = 1, \dots, n - 1$, then any transcendental entire solution f of (2.4) satisfies $\bar{\lambda}(f) = \sigma(f) = 1 + d_{n-1}$ or $\bar{\lambda}(f) = \sigma(f) = 1 + d_{n-2} - d_{n-1}$ or ... or $\bar{\lambda}(f) = \sigma(f) = 1 + d_j - d_{j+1}$ or $\bar{\lambda}(f) = \sigma(f) = 1 + d_{j-2} - d_{j-1}$ or ... or $\bar{\lambda}(f) = \sigma(f) = 1 + d_0 - d_1$ (or 0).

Remark 2. If the corresponding homogeneous equation of (2.4) has a polynomial solution $p(z)$, then (2.4) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (2.4), c is a constant). If the corresponding homogeneous equation of (2.4) has no polynomial solution, then (2.4) has at most one polynomial solution.



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3. Proof of Theorem 2.1

Assume that $f(z)$ is a transcendental entire solution of (2.3). First of all from the Wiman-Valiron theory (see [4] or [6]), it follows that there exists a set E_1 that has finite logarithmic measure, such that for all $j = 1, \dots, n$ we have

$$(3.1) \quad \frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1))$$

as $r \rightarrow +\infty, r \notin E_1$, where $|z| = r$ and $|f(z)| = M(r, f)$. Here $\nu_f(r)$ denotes the central index of f . Furthermore

$$(3.2) \quad \nu_f(r) = (1 + o(1)) \alpha r^\sigma$$

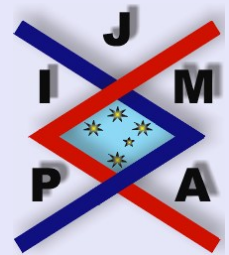
as $r \rightarrow +\infty$, where $\sigma = \sigma(f)$ and α is a positive constant. Now we divide equation (2.3) by f , and then substitute (3.1) and (3.2) into (2.3). This yields an equation whose right side is zero and whose left side consists of a sum of $(n + 1)$ terms whose absolute values are asymptotic as $(r \rightarrow +\infty, r \notin E_1)$ to the following $(n + 1)$ terms:

$$(3.3) \quad \alpha^n r^{n(\sigma-1)}, \beta_{n-1} r^{d_{n-1} + (n-1)(\sigma-1)}, \dots, \beta_j r^{d_j + j(\sigma-1)}, \dots, \beta_0 r^{d_0}$$

where $\beta_j = \alpha^j |b_j|$ and $a_j = b_j z^{d_j} (1 + o(1))$ for each $j = 0, \dots, n - 1$.

(i) If $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \dots, n - 1$, then

$$(3.4) \quad \sigma(f) \leq 1 + \max_{0 \leq k \leq n-1} \frac{d_k}{n-k} = 1 + \frac{d_0}{n}.$$



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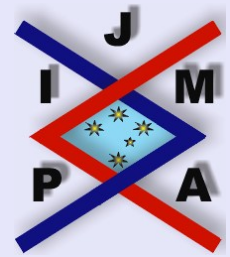


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Suppose that $\sigma(f) < 1 + \frac{d_0}{n}$, then we have

$$(3.5) \quad d_j + j(\sigma - 1) < \left(\frac{n-j}{n}\right) d_0 + j \frac{d_0}{n} = d_0$$

for all $j = 1, \dots, n-1$. Then the term in (3.3) with exponent d_0 is a dominant term as $(r \rightarrow +\infty, r \notin E_1)$. This is impossible. Hence $\sigma(f) = 1 + \frac{d_0}{n}$.

(ii) If $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \dots, n-2$, then we have

$$(3.6) \quad \frac{d_j}{n-j} < \frac{d_{n-1} - (n-j-1)}{n-j} < \frac{d_{n-1}}{n-j} < d_{n-1}$$

for all $j = 0, \dots, n-2$. Hence $\max_{0 \leq j \leq n-1} \frac{d_j}{n-j} = d_{n-1}$ and $\sigma(f) \leq 1 + d_{n-1}$.

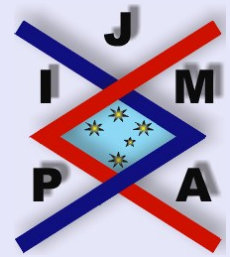
Suppose that $\sigma(f) < 1 + d_{n-1}$. We have for all $j = 0, \dots, n-2$,

$$(3.7) \quad \begin{aligned} d_j + j(\sigma - 1) &< d_{n-1} - (n-j-1) + j(\sigma - 1) \\ &< d_{n-1} - (n-j-1) + j(\sigma - 1) + (n-j-1)\sigma \\ &\leq d_{n-1} + (n-1)(\sigma - 1). \end{aligned}$$

Then the term in (3.3) with exponent $d_{n-1} + (n-1)(\sigma - 1)$ is a dominant term as $(r \rightarrow +\infty, r \notin E_1)$. This is impossible. Hence $\sigma(f) = 1 + d_{n-1}$.

(iii) If $d_j - 1 \leq d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \dots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, then we have in this case

$$(3.8) \quad \max_{0 \leq j \leq n-1} \frac{d_j}{n-j} = d_{n-1}.$$



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Hence $\sigma(f) \leq 1 + d_{n-1}$. Set

$$(3.9) \quad \sigma_j = 1 + d_{j-1} - d_j \quad (j = 1, \dots, n-1)$$

and

$$(3.10) \quad \sigma_n = 1 + d_{n-1}.$$

First, we prove that $\sigma_1 < \sigma_2 < \dots < \sigma_{n-1} < \sigma_n$. From the conditions, we have

$$(3.11) \quad d_{j-1} - d_j > \frac{d_{j-2} - d_j}{2} \quad (j = 2, \dots, n-1),$$

which yields

$$(3.12) \quad -(j-2)d_{j-1} - d_j > d_{j-2} - jd_{j-1}.$$

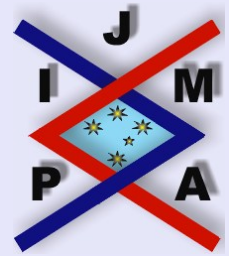
Adding $(j-1)d_{j-1}$ to both sides of (3.12) gives

$$(3.13) \quad d_{j-1} - d_j > d_{j-2} - d_{j-1} \quad (j = 2, \dots, n-1).$$

Hence $\sigma_{j-1} < \sigma_j$ for all $j = 2, \dots, n-1$. Furthermore, from the conditions, we have $d_{j-1} - d_j < d_{n-1}$ for all $j = 1, \dots, n-1$. Hence $\sigma_j < \sigma_n$ for all $j = 1, \dots, n-1$. Finally, we obtain that $\sigma_1 < \sigma_2 < \dots < \sigma_{n-1} < \sigma_n$. Next suppose $\sigma_j < \sigma < \sigma_{j+1}$ for some $j = 1, \dots, n-1$.

- (a) First we prove that if $\sigma > \sigma_j$ for some $j = 1, \dots, n-1$, and k is any integer satisfying $0 \leq k < j$, then $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$. Since

$$(3.14) \quad d_k + k(\sigma - 1) = d_j + j(\sigma - 1) + d_k - d_j + (k - j)(\sigma - 1),$$



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we obtain

$$(3.15) \quad d_k + k(\sigma - 1) < d_j + j(\sigma - 1) + d_k - d_j + (k - j)(\sigma_j - 1).$$

Now from the definition of σ_j in (3.9), we obtain

$$(3.16) \quad d_k - d_j + (k - j)(\sigma_j - 1) = (k - j) \left[d_{j-1} - d_j - \frac{d_k - d_j}{j - k} \right].$$

Since $0 \leq k < j$, it follows from the conditions that

$$(3.17) \quad d_{j-1} - d_j \geq \frac{d_k - d_j}{j - k}.$$

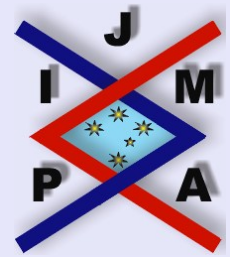
Then from (3.16) and (3.17), we obtain that

$$(3.18) \quad d_k - d_j + (k - j)(\sigma_j - 1) \leq 0.$$

Hence $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for all $0 \leq k < j$.

- (b) Now, we prove that if $\sigma < \sigma_{j+1}$ for some $j = 0, \dots, n-1$ and k is any integer satisfying $j < k \leq n-1$, then $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$. First, remark that if $k = j + 1$, then

$$\begin{aligned} d_{j+1} + (j + 1)(\sigma - 1) &= d_{j+1} + (\sigma - 1) + j(\sigma - 1) \\ &< d_{j+1} + (\sigma_{j+1} - 1) + j(\sigma - 1) \\ &= d_{j+1} + (d_j - d_{j+1}) + j(\sigma - 1) \\ &\leq d_j + j(\sigma - 1). \end{aligned}$$



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Hence

$$(3.19) \quad d_{j+1} + (j + 1)(\sigma - 1) < d_j + j(\sigma - 1).$$

We have,

$$(3.20) \quad \sigma < \sigma_{j+1} < \sigma_{j+2} < \dots < \sigma_{n-1} < \sigma_n.$$

Then

$$(3.21) \quad d_{j+2} + (j + 2)(\sigma - 1) < d_{j+1} + (j + 1)(\sigma - 1) \quad (\sigma < \sigma_{j+2})$$

$$\dots$$

$$d_{n-1} + (n - 1)(\sigma - 1) < d_{n-2} + (n - 2)(\sigma - 1) \quad (\sigma < \sigma_{n-1}).$$

Therefore from (3.20) and by combining the inequalities in (3.19) and (3.21), we obtain that $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for all $j < k \leq n - 1$. Furthermore

$$n(\sigma - 1) = (n - 1)(\sigma - 1) + (\sigma - 1) < (n - 1)(\sigma - 1) + d_{n-1}$$

since $\sigma < \sigma_n$ and from (3.21) and (3.19), we deduce that $n(\sigma - 1) < d_j + j(\sigma - 1)$. Then from a) and b), we obtain that if $\sigma_j < \sigma < \sigma_{j+1}$ for some $j = 1, \dots, n - 1$, then $n(\sigma - 1) < d_j + j(\sigma - 1)$ and $d_k + k(\sigma - 1) < d_j + j(\sigma - 1)$ for any $k \neq j$. It follows that the term in (3.3) with exponent $d_j + j(\sigma - 1)$ is a dominant term (as $r \rightarrow +\infty, r \notin E_1$). This is impossible. From b), it follows that if $\sigma < \sigma_1$, then $d_k + k(\sigma - 1) < d_0$ for all $0 < k \leq n - 1$ and

$n(\sigma - 1) < d_0$. Hence the term in (3.3) with exponent d_0 is a dominant term (as $r \rightarrow +\infty, r \notin E_1$). This is impossible.

Finally, we deduce that the possible orders of f are

$$1 + d_{n-1}, 1 + d_{n-2} - d_{n-1}, \dots, 1 + d_{j-1} - d_j, \dots, 1 + d_0 - d_1.$$

(iv) If $d_{j-1} = d_j - 1$ for all $j = 1, \dots, n - 1$, it is easy to see that (2.3) has possibly polynomial solutions. Now we discuss polynomial solutions of equation (2.3), if $f_1(z), \dots, f_n(z)$ are linearly independent polynomial solutions, then by the well-known identity

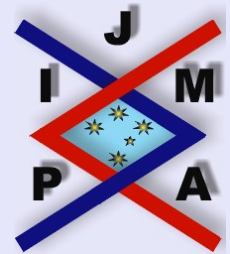
$$(3.22) \quad \begin{vmatrix} f_1 & f_2 & f_n \\ f_1' & f_2' & f_n' \\ \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_n^{(n-1)} \end{vmatrix} = C \exp \left\{ - \int_0^z a_{n-1}(s) ds \right\},$$

where $C \neq 0$ is some constant, we obtain a contradiction. Therefore any n polynomial solutions are linearly dependent, hence all polynomial solutions have the form $f_c(z) = cp(z)$, where p is a polynomial and c is an arbitrary constant.

Next, we give several examples that illustrate the sharpness of Theorem 2.1.

Example 3.1. Consider the differential equation

$$(3.23) \quad f''' - (6z + 1)f'' + 3z(3z + 1)f' - 2(z^3 + z^2 - 1)f = 0.$$



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Set

$$\begin{aligned}a_2(z) &= -(6z + 1), & d_2 &= 1; \\a_1(z) &= 3z(3z + 1), & d_1 &= 2; \\a_0(z) &= -2(z^3 + z^2 - 1), & d_0 &= 3.\end{aligned}$$

We have $\frac{d_0}{3} \geq \frac{d_1}{2}$ and $\frac{d_0}{3} \geq \frac{d_2}{1}$. Hence, by Theorem 2.1(i), all transcendental solutions of equation (3.23) are of order $1 + \frac{d_0}{3} = 2$. We see for example that $f(z) = e^{z^2}$ is a solution of (3.23) with $\sigma(f) = 2$.

Example 3.2. Consider the differential equation

$$(3.24) \quad f''' + zf'' + 2(z^2 - 8z - 1)f' - 3(9z^6 + 3z^5 + 2z^4 + 2z^3 + 2)f = 0.$$

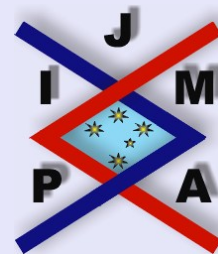
Set

$$\begin{aligned}a_2(z) &= z, & d_2 &= 1; \\a_1(z) &= 2(z^2 - 8z - 1), & d_1 &= 2; \\a_0(z) &= -3(9z^6 + 3z^5 + 2z^4 + 2z^3 + 2), & d_0 &= 6.\end{aligned}$$

We have $\frac{d_0}{3} > \frac{d_1}{2}$ and $\frac{d_0}{3} > \frac{d_2}{1}$. Hence, by Theorem 2.1(i), all transcendental solutions of equation (3.24) are of order $1 + \frac{d_0}{3} = 3$. Remark that $f(z) = e^{z^3}$ is a solution of (3.24) with $\sigma(f) = 3$.

Example 3.3. Consider the differential equation

$$(3.25) \quad f'''' - 2zf''' - 4(z^2 + 1)f'' + 6z^3f' + 4(z^4 - 1)f = 0.$$



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Set

$$\begin{aligned}a_3(z) &= -2z, & d_3 &= 1; \\a_2(z) &= -4(z^2 + 1), & d_2 &= 2; \\a_1(z) &= 6z^3, & d_1 &= 3; \\a_0(z) &= 4(z^4 - 1), & d_0 &= 4.\end{aligned}$$

We have $\frac{d_j}{4-j} \leq \frac{d_0}{4}$ for all $j = 1, 2, 3$. Hence, by Theorem 2.1(i), all transcendental solutions of equation (3.25) are of order $1 + \frac{d_0}{4} = 2$. Remark that $f(z) = e^{z^2}$ is a solution of (3.25) with $\sigma(f) = 2$.

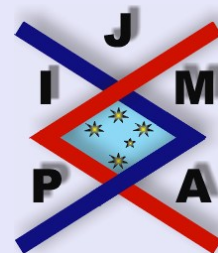
Example 3.4. Consider the differential equation

$$(3.26) \quad f''' + (z^2 + z - 1) f'' + (z^3 - z^2 - z + 1) f' - (z^3 + 1) f = 0.$$

Set

$$\begin{aligned}a_2(z) &= z^2 + z - 1, & d_2 &= 2; \\a_1(z) &= z^3 - z^2 - z + 1, & d_1 &= 3; \\a_0(z) &= -(z^3 + 1), & d_0 &= 3.\end{aligned}$$

We have $d_1 - 1 < d_0 < d_1 + d_2$ and $d_2 - 1 < d_1 < 2d_2$, $d_1 - d_2 > \frac{d_0 - d_2}{2}$. Hence, by Theorem 2.1(iii), all possible orders of solutions of equation (3.26) are $1 + d_2 = 3$, $1 + d_1 - d_2 = 2$, $1 + d_0 - d_1 = 1$. For example $f(z) = e^z$ is a solution of (3.26) with $\sigma(f) = 1$.



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Example 3.5. *The equation*

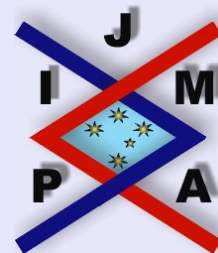
$$f''' + z^3 f'' - 2z^2 f' + 2zf = 0$$

has a polynomial solution $f_c(z) = c(z^2 + 2z)$ where c is a constant.

Example 3.6. *The equation*

$$f'''' - z(z^3 + 3z^2 + 2z + 1)f'''' - z(z^2 + 3z + 1)f''' + 2(z^2 + z + 1)f' + 6(z + 1)f = 0$$

has a polynomial solution $f_c(z) = c(z^3 + 3z^2)$ where c is a constant.



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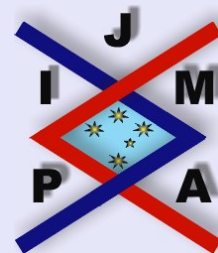
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4. Proof of Theorem 2.2

We assume that $f(z)$ is a transcendental entire solution of (2.4). We adopt the argument as used in the proof of Theorem 2.1, and notice that when z satisfies $|f(z)| = M(r, f)$ and $|z| \rightarrow +\infty$, $\left| \frac{b(z)}{f(z)} \right| \rightarrow 0$, we can prove that

1. if $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \dots, n-1$, then $\sigma(f) = \frac{d_0+n}{n}$;
2. if $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \dots, n-2$, then $\sigma(f) = 1 + d_{n-1}$;
3. if $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \dots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, then $\sigma(f) = 1 + d_{n-1}$ or $\sigma(f) = 1 + d_{n-2} - d_{n-1}$ or ... or $\sigma(f) = 1 + d_{j-1} - d_j$ or ... or $\sigma(f) = 1 + d_1 - d_2$ or $\sigma(f) = 1 + d_0 - d_1$.

We know that when $\frac{d_0}{n} \geq \frac{d_j}{n-j}$ for all $j = 1, \dots, n-1$ or $d_j < d_{n-1} - (n-j-1)$ for all $j = 0, \dots, n-2$ or $d_j - 1 < d_{j-1} < d_j + d_{n-1}$ for all $j = 1, \dots, n-1$ with $d_{j-1} - d_j = \max_{0 \leq k < j} \frac{d_k - d_j}{j-k}$ and $d_{j-1} - d_j > \frac{d_k - d_j}{j-k}$ for all $0 \leq k < j-1$, every solution $f \not\equiv 0$ of the corresponding homogeneous equation of (2.4) is transcendental, so that the equation (2.4) has at most one exceptional polynomial solution, in fact if f_1, f_2 ($f_2 \not\equiv f_1$) are polynomial solutions of (2.4), then $f_1 - f_2 \not\equiv 0$ is a polynomial solution of the corresponding homogeneous equation of (2.4), this is a contradiction. When $d_{j-1} = d_j - 1$ for some $j = 1, \dots, n-1$, if the corresponding homogeneous equation of (2.4) has no polynomial solution, then (2.4) has clearly at most one exceptional polynomial solution, if the corresponding homogeneous equation of (2.4) has



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a polynomial solution $p(z)$, then (2.4) may have a family of polynomial solutions $\{cp(z) + f_0(z)\}$ (f_0 is a polynomial solution of (2.4), c is a constant). Now we prove $\bar{\lambda}(f) = \sigma(f)$ for a transcendental solution f of (2.4). Since $b(z)$ is a polynomial which has only finitely many zeros, it follows that if z_0 is a zero of $f(z)$ and $|z_0|$ is sufficiently large, then the order of zero at z_0 is less than or equal to n from (2.4). Hence

$$(4.1) \quad N\left(r, \frac{1}{f}\right) \leq n \bar{N}\left(r, \frac{1}{f}\right) + O(\ln r).$$

By (2.4), we have

$$(4.2) \quad \frac{1}{f} = \frac{1}{b} \left(\frac{f^{(n)}}{f} + a_{n-1} \frac{f^{(n-1)}}{f} + \cdots + a_1 \frac{f'}{f} + a_0 \right).$$

Hence

$$(4.3) \quad m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^k m\left(r, \frac{f^{(j)}}{f}\right) + O(\ln r).$$

By $\sigma(f) < +\infty$, we have

$$(4.4) \quad m\left(r, \frac{f^{(j)}}{f}\right) = O(\ln r) \quad (j = 1, \dots, n).$$

Then we get from (4.1), (4.3) and (4.4),

$$(4.5) \quad \begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq n \bar{N}\left(r, \frac{1}{f}\right) + d(\log r) \end{aligned}$$



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where $d (> 0)$ is a constant. By (4.5), we have $\sigma(f) \leq \bar{\lambda}(f)$. On the other hand, we have

$$(4.6) \quad \bar{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right)$$

since $m\left(r, \frac{1}{f}\right)$ is a positive function. Hence

$$(4.7) \quad \bar{N}\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{1}{f}\right) = T(r, f) + O(1).$$

From (4.7), we obtain $\bar{\lambda}(f) \leq \sigma(f)$. Therefore, $\bar{\lambda}(f) = \sigma(f)$.

Next, we give several examples that illustrate the sharpness of Theorem 2.2.

Example 4.1. Consider the differential equation

$$(4.8) \quad f''' - (6z + 1)f'' + 3z(3z + 1)f' - 2(z^3 + z^2 - 1)f \\ = z(-2z^3 - 2z^2 + 9z + 5).$$

By Theorem 2.2(i), every entire transcendental solution of equation (4.8) is of order $1 + \frac{d_0}{3} = 2$. Remark that $f(z) = z + e^{z^2}$ is a solution of (4.8) with $\sigma(f) = \bar{\lambda}(f) = 2$.

Example 4.2. Consider the differential equation

$$(4.9) \quad f'''' - 2zf'''' - 4(z^2 + 1)f'' + 6z^3f' + 4(z^4 - 1)f \\ = 4(z^6 + 3z^4 - 3z^2 - 2).$$



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From Theorem 2.2(i), it follows that every entire transcendental solution of equation (4.9) is of order $1 + \frac{d_0}{4} = 2$. We have $f(z) = z^2 + e^{z^2}$ is a solution of (4.9) with $\sigma(f) = \bar{\lambda}(f) = 2$.

Example 4.3. Consider the differential equation

$$(4.10) \quad f''' + (z^2 + z - 1) f'' + (z^3 - z^2 - z + 1) f' - (z^3 + 1) f = z^4 - z^3 + z^2 + 2z - 1.$$

If f is a solution of equation (4.10), then by Theorem 2.2(iii), it follows that $\sigma(f) = \bar{\lambda}(f) = 3$ or $\sigma(f) = \bar{\lambda}(f) = 2$ or $\sigma(f) = \bar{\lambda}(f) = 1$. We have for example $f(z) = -z + e^z$ is a solution of (4.10) with $\sigma(f) = \bar{\lambda}(f) = 1$.

Example 4.4. The equation

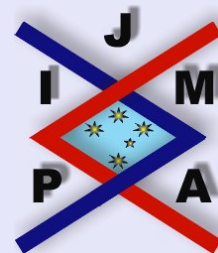
$$f''' + (z^3 + z^2 + z + 1) f'' - (2z^2 + 2z + 1) f' + 2(z + 1) f = 2(z + 1)$$

has a family of polynomial solutions $\{c(z^2 + 2z) + 1\}$ (c is a constant).

Example 4.5. The equation

$$f''' + (z^3 + z^2 + z + 1) f'' - (2z^2 + 2z + 1) f' + 2(z + 1) f = 4z + 3$$

has a family of polynomial solutions $\{c(z^2 + 2z) + z + 2\}$ (c is a constant).



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