SOME NORMALITY CRITERIA

INDRAJIT LAHIRI AND SHYAMALI DEWAN

Department of Mathematics
University of Kalyani
West Bengal 741235, India.
EMail: indrajit@cal2.vsnl.net.in

Department of Mathematics
University of Kalyani
West Bengal 741235, India.

Abstract

Contents

◀◀   ▶▶

◀   ▶

Home Page

Go Back

Close

Quit

Abstract

In the paper we prove some sufficient conditions for a family of meromorphic functions to be normal in a domain.

Contents

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 2 of 21

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

1. Introduction and Results

Let $\mathbb{C}$ be the open complex plane and $\mathcal{D} \subset \mathbb{C}$ be a domain. A family $\mathcal{F}$ of meromorphic functions defined in $\mathcal{D}$ is said to be normal, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j} such that f_{n_j} converges spherically, locally and uniformly in $\mathcal{D}$ to a meromorphic function or ∞.

$\mathcal{F}$ is said to be normal at a point $z_0 \in \mathcal{D}$ if there exists a neighbourhood of z_0 in which $\mathcal{F}$ is normal. It is well known that $\mathcal{F}$ is normal in $\mathcal{D}$ if and only if it is normal at every point of $\mathcal{D}$.

It is an interesting problem to find out criteria for normality of a family of analytic or meromorphic functions. In recent years this problem attracted the attention of a number of researchers worldwide.

In 1969 D. Drasin [5] proved the following normality criterion.

Theorem A. *Let $\mathcal{F}$ be a family of analytic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, b be two finite numbers. If for every $f \in \mathcal{F}$, $f' - af^n - b$ has no zero then $\mathcal{F}$ is normal, where $n(\geq 3)$ is an integer.*

Chen-Fang [2] and Ye [21] independently proved that *Theorem A* also holds for $n = 2$. A number of authors {cf. [3, 11, 12, 13, 16, 24]} extended *Theorem A* to a family of meromorphic functions in a domain. Their results can be combined in the following theorem.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, b be two finite numbers. If for every $f \in \mathcal{F}$, $f' - af^n - b$ has no zero then $\mathcal{F}$ is normal, where $n(\geq 3)$ is an integer.*

Li [12], Li [13] and Langley [11] proved *Theorem B* for $n \geq 5$, Pang [16] proved for $n = 4$ and Chen-Fang [3], Zalcman [24] proved for $n = 3$. Fang-
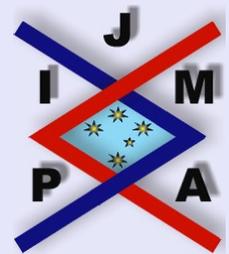
Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

Page 3 of 21

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

Yuan [6] showed that *Theorem B* does not, in general, hold for $n = 2$. For the case $n = 2$ they [6] proved the following result.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, b be two finite numbers. If $f' - af^2 - b$ has no zero and f has no simple and double pole for every $f \in \mathcal{F}$ then $\mathcal{F}$ is normal.*

Fang-Yuan [6] mentioned the following example from which it appears that the condition for each $f \in \mathcal{F}$ not to have any simple and double pole is necessary for *Theorem C*.

Example 1.1. *Let $f_n(z) = nz(z\sqrt{n} - 1)^{-2}$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$. Then each f_n has only a double pole and a simple zero. Also $f_n' + f_n^2 = n(z\sqrt{n} - 1)^{-4} \neq 0$. Since $f_n^{\#}(0) = n \to \infty$ as $n \to \infty$, it follows from Marty's criterion that $\{f_n\}$ is not normal in $\mathcal{D}$.*

However, the following example suggests that the restriction on the poles of $f \in \mathcal{F}$ may be relaxed at the cost of some restriction imposed on the zeros of $f \in \mathcal{F}$.

Example 1.2. *Let $f_n(z) = nz^{-2}$ for $n = 3, 4, \ldots$ and $\mathcal{D} : |z| < 1$. Then each f_n has only a double pole and no simple zero. Also we see that $f_n' + f_n^2 = n(n - 2z)z^{-4} \neq 0$ in $\mathcal{D}$. Since*

$$f_n^{\#}(z) = \frac{2n|z|}{|z|^2 + n^2} \leq \frac{2}{n} < 1$$

in $\mathcal{D}$, it follows from Marty's criterion that the family $\{f_n\}$ is normal in $\mathcal{D}$.

Now we state the first theorem of the paper.
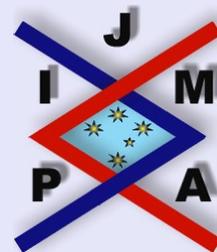
J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ such that no $f \in \mathcal{F}$ has any simple zero and simple pole. Let*

$$E_f = \{z : z \in \mathcal{D} \ \text{and} \ f'(z) - af^2(z) = b\},$$

where $a(\neq 0)$, b are two finite numbers.

If there exists a positive number M such that for every $f \in \mathcal{F}$, $|f(z)| \leq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.

The following examples together with *Example 1.1* show that the condition of *Theorem 1.1* on the zeros and poles are necessary.

Example 1.3. *Let $f_n(z) = n \tan nz$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < \pi$. Then f_n has only simple zeros and simple poles. Also we see that $f_n' - f_n^2 = n^2 \neq 0$. Since $f_n^{\#}(0) = n^2 \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal.*

Example 1.4. *Let $f_n(z) = (1 + e^{2nz})^{-1}$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$. Then f_n has no simple zero and no multiple pole. Also we see that $f_n' + f_n^2 \neq 1$. Since $f_n^{\#}(0) = \frac{2n}{3} \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal.*

Drasin [18, p. 130] also proved the following normality criterion which involves differential polynomials.

Theorem D. *Let $\mathcal{F}$ be a family of analytic functions in a domain $\mathcal{D}$ and $a_0, a_1, \ldots, a_{k-1}$ be finite constants, where k is a positive integer. Let*

$$H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f^{(1)} + a_0 f.$$

If for every $f \in \mathcal{F}$
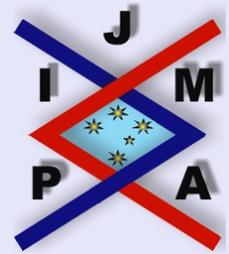
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

(i) f *has no zero,*

(ii) $H(f) - 1$ *has no zero of multiplicity less than* $k + 2$,

then $\mathcal{F}$ *is normal.*

Recently Fang-Yuan [6] proved that *Theorem D* remains valid even if $H(f) - 1$ has only multiple zeros for every $f \in \mathcal{F}$. In the next theorem we extend *Theorem D* to a family of meromorphic functions which also improves a result of Fang-Yuan [6].

Theorem 1.2. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* $\mathcal{D}$ *and*

$$H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f^{(1)} + a_0 f,$$

where $a_0, a_1, \ldots, a_{k-1}$ *are finite constants and* k *is a positive integer.*
   Let
$$E_f = \{z : z \in \mathcal{D} \text{ and } z \text{ is a simple zero of } H(f) - 1\}.$$
If for every $f \in \mathcal{F}$

(i) f *has no pole of multiplicity less than* $3 + k$,

(ii) f *has no zero,*

(iii) there exists a positive constant M *such that* $|f(z)| \geq M$ *whenever* $z \in E_f$,

then $\mathcal{F}$ *is normal.*

The following examples show that conditions (ii) and (iii) of *Theorem 1.2* are necessary, leaving the question of necessity of the condition (i) as open.
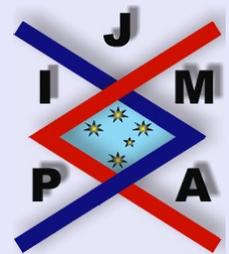
Some Normality Criteria

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 6 of 21

Example 1.5. *Let $f_n(z) = nz$ for $n = 2, 3, \ldots$, $\mathcal{D} : |z| < 1$, $H(f) = f' - f$ and $M = \frac{1}{2}$. Then each f_n has a zero at $z = 0$ and $E_{f_n} = \{1 - \frac{1}{n}\}$ for $n = 2, 3, \ldots$. So $|f(1 - \frac{1}{n})| = n - 1 \geq M$ for $n = 2, 3, \ldots$. Since $f_n^{\#}(0) = n \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathcal{D}$.*

Example 1.6. *Let $f_n(z) = e^{nz}$ for $n = 2, 3, \ldots$, $\mathcal{D} : |z| < 1$ and $H(f) = f' - f$. Then each f_n has no zero and $E_{f_n} = \{z : z \in \mathcal{D}$ and $(n-1)e^{nz} = 1\}$ for $n = 2, 3, \ldots$. Also we see that for $z \in E_{f_n}$, $|f_n(z)| = \frac{1}{n-1} \to 0$ as $n \to \infty$. Since $f_n^{\#}(0) = \frac{n}{2} \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathcal{D}$.*

In connection to *Theorem A* Chen-Fang [3] proposed the following conjecture:

Conjecture 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain$\mathcal{D}$. If for every function $f \in \mathcal{F}$, $f^{(k)} - af^n - b$ has no zero in $\mathcal{D}$ then $\mathcal{F}$ is normal, where $a(\neq 0)$, b are two finite numbers and k, $n(\geq k+2)$ are positive integers.*

In response to this conjecture Xu [23] proved the following result.

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, b be two finite constants. If k and n are positive integers such that $n \geq k + 2$ and for every $f \in \mathcal{F}$*

(i) $f^{(k)} - af^n - b$ has no zero,

(ii) f has no simple pole,

then $\mathcal{F}$ is normal.
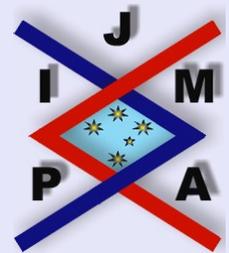
Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

The condition (ii) of *Theorem E* can be dropped if we choose $n \geq k + 4$ (cf. [15, 17]). Also some improvement of *Theorem E* can be found in [22]. In the next theorem we investigate the situation when the power of f is negative in condition (i) of *Theorem E*.

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, b be two finite numbers. Suppose that $E_f = \{z : z \in \mathcal{D} \ and \ f^{(k)}(z) + af^{-n}(z) = b\}$, where $k, n(\geq k)$ are positive integers.*
If for every $f \in \mathcal{F}$

(i) *f has no zero of multiplicity less than k,*

(ii) *there exists a positive number M such that for every $f \in \mathcal{F}$, $|f(z)| \geq M$ whenever $z \in E_f$,*

then $\mathcal{F}$ is normal.

Following examples show that the conditions of *Theorem 1.3* are necessary.

Example 1.7. *Let $f_p(z) = pz^2$ for $p = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$, $n = k = 3$, $a = 1$, $b = 0$. Then f_p has only a double zero and $E_{f_p} = \emptyset$. Since $f_p(0) = 0$ and for $z \neq 0$, $f_p(z) \to \infty$ as $p \to \infty$, it follows that the family $\{f_p\}$ is not normal.*

Example 1.8. *Let $f_p(z) = pz$ for $p = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$, $n = k = 1$. Then f_p has simple zero at the origin and for any two finite numbers $a(\neq 0)$, b, $E_{f_p} = \{a/p(b - p)\}$ so that $|f_p(z)| \to 0$ as $p \to \infty$ whenever $z \in E_{f_p}$. Since $f_p^{\#}(0) = p \to \infty$ as $p \to \infty$, by Marty's criterion the family $\{f_p\}$ is not normal.*

For the standard definitions and notations of the value distribution theory we refer to [8, 18].
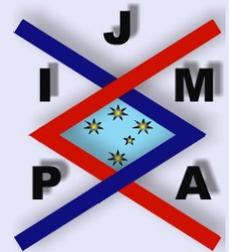
Some Normality Criteria

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 8 of 21

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[1] Let f be a transcendental meromorphic function of finite order in $\mathbb{C}$. If f has no simple zero then f' assumes every non-zero finite value infinitely often.*

Lemma 2.2. *[10] Let f be a nonconstant rational function in $\mathbb{C}$ having no simple zero and simple pole. Then f' assumes every non-zero finite value.*

The following lemma can be proved in the line of [9].

Lemma 2.3. *Let f be a meromorphic function in $\mathbb{C}$ such that $f^{(k)} \not\equiv 0$. Suppose that $\psi = f^n f^{(k)}$, where k, n are positive integers. If $n > k = 2$ or $n \geq k \geq 3$ then*

$$\left\{ 1 - \frac{1+k}{n+k} - \frac{n(1+k)}{(n+k)(n+k+1)} \right\} T(r, \psi) \leq \overline{N}(r, a; \psi) + S(r, \psi),$$

where $a(\neq 0, \infty)$ is a constant.

Lemma 2.4. *[19] Let f be a transcendental meromorphic function in $\mathbb{C}$ and $\psi = f^n f^{(2)}$, where $n(\geq 2)$ is an integer. Then*

$$\limsup_{r \to \infty} \frac{\overline{N}(r, a; \psi)}{T(r, \psi)} > 0,$$

where $a(\neq 0, \infty)$ is a constant.

The following lemma is a combination of the results of [3, 7, 14].
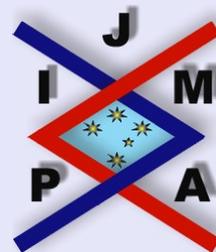
J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au
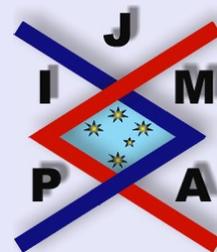
Lemma 2.5. *Let f be a transcendental meromorphic function in $\mathbb{C}$. Then $f^n f'$ assumes every non-zero finite value infinitely often, where $n(\geq 1)$ is an integer.*

Lemma 2.6. *Let f be a non-constant rational function in $\mathbb{C}$. Then $f^n f'$ assumes every non-zero finite value.*

Proof. Let $g = f^{n+1}/(n+1)$. Then g is a nonconstant rational function having no simple zero and simple pole. So by *Lemma 2.2* $g' = f^n f'$ assumes every non-zero finite value. This proves the lemma. □

Lemma 2.7. *Let f be a rational function in $\mathbb{C}$ such that $f^{(2)} \not\equiv 0$. Then $\psi = f^2 f^{(2)}$ assumes every non-zero finite value.*

Proof. Let $f = p/q$, where p, q are polynomials of degree m, n respectively and p, q have no common factor.

Let a be a non-zero finite number. We now consider the following cases.

Case 1. *Let $m = n$. Then $f = \alpha + p_1/q$, where α is a constant and p_1 is a polynomial of degree $m_1 < n$.*

Now

$$f' = \frac{p_1' q - p_1 q'}{q^2} = \frac{p_2}{q_2}, \;\; say,$$

where p_2 and q_2 are polynomials of degree $m_2 = m_1 + n - 1$ and $n_2 = 2n$. Also we note that $m_2 < n_2$. Hence

$$f'' = \frac{p_2' q_2 - p_2 q_2'}{q_2^2} = \frac{p_3}{q_3}, \;\; say,$$

Some Normality Criteria

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 10 of 21

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

*where p_3 and q_3 are polynomials of degree $m_3 = m_2 + n_2 - 1 = m_1 + 3n - 2$
and $n_3 = 2n_2 = 4n$. Also we see that $m_3 < n_3$.*

*Let $\psi = f^2 f^{(2)} = P/Q$. Then P, Q are polynomials of degree $2m + m_3$ and
$2n + n_3$ respectively and $2m + m_3 < 2n + n_3$. Therefore ψ is nonconstant.*

*Now $\psi - a = (P - aQ)/Q$ and the degree of $P - aQ$ is equal to the degree of
Q. If $\psi - a$ has no zero then $P - aQ$ and Q share 0 CM (counting multiplicites)
and so $P - aQ \equiv AQ$, where A is a constant. Therefore $\psi = A - a$, which is
impossible. So $\psi - a$ must have some zero.*

Case 2. *Let $m = n + 1$. Then*

$$f = \alpha z + \beta + \frac{p_1}{q},$$

where α, β are constants and p_1 is a polynomial of degree $m_1 < n$.

*Now $f'' = p_3/q_3$, where p_3 and q_3 are polynomials of degree $m_3 = m_1 +
3n - 2$ and $n_3 = 4n$ respectively and $m_3 < n_3$.*

*If $\psi = P/Q$ then P, Q are polynomials of degree $2m + m_3$ and $2n + n_3$
respectively. We see that $2m + m_3 = 5n + m_1 < 6n = 2n + n_3$ and so ψ is
nonconstant. Therefore as Case 1 $\psi - a$ must have some zero.*

Case 3. *Let $m \neq n, n + 1$. Then*

$$f' = \frac{pq' - p'q}{q^2} = \frac{p_4}{q_4}, \quad say,$$

*where p_4, q_4 are polynomials of degree $m_4 = m + n - 1$ and $n_4 = 2n$. Also we
note that $m_4 \neq n_4$.*
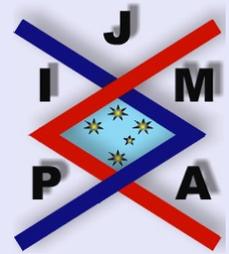
Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Hence

$$f'' = \frac{p_4' q_4 - p_4 q_4'}{q_4^2} = \frac{p_5}{q_5}, \quad \text{say,}$$

where p_5, q_5 are polynmials of degree $m_5 = m_4 + n_4 - 1 = m + 3n - 2$ and $n_5 = 2n_4 = 4n$.

If $\psi = P/Q$ then P, Q are polynomials of degree $2m + m_5$ and $2n + n_5$ respectively. Clearly $2m + m_5 \neq 2n + n_5$ because otherwise $m = n + 2/3$, which is impossible. So ψ is nonconstant. Also we see that $\psi - a = (P - aQ)/Q$, where the degree of $P - aQ$ is not less than that of Q. If $\psi - a$ has no zero then as per Case 1 ψ becomes a constant, which is impossible. So $\psi - a$ must have some zero. This proves the lemma.

$\square$

Lemma 2.8. *Let f be a meromorphic function in $\mathbb{C}$ such that $f^{(k)} \not\equiv 0$ and $a(\neq 0)$ be a finite constant. Then $f^{(k)} + af^{-n}$ must have some zero, where k and $n(\geq k)$ are positive integers.*

Proof. First we assume that $k = 1$. Then by *Lemmas 2.5* and *2.6* we see that $f^n f' + a$ must have some zero. Since a zero of $f^n f' + a$ is not a pole or a zero of f, it follows that a zero of $f^n f' + a$ is a zero of $f' + af^{-n}$.

Now we assume that $k = 2$. Then by *Lemmas 2.3, 2.4* and *2.7* we see that $f^n f^{(2)} + a$ must have some zero. As the preceding paragraph a zero of $f^n f^{(2)} + a$ is a zero of $f^{(2)} + af^{-n}$.

Finally we assume that $k \geq 3$. Then by *Lemma 2.3* $f^n f^{(k)} + a$ must have some zero. Since a zero of $f^n f^{(k)} + a$ is a zero of $f^{(k)} + af^{-n}$, the lemma is proved. $\square$
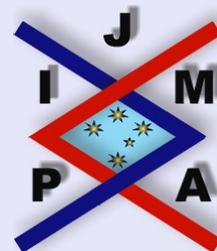
Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Lemma 2.9. *Let f be a nonconstant meromorphic function in $\mathbb{C}$ such that f has no zero and has no pole of multiplicity less than $3 + k$. Then $f^{(k)} - 1$ must have some simple zero, where k is a positive integer.*

Proof. Since $N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f)$ and $m(r, f^{(k)}) \leq m(r, f) + S(r, f)$, we get

$$
\begin{aligned}
T(r, f^{(k)}) &\leq T(r, f) + k\overline{N}(r, f) + S(r, f) \\
&\leq T(r, f) + \frac{k}{3 + k}N(r, f) + S(r, f) \\
&\leq \frac{3 + 2k}{3 + k}T(r, f) + S(r, f).
\end{aligned}
$$

Since f has no zero and no pole of multiplicity less than $3+k$, we get by Milloux inequality ([8, p. 57])

$$
\begin{aligned}
T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, 1; f^{(k)}) + S(r, f) \\
&\leq \frac{1}{3 + k}T(r, f) + \overline{N}(r, 1; f^{(k)}) + S(r, f).
\end{aligned}
$$

If possible, suppose that $f^{(k)} - 1$ has no simple zero. Then we get from above

$$
\begin{aligned}
T(r, f) &\leq \frac{1}{3 + k}T(r, f) + \frac{1}{2}N(r, 1; f^{(k)}) + S(r, f) \\
&\leq \left\{ \frac{1}{3 + k} + \frac{3 + 2k}{2(3 + k)} \right\} T(r, f) + S(r, f)
\end{aligned}
$$
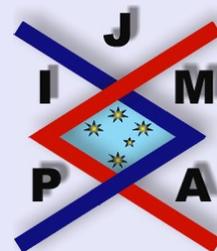
Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

and so

$$\frac{1}{2(3+k)}T(r,f) \leq S(r,f),$$

a contradiction. This proves the lemma. □

Lemma 2.10. *[4, 20] Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and let the zeros of f be of multiplicity not less than k (a positive integer) for each $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$ then for $0 \leq \alpha < k$ there exist a sequence of complex numbers $z_j \to z_0$, a sequence of functions $f_j \in \mathcal{F}$, and a sequence of positive numbers $\rho_j \to 0$ such that*

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Moreover the order of g is not greater than two and the zeros of g are of multiplicity not less than k.

Note 1. *If each $f \in \mathcal{F}$ has no zero then g also has no zero and in this case we can choose α to be any finite real number.*
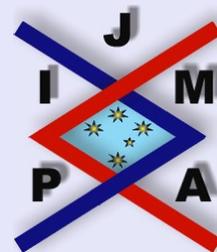
J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

3. Proofs of the Theorems

In this section we discuss the proofs of the theorems.

Proof of Theorem 1.1. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then $\mathcal{F}_1 = \{1/f : f \in \mathcal{F}\}$ is not normal at $z_0 \in \mathcal{D}$. Let $\alpha = 1$. Then by *Lemma 2.10* there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-1} f_j^{-1}(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic fucntion $g(\zeta)$ in $\mathbb{C}$. Also the order of g does not exceed two and g has no simple zero. Again by Hurwitz's theorem g has no simple pole.

By *Lemmas 2.1* and *2.2* we see that there exists $\zeta_0 \in \mathbb{C}$ such that

$$(3.1) \qquad\qquad g'(\zeta_0) + a = 0.$$

Since ζ_0 is not a pole of g, it follows that $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighbourhood of ζ_0. We also see that $\frac{-1}{g^2(\zeta)}\{g'(\zeta) + a\}$ is the uniform limit of $\rho_j^2 \{f_j' - a f_j^2 - b\}$ in some neighbourhood of ζ_0.

In view of (3.1) and Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that $f_j'(\zeta_j) - a f_j^2(\zeta_j) - b = 0$. So by the given condition

$$|g_j(\zeta_j)| = \frac{1}{\rho_j} \cdot \frac{1}{|f_j(z_j + \rho_j \zeta_j)|} \geq \frac{1}{\rho_j M}.$$

Since ζ_0 is not a pole of g, there exists a positive number K such that in some neighbourhood of ζ_0 we get $|g(\zeta)| \leq K$.
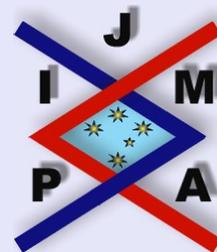
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

Page 15 of 21

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

Since $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighbourhood of ζ_0, we get for all large values of j and for all ζ in that neighbourhood of ζ_0

$$|g_j(\zeta) - g(\zeta)| < 1.$$

Since $\zeta_j \to \zeta$, we get for all large values of j

$$K \geq |g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| > \frac{1}{\rho_j M} - 1,$$

which is a contradiction. This proves the theorem. $\qquad \square$

Proof of Theorem 1.2. Let $\alpha = k$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by *Lemma 2.10* and *Note 1* there exists a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Now by conditions (i) and (ii) and by Hurwitz's theorem we see that $g(\zeta)$ has no zero and has no pole of multiplicity less than $3 + k$.

Now by *Lemma 2.9* $g^{(k)}(\zeta) - 1$ has a simple zero at a point $\zeta_0 \in \mathbb{C}$. Since ζ_0 is not a pole of $g(\zeta)$, in some neighbourhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$.

Since

$$g_j^{(k)}(\zeta) - 1 + \sum_{i=0}^{k-1} a_i \rho_j^{k-i} g_j^{(i)}(\zeta) = f_j^{(k)}(z_j + \rho_j \zeta) + \sum_{i=0}^{k-1} a_i f_j^{(i)}(z_j + \rho_j \zeta) - 1$$
$$= H(f_j(z_j + \rho_j \zeta)) - 1$$
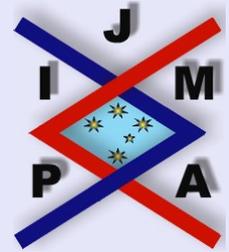
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

and $\sum_{i=0}^{k-1} a_i \rho_j^{k-i} g_j^{(i)}(\zeta)$ converges uniformly to zero in some neighbourhood of ζ_0, it follows that $g^{(k)}(\zeta) - 1$ is the uniform limit of $H(f_j(z_j + \rho_j\zeta)) - 1$.

Since ζ_0 is a simple zero of $g^{(k)}(\zeta) - 1$, by Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that ζ_j is a simple zero of $H(f_j(z_j + \rho_j\zeta)) - 1$. So by the given condition $|f_j(z_j + \rho_j\zeta_j)| \geq M$ for all large values of j.

Hence for all large values of j we get $|g_j(\zeta_j)| \geq M/\rho_j^k$ and as the last part of the proof of *Theorem 1.1* we arrive at a contradiction. This proves the theorem.

□

Proof of Theorem 1.3. Let $\alpha = k/(1+n) < 1$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by *Lemma 2.10* there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j\zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Also g has no zero of multiplicity less than k. So $g^{(k)} \not\equiv 0$ and by *Lemma 2.8* we get

$$(3.2) \qquad g^{(k)}(\zeta_0) + \frac{a}{g^n(\zeta_0)} = 0$$

for some $\zeta_0 \in \mathbb{C}$.

Clearly ζ_0 is neither a zero nor a pole of g. So in some neighbourhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$.
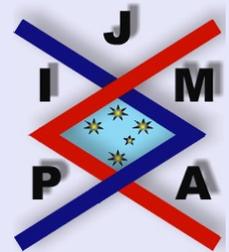
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

Now in some neighbourhood of ζ_0 we see that $g^{(k)}(\zeta) + ag^{-n}(\zeta)$ is the uniform limit of

$$g_j^{(k)} + ag_j^{-n}(\zeta) - \rho_j^{n\alpha}b = \rho_j^{\frac{nk}{1+n}} \left\{ f_j^{(k)}(z_j + \rho_j\zeta) + af_j^{-n}(z_j + \rho_j\zeta) - b \right\}.$$

By (3.2) and Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j

$$f_j^{(k)}(z_j + \rho_j\zeta_j) + af_j^{-n}(z_j + \rho_j\zeta_j) = b.$$

Therefore for all large values of j it follows from the given condition $|g_j(\zeta_j)| \geq M/\rho_j^\alpha$ and as in the last part of the proof of *Theorem 1.1* we arrive at a contradiction. This proves the theorem. □

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **5**(2) Art. 35, 2004

http://jipam.vu.edu.au

References

[1] W. BERGWEILER AND A. EREMENKO, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana*, **11**(2) (1995), 355–373.

[2] H. CHEN AND M. FANG, On a theorem of Drasin, *Adv. in Math*. (China), **20** (1991), 504.

[3] H. CHEN AND M. FANG, The value distribution of $f^n f'$, *Science in China* (Ser.A), **38**(7) (1995), 121–127.

[4] H. CHEN, Yosida functions and Picard values of integral functions and their derivatives, *Bull. Austral. Math. Soc.*, **54** (1996), 373–384.

[5] D. DRASIN, Normal families and Nevanlinna theory, *Acta Math.*, **122** (1969), 231–263.

[6] M. FANG AND W. YUAN, On the normality for families of meromorphic functions, *Indian J. Math*., **43**(3) (2001), 341–351.

[7] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70** (1959), 9–42.

[8] W.K. HAYMAN, *Meromorphic Functions*, The Clarendon Press (1964).

[9] I. LAHIRI AND S. DEWAN, Value distribution of the product of a meromorphic function and its derivative, *Kodai Math. J*., **26**(1) (2003), 95–100.
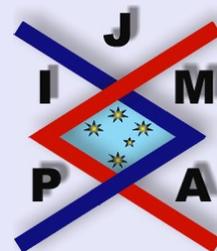
Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

[10] I. LAHIRI, A simple normality criterion leading to a counterexample to the converse of Bloch principle, *New Zealand J. Math.*, (to appear).

[11] J.K. LANGLEY, On normal families and a result of Drasin, *Proc. Royal Soc. Edinburgh*, **98A** (1984), 385–393.

[12] S. LI, The normality criterion of a class of the functions, *J. East China Normal Univ.*, **2** (1984), 156–158.

[13] X. LI, The proof of Hayman's conjecture on normal families, *Sci. Sinica* (Ser. A), **28**(6) (1985), 569–603.

[14] E. MUES, Über ein problem von Hayman, *Math. Z.*, **164** (1979), 239–259.

[15] X. PANG, Criteria for normality about differential polynomial, *Chin. Sci. Bull.*, **22** (1988), 1690–1693.

[16] X. PANG, On normal criterion of meromorphic fucntions, *Sci. Sinica*, **33**(5) (1990), 521–527.

[17] W. SCHWICK, Normality criteria for families of meromorphic functions, *J. Analyse Math.*, **52** (1989), 241–289.

[18] J.L. SCHIFF, *Normal Families*, Springer-Verlag (1993).

[19] N. STEINMETZ, Über die nullste von differential polynomen, *Math. Z.*, **176** (1981), 255–264.
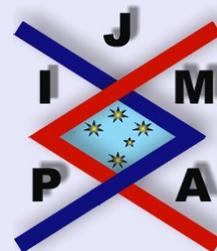
Some Normality Criteria

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 20 of 21

[20] Y. WANG AND M. FANG, Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica* (N.S.), **14**(1) (1998), 17–26.

[21] Y. YE, A new criterion and its application, *Chin. Ann. Math., Ser. A* (Supplement), **2** (1991), 44–49.

[22] Y. YE AND X. HUA, Normality criteria of families of meromorphic functions, *Indian J. Pure Appl. Math.*, **31**(1) (2000), 61–68.

[23] Y. XU, Normal families of meromorphic functions (preprint).

[24] L. ZALCMAN, On some problem of Hayman (preprint).

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit