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GENERALIZED NONLINEAR MIXED QUASI-VARIATIONAL INEQUALITIES INVOLVING MAXIMAL η-MONOTONE MAPPINGS

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ABSTRACT. In this paper, we introduce and study a new class of generalized nonlinear mixed quasi-variational inequalities involving maximal η-monotone mapping. Using the resolvent operator technique for maximal η-monotone mapping, we prove the existence of solution for this kind of generalized nonlinear multi-valued mixed quasi-variational inequalities without compactness and the convergence of iterative sequences generated by the new algorithm. We also discuss the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm for solving a class of generalized nonlinear mixed quasi-variational inequalities.

Key words and phrases: Mixed quasi-variational inequality, Maximal η-monotone mapping, Resolvent operator, Convergence, stability.

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1. INTRODUCTION

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques. These have been used to study wider classes of unrelated problems arising in optimization and control, economics and finance, transportation and electrical networks, operations research and engineering sciences in a general and unified framework, see [1] – [15], [18] – [27] and the references therein. An important and useful generalization of variational inequality is called the variational inclusion. It is well known that one of the most important and interesting problems in the theory of variational inequalities is the development of an efficient and implementable algorithm for solving variational inequalities. For the past years, many numerical methods have been developed for solving various classes of variational inequalities, such as the projection method and its variant forms, linear approximation, descent, and Newton's methods.

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Recently, Huang and Fang [10] introduced a new class of maximal η-monotone mappings and proved the Lipschitz continuity of the resolvent operator for maximal η-monotone mappings in Hilbert spaces. They also introduced and studied a new class of generalized variational inclusions involving maximal η-monotone mappings and constructed a new algorithm for solving this class of generalized variational inclusions by using the resolvent operator technique for maximal η-monotone mappings.

The main purpose of this paper is to introduce and study a new class of generalized nonlinear mixed quasi-variational inequalities involving maximal η-monotone mappings. Using the resolvent operator technique for maximal η-monotone mappings, we prove the existence of a solution for this kind of generalized nonlinear multivalued mixed quasi-variational inequalities without compactness and the convergence of iterative sequences generated by the new algorithm. We also discuss the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm for solving a class of generalized nonlinear mixed quasi-variational inequalities. The results shown in this paper improve and extend the previously known results in this area.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. Let $2^H, CB(H),$ and $\mathcal{H}(\cdot,\cdot)$ denote the family of all the nonempty subsets of H, the family of all the nonempty closed bounded subsets of H, and the Hausdorff metric on $CB(H)$, respectively. Let $\eta, N : H \times H \to H$ be two single-valued mappings with two variables and $g : H \to H$ be a single-valued mapping. Let $S, T, G : H \to CB(H)$ be three multivalued mappings and $M : H \times H \to 2^H$ be a multivalued mapping such that for each $t \in H$, $M(\cdot, t)$ is maximal η-monotone with $\mathrm{Ran}(g) \bigcap \mathrm{Dom}\, M(\cdot, t) \neq \emptyset$. Now we consider the following problem:

Find $u \in H, x \in Su, y \in Tu$, and $z \in Gu$ such that $g(u) \in \mathrm{Dom}(M(\cdot, z))$ and

$$0 \in N(x, y) + M(g(u), z)). \tag{2.1}$$

Problem (2.1) is called a generalized nonlinear multivalued mixed quasi-variational inequality.

Some special cases of the problem (2.1):

(I) If $\eta(x, y) = x - y$ for all x, y in H and G is the identity mapping, then problem (2.1) reduces to finding $u \in H$, $x \in Su$, $y \in Tu$ such that $g(u) \in \mathrm{Dom}(M(\cdot, u))$ and

$$0 \in N(x, y) + M(g(u), u). \tag{2.2}$$

Problem (2.2) is called the multivalued quasi-variational inclusion, which was studied by Noor [18] – [22].

(II) If S, T are single-valued mappings and G is the identity mapping, then problem (2.1) is equivalent to finding $u \in H$ such that $g(u) \in \mathrm{Dom}(M(\cdot, u))$ and

$$0 \in N(Su, Tu) + M(g(u), u)). \tag{2.3}$$

Problem (2.3) is called a generalized nonlinear mixed quasi-variational inequality.

(III) If $M(\cdot, t) = \partial\varphi(\cdot, t)$, where $\varphi : H \times H \to R \bigcup \{+\infty\}$ is a functional such that for each fixed t in H, $\varphi(\cdot, t) : H \to R \bigcup \{+\infty\}$ is lower semicontinuous and η-subdifferentiable on H, and $\partial\varphi(\cdot, t)$ denotes the η-subdifferential of $\varphi(\cdot, t)$, then problem (2.1) reduces to the following problem:

Find $u \in H, x \in Su$ and $y \in Tu$ such that

$$\langle N(x, y), \eta(v, g(u))\rangle \geq \varphi(g(u), z) - \varphi(v, z) \tag{2.4}$$

for all v in H, which which appears to be a new one. Furthermore, if $N(x, y) = x - y$ for all x, y in H, S, T are single-valued mappings and G is the identity mapping, then problem (2.4) reduces to the general quasi-variational-like inclusion considered by Ding and Luo [3].

(IV) If $S, T : H \to H$ are single-valued mappings, g is an identity mapping, $N(x, y) = x - y$ for all x, y in H, and $M(\cdot, t) = \partial\varphi$ for all t in H, where $\partial\varphi$ denotes the η-subdifferential of a proper convex lower semicontinuous function $\varphi : H \to R \bigcup\{+\infty\}$, then problem (2.1) reduces to the following problem:

Find $u \in H$ such that

$$\langle Su - Tu, \eta(v, u)\rangle \geq \varphi(u) - \varphi(v) \tag{2.5}$$

for all v in H, which is called the strongly nonlinear variational-like inclusion problem considered by Lee et al. [15].

(V) If G is an identity mapping, $\eta(x, y) = x - y$ and $M(\cdot, t) = \partial\varphi$ for each $t \in H$, where $\varphi : H \to R \bigcup\{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H) \bigcap \mathrm{Dom}(\partial\varphi(\cdot, t)) \neq \emptyset$ for each $t \in H$ and $\partial\varphi(\cdot, t)$ denotes the subdifferential of function $\varphi(\cdot, t)$, then problem (2.1) reduces to finding $u \in H, x \in Su$ and $y \in Tu$ such that $g(u) \in \mathrm{Dom}(\partial\varphi(\cdot, t))$ and

$$\langle N(x, y), v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v) \tag{2.6}$$

for all v in H. Furthermore, if $N(x, y) = x - y$ for all x, y in H, and g is an identity mapping, then the problem (2.6) is equivalent to the set-valued nonlinear generalized variational inclusion considered by Huang [6] and, in turn, includes the variational inclusions studied by Hassouni and Moudafi [5] and Kazmi [14] as special cases.

For a suitable choice of N, η, M, S, T, G, g, and for the space H, one can obtain a number of known and new classes of variational inclusions, variational inequalities, and corresponding optimization problems from the general set-valued variational inclusion problem (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in the mathematical, physical, and engineering sciences in a general and unified framework.

Definition 2.1. Let T be a selfmap of H, $x_0 \in H$ and let $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in H. Suppose that $\{x \in H : Tx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of T. Let $\{y_n\} \subset H$ and let $\epsilon_n = ||y_{n+1} - f(T, y_n)||$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.

Lemma 2.1 ([16]). *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers satisfying the following conditions: there exists n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n,$$

for all $n \geq n_0$, where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim\limits_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.

Definition 2.2. A mapping $g : H \to H$ is said to be

(i) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle g(u_1) - g(u_2), u_1 - u_2\rangle \geq \alpha\|u_1 - u_2\|^2,$$

for all $u_i \in H$, $i = 1, 2$;

(ii) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|g(u_1) - g(u_2)\| \leq \beta\|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$.

Definition 2.3. A multivalued mapping $S : H \to CB(H)$ is said to be

(i) $\mathcal{H}$-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\mathcal{H}(Su_1, Su_2) \leq \gamma\|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$;

(ii) strongly monotone with respect to the first argument of $N(\cdot, \cdot) : H \times H \to H$, if there exists a constant $\mu > 0$ such that

$$\langle N(x_1, \cdot) - N(x_2, \cdot), u_1 - u_2\rangle \geq \mu\|u_1 - u_2\|^2,$$

for all $x_i \in Su_i$, $u_i \in H$, $i = 1, 2$.

Definition 2.4. A mapping $N(\cdot, \cdot) : H \times H \to H$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $\nu > 0$ such that

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \nu\|u_1 - u_2\|,$$

for all $u_i \in H$, $i = 1, 2$.

In a similar way, we can define Lipschitz continuity of $N(\cdot, \cdot)$ with respect to the second argument.

Definition 2.5. Let $\eta : H \times H \to H$ be a single-valued mapping. A multivalued mapping $M : H \to 2^H$ is said to be

(i) η-monotone if

$$\langle x - y, \eta(u, v)\rangle \geq 0 \qquad \text{for all } u, v \in H, x \in Mu, y \in Mv;$$

(ii) strictly η-monotone if

$$\langle x - y, \eta(u, v)\rangle \geq 0 \qquad \text{for all } u, v \in H, x \in Mu, y \in Mv$$

and equality holds if and only if $u = v$;

(iii) strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle x - y, \eta(u, v)\rangle \geq r\|u - v\|^2 \qquad \text{for all } u, v \in H, x \in Mu, y \in Mv;$$

(iv) maximal η-monotone if M is η-monotone and $(I + \lambda M)(H) = H$ for any $\lambda > 0$.

Remark 2.2.

(1) If $\eta(u, v) = u - v$ for all u, v in H, then (i)-(iv) of Definition 2.5 reduce to the classical definitions of monotonicity, strict monotonicity, strong monotonicity, and maximal monotonicity, respectively.

(2) Huang and Fang gave one example of maximal η-monotone mapping in [10].

Lemma 2.3 ([10]). *Let $\eta : H \times H \to H$ be strictly monotone and $M : H \to 2^H$ be a maximal η-monotone mapping. Then the following conclusions hold:*

(1) $\langle x-y, \eta(u, v)\rangle \geq 0$ *for all* $(v, y) \in Graph(M)$ *implies that* $(u, x) \in Graph(M)$*, where* $Graph(M) = \{(u, x) \in H \times H : x \in Mu\}$*;*

(2) *the inverse mapping* $(I + \lambda M)^{-1}$ *is single-valued for any* $\lambda > 0$.

Based on Lemma 2.3, we can define the resolvent operator for a maximal η-monotone mapping M as follows:

$$J_\rho^M(z) = (I + \rho M)^{-1}(z) \qquad \text{for all } z \in H, \tag{2.7}$$

where $\rho > 0$ is a constant and $\eta : H \times H \to H$ is a strictly monotone mapping.

Lemma 2.4 ([10]). *Let $\eta : H \times H \to H$ be strongly monotone and Lipschtiz continuous with constants $\delta > 0$ and $\tau > 0$, respectively. Let $M : H \to 2^H$ be a maximal η-monotone mapping. Then the resolvent operator J_ρ^M for M is Lipschitz continuous with constant τ/δ, i.e.,*

$$\|J_\rho^M(u) - J_\rho^M(v)\| \leq \frac{\tau}{\delta}\|u - v\| \qquad \textit{for all } u, v \in H.$$

3. ITERATIVE ALGORITHMS

We first transfer problem (2.1) into a fixed point problem.

Lemma 3.1. *For given $u \in H$, $x \in Su$, $y \in Tu$, and $z \in Gu$, (u, x, y, z) is a solution of the problem (2.1) if and only if*

$$g(u) = J_\rho^{M(\cdot,z)}(g(u) - \rho N(x, y)), \tag{3.1}$$

where $J_\rho^{M(\cdot,z)} = (I + \rho M(\cdot, z))^{-1}$ and $\rho > 0$ is a constant.

Proof. This directly follows from the definition of $J_\rho^{M(\cdot,u)}$. □

Remark 3.2. Equality (3.1) can be written as

$$u = (1 - \lambda)u + \lambda[u - g(u) + J_\rho^{M(\cdot,z)}(g(u) - \rho N(x, y))],$$

where $0 < \lambda \leq 1$ is a parameter and $\rho > 0$ is a constant. This fixed point formulation enables us to suggest the following iterative algorithm for problem (2.1) as follows:

Algorithm 3.1. Let $\eta, N : H \times H \to H, g : H \to H$ be single-valued mappings and $S, T, G : H \to CB(H)$ be multivalued mappings. Let $M : H \times H \to 2^H$ be such that for each fixed $t \in H$, $M(\cdot, t) : H \to 2^H$ is a maximal η-monotone mapping satisfying $g(u) \in \mathrm{Dom}(M(\cdot, z))$. For given $\lambda \in [0, 1]$, $u_0 \in H$, $x_0 \in Su_0$, $y_0 \in Tu_0$ and $z_0 \in Gu_0$, let

$$u_1 = (1 - \lambda)u_0 + \lambda \left[u_0 - g(u_0) + J_\rho^{M(\cdot,z_0)}(g(u_0) - \rho N(x_0, y_0))\right].$$

By Nadler [17], there exist $x_1 \in Su_1$, $y_1 \in Tu_1$ and $z_1 \in Gu_1$ such that

$$\|x_0 - x_1\| \leq (1 + 1)\mathcal{H}(Su_0, Su_1),$$
$$\|y_0 - y_1\| \leq (1 + 1)\mathcal{H}(Tu_0, Tu_1),$$
$$\|z_0 - z_1\| \leq (1 + 1)\mathcal{H}(Gu_0, Gu_1).$$

Let

$$u_2 = (1 - \lambda)u_1 + \lambda \left[u_1 - g(u_1) + J_\rho^{M(\cdot,z_1)}(g(u_1) - \rho N(x_1, y_1))\right].$$

By induction, we can obtain sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ satisfying

$$\begin{cases} u_{n+1} = (1 - \lambda)u_n + \lambda \left[u_n - g(u_n) + J_\rho^{M(\cdot,z_n)}(g(u_n) - \rho N(x_n, y_n))\right], \\ x_n \in Su_n, \quad \|x_n - x_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(Su_n, Su_{n+1}), \\ y_n \in Tu_n, \quad \|y_n - y_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(Tu_n, Tu_{n+1}), \\ z_n \in gu_n, \quad \|z_n - z_{n+1}\| \leq (1 + (1 + n)^{-1})\mathcal{H}(Gu_n, Gu_{n+1}), \end{cases} \tag{3.2}$$

for $n = 1, 2, 3, \ldots$, where $0 < \lambda \leq 1$ and $\rho > 0$ are both constants.

Now we construct a new pertured iterative algorithm for solving the generalized nonlinear mixed quasi-variational inequality (2.3) as follows:

Algorithm 3.2. Let $\eta, N : H \times H \to H$ and $S, T : H \to H$ be single-valued mappings. Let $M : H \times H \to 2^H$ be such that for each fixed $t \in H$, $M(\cdot, t) : H \to 2^H$ is a maximal η-monotone mapping satisfying $g(u) \in \mathrm{Dom}(M(\cdot, u))$. For given $u_0 \in H$, the perturbed iterative sequence $\{u_n\}$ is defined by

$$\tag{3.3} \begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n[v_n - g(v_n) + J_\rho^{M(\cdot,v_n)}(g(v_n) - \rho N(Sv_n, Tv_n))] + \alpha_n e_n, \\ \\ v_n = (1-\beta_n)u_n + \beta_n[u_n - g(u_n) + J_\rho^{M(\cdot,u_n)}(g(u_n) - \rho N(Su_n, Tu_n))] + \beta_n f_n, \end{cases}$$

for $n = 0, 1, 2, \dots$, where $\{e_n\}$ and $\{f_n\}$ are two sequences of the elements of H introduced to take into account possible inexact computation and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

$$0 \le \alpha_n,\, \beta_n \le 1 (n \ge 0), \qquad \text{and} \qquad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{y_n\}$ be any sequence in H and define $\{\epsilon_n\}$ by

$$\tag{3.4} \begin{cases} \epsilon_n = \Big\| y_{n+1} - \Big\{ (1-\alpha_n)y_n + \alpha_n \Big[x_n - g(x_n) \\ \qquad\qquad + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n)) \Big] + \alpha_n e_n \Big\} \Big\|, \\ \\ x_n = (1-\beta_n)y_n + \beta_n \Big[y_n - g(y_n) + J_\rho^{M(\cdot,y_n)}(g(y_n) - \rho N(Sy_n, Ty_n)) \Big] + \beta_n f_n, \end{cases}$$

for $n = 0, 1, 2, \dots$.

4. MAIN RESULTS

In this section, we first prove the existence of a solution of problem (2.1) and the convergence of an iterative sequence generated by Algorithm 3.1.

Theorem 4.1. *Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants δ and τ, respectively. Let $S, T, G : H \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constants α, β, γ, respectively, $g : H \to H$ be μ-Lipschitz continuous and ν-strongly monotone. Let $N : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with constants ξ and ζ, respectively, and $S : H \to CB(H)$ be strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ with constant r. Let $M : H \times H \to 2^H$ be a multivalued mapping such that for each fixed $t \in H$, $M(\cdot, t)$ is maximal η-monotone. Suppose that there exist constants $\rho > 0$ and $\kappa > 0$ such that for each $x, y, z \in H$,*

$$\tag{4.1} \left\| J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z) \right\| \le \kappa \|x - y\|,$$

and

$$\tag{4.2} \begin{cases} \left| \rho - \frac{\tau r - \delta(1-h)\zeta\beta}{\tau(\xi^2\alpha^2 - \zeta^2\beta^2)} \right| < \frac{\sqrt{[\tau r - \delta(1-h)\zeta\beta]^2 - (\xi^2\alpha^2 - \zeta^2\beta^2)(\tau^2 - \delta^2(1-h)^2)}}{\tau(\xi^2\alpha^2 - \zeta^2\beta^2)}, \\ \\ \tau r > \delta(1-h)\zeta\beta + \sqrt{(\xi^2\alpha^2 - \zeta^2\beta^2)(\tau^2 - \delta^2(1-h)^2)}, \quad \xi\alpha > \zeta\beta, \\ \\ h = (1 + \delta\tau^{-1})\sqrt{1 - 2\nu + \mu^2} + \kappa\gamma,\ \rho\tau\zeta\beta < \delta(1-h), \quad h < 1. \end{cases}$$

Then the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.1 converge strongly to u^*, x^*, y^* and z^*, respectively and (u^*, x^*, y^*, z^*) is a solution of problem (2.1).

Proof. It follows from (3.2), (4.1) and Lemma 2.4 that

$$\begin{aligned}
&\|u_{n+1}-u_n\|\\
&=\Big\|(1-\lambda)(u_n-u_{n-1})+\lambda\big[[u_n-u_{n-1}-(g(u_n)-g(u_{n-1}))\\
&\qquad +J_\rho^{M(\cdot,z_n)}(g(u_n)-\rho N(x_n,y_n))-J_\rho^{M(\cdot,z_{n-1})}(g(u_{n-1})-\rho N(x_{n-1},y_{n-1}))\big]\Big\|\\
&\le (1-\lambda)\|u_n-u_{n-1}\|+\lambda\|u_n-u_{n-1}-(g(u_{n+1})-g(u_n))\|\\
&\qquad +\lambda\left\|J_\rho^{M(\cdot,z_n)}(g(u_n)-\rho N(x_n,y_n))-J_\rho^{M(\cdot,z_n)}(g(u_{n-1})-\rho N(x_{n-1},y_{n-1}))\right\|\\
&\qquad +\lambda\Big\|J_\rho^{M(\cdot,z_n)}(g(u_{n-1})-\rho N(x_{n-1},y_{n-1}))\\
&\qquad\qquad -J_\rho^{M(\cdot,z_{n-1})}(g(u_{n-1})-\rho N(x_{n-1},y_{n-1}))\Big\|\\
&\le (1-\lambda)\|u_n-u_{n-1}\|+\lambda\|u_n-u_{n-1}-(g(u_n)-g(u_{n-1}))\|\\
&\qquad +\lambda\frac{\tau}{\delta}\|g(u_n)-g(u_{n-1})-\rho(N(x_n,y_n)-N(x_{n-1},y_{n-1}))\|+\lambda\kappa\|z_n-z_{n-1}\|\\
&\le (1-\lambda)\|u_n-u_{n-1}\|+\lambda\left(1+\frac{\tau}{\delta}\right)\|u_n-u_{n-1}-(g(u_n)-g(u_{n-1}))\|\\
&\qquad +\lambda\frac{\tau}{\delta}\|u_n-u_{n-1}-\rho(N(x_n,y_n)-N(x_{n-1},y_n))\|\\
&\qquad +\lambda\rho\frac{\tau}{\delta}\|N(x_{n-1},y_n)-N(x_{n-1},y_{n-1})\|+\lambda\kappa\|z_n-z_{n-1}\|. \qquad (4.3)
\end{aligned}$$

Since g is strongly monotone and Lipschitz continuous, we obtain

$$\begin{aligned}
&\|u_n-u_{n-1}-(g(u_n)-g(u_{n-1}))\|^2\\
&\qquad =\|u_n-u_{n-1}\|^2-2\langle u_n-u_{n-1},g(u_n)-g(u_{n-1})\rangle+\|g(u_n)-g(u_{n-1})\|^2\\
&\qquad \le (1-2\nu+\mu^2)\|u_n-u_{n-1}\|^2. \qquad (4.4)
\end{aligned}$$

Since S is $\mathcal{H}$-Lipschitz continuous and strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ and N is Lipschitz continuous with respect to the first argument, we have

$$\begin{aligned}
&\|u_n-u_{n-1}-\rho(N(x_n,y_n)-N(x_{n-1},y_n))\|^2\\
&\qquad =\|u_n-u_{n-1}\|^2-2\rho\langle u_n-u_{n-1},N(x_n,y_n)-N(x_{n-1},y_n)\rangle\\
&\qquad\qquad +\rho^2\|N(x_n,y_n)-N(x_{n-1},y_n)\|^2\\
&\qquad \le (1-2\rho r+\rho^2\xi^2(1+n^{-1})^2\alpha^2)\|u_n-u_{n-1}\|^2. \qquad (4.5)
\end{aligned}$$

Further, since T,G are $\mathcal{H}$-Lipschitz continuous and N is Lipschitz continuous with respect to the second argument, we get

$$\|N(x_{n-1},y_n)-N(x_{n-1},y_{n-1})\|\le\zeta\|y_n-y_{n-1}\|\le\zeta\beta(1+n^{-1})\|u_n-u_{n-1}\|, \qquad (4.6)$$

$$\|z_n-z_{n-1}\|\le\gamma(1+n^{-1})\|u_n-u_{n-1}\|. \qquad (4.7)$$

By (4.3) – (4.7), we obtain

$$\begin{aligned}
\|u_n-u_{n-1}\| &\le (1-\lambda+\lambda(1+\tau\delta^{-1})\sqrt{1-2\nu+\mu^2}\\
&\qquad +\lambda\tau\delta^{-1}\sqrt{1-2\rho r+\rho^2\xi^2(1+n^{-1})^2\alpha^2}\\
&\qquad +\lambda\rho\tau\delta^{-1}\zeta\beta(1+n^{-1})+\lambda\kappa\gamma(1+n^{-1})\\
&=(1-\lambda+\lambda h_n+\lambda t_n(\rho))\|u_n-u_{n-1}\|\\
&=\theta_n\|u_n-u_{n-1}\|, \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned}
h_n &= (1+\tau\delta^{-1})\sqrt{1-2\nu+\mu^2}+\kappa\gamma(1+n^{-1}),\\
t_n(\rho) &= \tau\delta^{-1}\sqrt{1-2\rho r+\rho^2\xi^2(1+n^{-1})^2\alpha^2}+\rho\tau\delta^{-1}\zeta\beta(1+n^{-1}) \quad \text{and}\\
\theta_n &= 1-\lambda+\lambda h_n+\lambda t_n(\rho).
\end{aligned}$$

Letting $\theta = 1-\lambda+\lambda h+\lambda t(\rho)$, where

$$h = (1+\tau\delta^{-1})\sqrt{1-2\nu+\mu^2}+\kappa\gamma \quad \text{and} \quad t(\rho) = \tau\delta^{-1}\sqrt{1-2\rho r+\rho^2\xi^2\alpha^2}+\rho\tau\delta^{-1}\zeta\beta,$$

we have that $h_n \to h,\ t_n(\rho) \to t(\rho)$ and $\theta_n \to \theta$ as $n \to \infty$. It follows from condition (4.2) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore, (4.8) implies that $\{u_n\}$ is a Cauchy sequence in H and so we can assume that $u_n \to u^* \in H$ as $n \to \infty$. By the Lipschitz continuity of S, T and G we obtain

$$\begin{aligned}
\|x_n - x_{n-1}\| &\le (1+(1+n)^{-1})\mathcal{H}(Su_n, Su_{n-1}) \le \alpha(1+(1+n)^{-1})\|u_n-u_{n-1}\|,\\
\|y_n - y_{n-1}\| &\le (1+(1+n)^{-1})\mathcal{H}(Tu_n, Tu_{n-1}) \le \beta(1+(1+n)^{-1})\|u_n-u_{n-1}\|,\\
\|z_n - z_{n-1}\| &\le (1+(1+n)^{-1})\mathcal{H}(Gu_n, Gu_{n-1}) \le \gamma(1+(1+n)^{-1})\|u_n-u_{n-1}\|.
\end{aligned}$$

It follows that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in H. We can assume that $x_n \to x^*$, $y_n \to y^*$ and $z_n \to z^*$, respectively. Note that for $x_n \in Su_n$, we have

$$\begin{aligned}
d(x^*, Su^*) &\le \|x^* - x_n\| + d(x_n, Su^*)\\
&\le \|x^* - x_n\| + \mathcal{H}(Su_n, Su^*)\\
&\le \|x^* - x_n\| + \alpha\|u_n - u^*\| \to 0,
\end{aligned}$$

as $n \to \infty$. Hence we must have $x^* \in Su^*$. Similarly, we can show that $y^* \in Tu^*$ and $z^* \in Gu^*$. From

$$u_{n+1} = (1-\lambda)u_n + \lambda\left[u_n - g(u_n) + J_\rho^{M(\cdot,z_n)}(g(u_n) - \rho N(x_n, y_n))\right],$$

it follows that

$$g(u^*) = J_\rho^{M(\cdot,z^*)}(g(u^*) - \rho N(x^*, y^*)).$$

By Lemma 3.1, (u^*, x^*, y^*, z^*) is a solution of problem (2.1). This completes the proof. □

Remark 4.2. For an appropriate and suitable choice of the mappings η, N, S, T, G, g, M and the space H, we can obtain several known results in [1], [3], [5] – [8], [14], [18] – [22], [24] – [26] as special cases of Theorem 4.1.

Now we prove the convergence and stability of the iterative sequence generated by the Algorithm 3.2.

Theorem 4.3. *Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants δ and τ, respectively. Let $S, T : H \to H$ be Lipschitz continuous with constants α, β, respectively, $g : H \to H$ be μ-Lipschitz continuous and ν-strongly monotone. Let $N : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with constants ξ and ζ, respectively, and $S : H \to H$ be strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ with constant r. Let $M : H \times H \to 2^H$ be a multivalued mapping such that for each fixed $t \in H$, $M(\cdot, t)$ is maximal η-monotone. Suppose that there exist constants $\rho > 0$ and $\kappa > 0$ such that for each $x, y, z \in H$,*

$$\left\| J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z) \right\| \le \kappa\|x - y\|, \tag{4.9}$$

and

$$(4.10)\qquad \begin{cases} \left|\rho - \frac{\tau r-\delta(1-h)\zeta\beta}{\tau(\xi^2\alpha^2-\zeta^2\beta^2)}\right| < \frac{\sqrt{[\tau r-\delta(1-h)\zeta\beta]^2-(\xi^2\alpha^2-\zeta^2\beta^2)(\tau^2-\delta^2(1-h)^2)}}{\tau(\xi^2\alpha^2-\zeta^2\beta^2)}, \\ \tau r > \delta(1-h)\zeta\beta + \sqrt{(\xi^2\alpha^2-\zeta^2\beta^2)(\tau^2-\delta^2(1-h)^2)}, \quad \xi\alpha > \zeta\beta, \\ h = (1+\delta\tau^{-1})\sqrt{1-2\nu+\mu^2}+\kappa,\ \rho\tau\zeta\beta < \delta(1-h), \quad h<1. \end{cases}$$

If $\lim_{n\to\infty}\|e_n\| = 0,\ \lim_{n\to\infty}\|f_n\| = 0$*, then*

(I) The sequence $\{u_n\}$ defined by Algorithm 3.2 converges strongly to the unique solution u^* of problem (2.3).
(II) If $\sum_{n=0}^{\infty}\epsilon_n < \infty$, then $\lim_{n\to\infty} y_n = u^*$.
(III) If $\lim_{n\to\infty} y_n = u^*$, then $\lim_{n\to\infty}\epsilon_n = 0$.

Proof. (I) It follows from Theorem 4.1 that there exists $u^* \in H$ which is a solution of problem (2.3) and so

$$(4.11)\qquad g(u^*) = J_\rho^{M(\cdot,u^*)}(g(u^*) - \rho N(Su^*, Tu^*)).$$

From (4.9), (4.11) and Algorithm 3.2, it follows that

$$\begin{aligned}
&\|u_{n+1}-u^*\| \\
&= \Big\|(1-\alpha_n)(u_n-u^*) - \alpha_n\Big[v_n - u^* - (g(v_n)-g(u^*)) \\
&\qquad + J_\rho^{M(\cdot,v_n)}(g(v_n)-\rho N(Sv_n,Tv_n)) \\
&\qquad - J_\rho^{M(\cdot,u^*)}(g(u^*)-\rho N(Su^*,Tu^*))\Big] + \alpha_n e_n\Big\| \\
&\le (1-\alpha_n)\|u_n-u^*\| + \alpha_n\|v_n-u^*-(g(v_n)-g(u^*))\| + \alpha_n\|e_n\| \\
&\qquad + \alpha_n\Big\|J_\rho^{M(\cdot,v_n)}(g(v_n)-\rho N(Sv_n,Tv_n)) - J_\rho^{M(\cdot,v_n)}(g(u^*)-\rho N(Su^*,Tu^*))\Big\| \\
&\qquad + \alpha_n\left\|J_\rho^{M(\cdot,v_n)}(g(u^*)-\rho N(Su^*,Tu^*)) - J_\rho^{M(\cdot,u^*)}(g(u^*)-\rho N(Su^*,Tu^*))\right\| \\
&\le (1-\alpha_n)\|u_n-u^*\| + \alpha_n\|v_n-u^*-(g(v_n)-g(u^*))\| + \alpha_n\|e_n\| \\
&\qquad + \alpha_n\frac{\tau}{\delta}\|g(v_n)-g(u^*)-\rho(N(Sv_n,Tv_n)-N(Su^*,Tu^*))\| + \alpha_n\kappa\|v_n-u^*\| \\
&\le (1-\alpha_n)\|u_n-u^*\| + \alpha_n\left(1+\frac{\tau}{\delta}\right)\|u_n-u^*-(g(v_n)-g(u^*))\| + \alpha_n\|e_n\| \\
&\qquad + \alpha_n\frac{\tau}{\delta}\|v_n-u^*-\rho(N(Sv_n,Tv_n)-N(Su^*,Tv_n))\| \\
&\qquad + \alpha_n\rho\frac{\tau}{\delta}\|N(Su^*,Tv_n)-N(Su^*,Tu^*)\| + \alpha_n\kappa\|v_n-u^*\|. \qquad (4.12)
\end{aligned}$$

By the Lipschitz continuity of N, S, T, g and the strong monotonicity of S and g, we obtain

$$(4.13)\qquad \|v_n-u^*-(g(v_n)-g(u^*))\|^2 \le (1-2\nu+\mu^2)\|v_n-u^*\|^2,$$

$$(4.14)\qquad \|v_n-u^*-\rho(N(Sv_n,Tv_n)-N(Su^*,Tv_n))\|^2 \le (1-2\rho r+\rho^2\xi^2\alpha^2)\|v_n-u^*\|^2,$$

$$(4.15)\qquad \|N(Su^*,Tv_n)-N(Su^*,Tu^*)\| \le \zeta\beta\|v_n-u^*\|.$$

It follows from (4.12) – (4.15) that

$$(4.16)\qquad \|u_{n+1}-u^*\| \le (1-\alpha_n)\|u_n-u^*\| + \theta\alpha_n\|v_n-u^*\| + \alpha_n\|e_n\|,$$

where

$$\theta = \kappa + (1+\tau\delta^{-1})\sqrt{1-2\nu+\mu^2} + \tau\delta^{-1}\sqrt{1-2\rho r+\rho^2\xi^2\alpha^2} + \rho\tau\delta^{-1}\zeta\beta.$$

Similarly, we have

$$\|v_n - u^*\| \le (1-\beta_n)\|u_n - u^*\| + \theta\beta_n\|u_n - u^*\| + \beta_n\|f_n\|. \tag{4.17}$$

From (4.16) and (4.17), we have

$$\|u_{n+1} - u^*\| \le [1-\alpha_n(1-\theta)(1+\theta\beta_n)]\|u_n - u^*\| + \alpha_n\beta_n\theta\|f_n\| + \alpha_n\|e_n\|$$

Condition (4.10) implies that $0 < \theta < 1$, and so

$$\|u_{n+1} - u^*\| \le [1-\alpha_n(1-\theta)]\|u_n - u^*\| + \alpha_n(1-\theta)d_n, \tag{4.18}$$

where $d_n = (\beta_n\theta\|f_n\| + \|e_n\|)(1-\theta)^{-1} \to 0$, as $n \to \infty$. It follows from (4.18) and Lemma 2.1 that $u_n \to u^*$ as $n \to \infty$.

Now we prove that u^* is a unique solution of problem (2.3). In fact, if u is also a solution of problem (2.3), then

$$g(u) = J_\rho^{M(\cdot,u)}(g(u) - \rho N(Su, Tu)),$$

and, as the proof of (4.16), we have

$$\|u^* - u\| \le \theta\|u^* - u\|,$$

where $0 < \theta < 1$ and so $u^* = u$. This completes the proof of Conclusion (I).

Next we prove Conclusion (II). Using (3.4) we obtain

$$\begin{aligned}
&\|y_{n+1} - u^*\| \\
&\le \Big\|y_{n+1} - \big\{(1-\alpha_n)y_n + \alpha_n\big[\,x_n - g(x_n) \\
&\qquad + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\big] + \alpha_n e_n\big\}\Big\| \\
&\qquad + \Big\|(1-\alpha_n)y_n + \alpha_n\big[x_n - g(x_n) \\
&\qquad + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\big] + \alpha_n e_n - u^*\Big\| \\
&= \Big\|(1-\alpha_n)y_n + \alpha_n\big[x_n - g(x_n) \\
&\qquad + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\big] + \alpha_n e_n - u^*\Big\| + \epsilon_n.
\end{aligned} \tag{4.19}$$

As the proof of inequality (4.18), we have

$$\begin{aligned}
&\Big\|(1-\alpha_n)y_n + \alpha_n\left[x_n - g(x_n) + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\right] + \alpha_n e_n - u^*\Big\| \\
&\qquad\qquad \le (1-\alpha_n(1-\theta))\|y_n - u^*\| + \alpha_n(1-\theta)d_n.
\end{aligned} \tag{4.20}$$

It follows from (4.19) and (4.20) that

$$\|y_{n+1} - u^*\| \le (1-\alpha_n(1-\theta))\|y_n - u^*\| + \alpha_n(1-\theta)d_n + \epsilon_n. \tag{4.21}$$

Since $\sum_{n=0}^{\infty}\epsilon_n < \infty$, $d_n \to 0$, and $\sum_{n=0}^{\infty}\alpha_n < \infty$. It follows that (4.21) and Lemma 2.1 that $\lim\limits_{n\to\infty} y_n = u^*$.

Now we prove Conclusion (III). Suppose that $\lim\limits_{n\to\infty} y_n = u^*$. Then we have

$$\begin{aligned}
\epsilon_n &= \Big\|y_{n+1} - (1-\alpha_n)y_n + \alpha_n\big[\,x_n \\
&\qquad - g(x_n) + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\big] + \alpha_n e_n\Big\| \\
&\le \|y_{n+1} - u^*\| + \Big\|(1-\alpha_n)y_n + \alpha_n\big[x_n \\
&\qquad - g(x_n) + J_\rho^{M(\cdot,x_n)}(g(x_n) - \rho N(Sx_n, Tx_n))\big] + \alpha_n e_n - u^*\Big\| \\
&\le \|y_{n+1} - u^*\| + (1-\alpha_n(1-\theta))\|y_n - u^*\| + \alpha_n(1-\theta)d_n \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

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