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FACTORIZATION OF INEQUALITIES

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ABSTRACT. We give a Bennett-type factorization of the space $ces(p_n)$ for monotone nonincreasing sequence $\{p_n\}$. If the sequence $\{p_n\}$ is nondecreasing and bounded then certain converse assertion is proved.

Key words and phrases: Inequalities for sums, factorization, space $\ell(p_n)$.

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1. INTRODUCTION

G. Bennett raised the interesting problem of the factorization of inequalities, and in his basic work [1] he gave a systematic treatment of the factorization of several classic and latest inequalities. In his essay we can also find the precise definition of factorization of inequalities and an explanation of its benefits.

In some previous papers we also studied such problems (see e.g. [3] – [7]).

Now we recall only one sample result.

It is well known that the classical Hardy inequality, in its crudest form, asserts that

$$l_p \subseteq ces(p), \quad p > 1, \tag{1.1}$$

where

$$l_p := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

$$ces(p) := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\},$$

and $\mathbf{x} := \{x_n\}$ is a sequence of real numbers.

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G. Bennett [1] gave the factorization of (1.1) as follows:

Theorem 1.1. *Let $p > 1$. A sequence $\mathbf{x}$ belongs to $ces(p)$ if and only if it admits a factorization*

$$\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \quad (x_n = y_n \cdot z_n) \tag{1.2}$$

with

$$\mathbf{y} \in l_p \quad \textit{and} \quad \sum_{k=1}^{n} |z_k|^{p^*} = \mathcal{O}(n), \quad p^* := \frac{p}{p-1}. \tag{1.3}$$

This theorem may be stated succinctly as:

$$ces(p) = l_p \cdot g(p^*) \quad (p > 1),$$

if

$$g(p) := \left\{ \mathbf{x} : \sum_{k=1}^{n} |x_k|^p = \mathcal{O}(n) \right\}.$$

It is clear that Theorem 1.1 contains Hardy's inequality, namely if $\mathbf{x} \in l_p$, then $\mathbf{x}$ may be factorized as in (1.2) such that $\mathbf{y}$ and $\mathbf{z}$ satisfy (1.3) by taking $\mathbf{y} = \mathbf{x}$ and $\mathbf{z} = \mathbf{1} = (1, 1, \ldots)$; and by Theorem 1.1 then $\mathbf{x} \in ces(p)$.

In [7] we considered the problem of factorization of the set

$$\lambda(\varphi) := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} \lambda_n \varphi \left(\sum_{k=1}^{n} |x_k| \right) < \infty \right\},$$

where $\{\lambda_n\}$ is a sequence of nonnegative terms having infinitely many positive ones, and φ is a nonnegative function on $[0, \infty)$, $\varphi(0) = 0$, $\varphi(x)x^{-p}$ is nondecreasing. If $\varphi(x) = x^p$ and $\lambda_n = n^{-p}$ then, clearly $\lambda(\varphi) \equiv ces(p)$. In this theme several open problems still remain.

A great number of mathematicians have investigated the following generalizations of the spaces l_p and $ces(p)$:

$$l(p_n) := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty \right\}$$

and

$$ces(p_n) := \left\{ \mathbf{x} : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{p_n} < \infty \right\},$$

where $\mathbf{p} := \{p_n\}$ is a sequence of positive numbers.

A good survey and some new results of this type can be found in the paper [2] by P.D. Johnson and R.N. Mohapatra.

The aim of the present paper is to find certain factorization of the space $ces(p_n)$. Unfortunately we can do this only if the sequence $\mathbf{p} := \{p_n\}$ is monotone decreasing. In the case if $\mathbf{p}$ is monotone increasing we can give sufficient conditions for the sequences $\mathbf{y}$ and $\mathbf{z}$ such that their product sequence $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ should belong to $ces(p_n)$. Hence it is clear that if $p_n = p$ for all n, then we get a necessary and sufficient condition as in Theorem 1.1.

Naturally many similar problems can be raised (and solved if you have enough stoicism and calculation ability) if you want to give the analogies of the results given in the special case $p_n = p$ regarding all the factorizations of inequalities (see e.g. only [1], [6] and [7]).

To present our theorem easily we need one more definition:

$$g(p_n) := \left\{ \mathbf{x} : \sum_{k=1}^{n} |x_k|^{p_k} \le K^{p_n - 1} n \right\},$$

where $K = K(\mathbf{p}) \geq 1$ is a constant depending only on the sequence $\mathbf{p}$.

2. THE RESULTS

Our theorem reads as follows:

Theorem 2.1. (i) *If* $\mathbf{p} := \{p_n\}$ *is a nonincreasing sequence of numbers, all* $p_n > 1$*; and* $\mathbf{x} \in ces(p_n)$*, then* $\mathbf{x}$ *admits a factorization (1.2) with*

$$\mathbf{y} \in l(p_n) \quad \textit{and} \quad \mathbf{z} \in g(p_n^*). \tag{2.1}$$

(ii) *Conversely, if* $\mathbf{p}$ *is a nondecreasing and bounded sequence of numbers,* $p_0 > 1$*, furthermore (2.1) holds, then the product sequence* $\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \in ces(p_n)$*.*

Part (ii) of our theorem clearly implies that

$$l(p_n) \subseteq ces(p_n) \quad (1 < p_0 \leq \ldots \leq p_n),$$

since if $\mathbf{x} \in l(p_n)$ then $\mathbf{x}$ can be factorized as in (1.2) with (2.1) by taking $\mathbf{y} = \mathbf{x}$ and $\mathbf{z} = \mathbf{1} = (1, 1, \ldots)$, and thus we get that $\mathbf{x} \in ces(p_n)$.

In order to prove our theorem we require the following lemma.

Lemma 2.2. *Let* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *be sequences with nonnegative terms and suppose that* w_k *decreases with* k*. If*

$$\sum_{k=1}^{n} u_k \leq \sum_{k=1}^{n} v_k \quad (n = 1, 2, \ldots),$$

then

$$\sum_{k=1}^{n} u_k w_k \leq \sum_{k=1}^{n} v_k w_k \quad (n = 1, 2, \ldots).$$

This result is well known, see e.g. Lemma 3.6 in [1].

3. PROOF OF THEOREM 2.1

(i) We assume that $\mathbf{x} \neq \mathbf{0} := (0, 0, \ldots)$, otherwise the statement is trivial.

If $\mathbf{x} \in ces(p_n)$, we set

$$b_n := \sum_{k=n}^{\infty} k^{-p_k} \left(\sum_{i=1}^{k} |x_i| \right)^{p_k - 1}, \tag{3.1}$$

and we note that b_n monotonically tends to zero. Indeed, using the well-known inequality

$$ab \leq \frac{a^p}{p} + \frac{b^{p^*}}{p^*}, \quad p > 1,$$

we have that

$$\begin{aligned} b_n &= \sum_{k=n}^{\infty} \frac{1}{k} \left(\frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^{p_k - 1} \\ &\leq \sum_{k=n}^{\infty} \frac{1}{p_k k^{p_k}} + \sum_{k=n}^{\infty} \frac{1}{p_k^*} \left(\frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^{p_k}, \end{aligned} \tag{3.2}$$

where the second sum clearly tends to zero because $\mathbf{x} \in ces(p_n)$ and $p_k > 1$. The first sums also tends to zero, namely if $\sum_{i=1}^{\infty} |x_i| \le 1$, then, by $p_k \le p_0$, the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^{p_k} \ge \sum_{k=1}^{\infty} \frac{1}{k^{p_k}} \left(\sum_{i=1}^{k} |x_i| \right)^{p_0}$$

and $\mathbf{x} \neq \mathbf{0}$ imply this assertion.

Now we factorize $\mathbf{x}$ as follows

$$\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \quad (x_n = y_n z_n),$$

where

$$y_n := (|x_n| b_n)^{1/p_n} sign(x_n) \tag{3.3}$$

and

$$z_n := |x_n|^{1/p_n^*} b_n^{-1/p_n} (\ge 0). \tag{3.4}$$

Thus, by (3.1) and (3.3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} |y_n|^{p_n} &= \sum_{n=1}^{\infty} |x_n| \sum_{k=n}^{\infty} k^{-p_k} \left(\sum_{i=1}^{k} |x_i| \right)^{p_k - 1} \\ &= \sum_{k=1}^{\infty} k^{-p_k} \left(\sum_{i=1}^{k} |x_i| \right)^{p_k - 1} \sum_{n=1}^{k} |x_n| \\ &= \sum_{k=1}^{\infty} k^{-p_k} \left(\sum_{i=1}^{k} |x_i| \right)^{p_k}, \end{aligned}$$

thus the condition $\mathbf{x} \in ces(p_n)$ implies $\mathbf{y} \in l(p_n)$.

On the other hand, by Hölder's inequality,

$$\begin{aligned} \left(\sum_{k=1}^{m} z_k^{p_k^*} \right)^{p_m} &= \left(\sum_{k=1}^{m} |x_k|^{\frac{1}{p_m^*} + \frac{1}{p_m}} b_k^{-p_k^*/p_k} \right)^{p_m} \\ &\le \left(\sum_{k=1}^{m} |x_k| \right)^{\frac{p_m}{p_m^*}} \left(\sum_{k=1}^{m} |x_k| b_k^{-\frac{p_m p_k^*}{p_k}} \right). \end{aligned} \tag{3.5}$$

Consequently, for $m = 1, 2, \ldots$, we have by (3.1), (3.4) and (3.5),

$$\begin{aligned} \sum_{n=m}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{m} z_k^{p_k^*} \right)^{p_m} &\le \sum_{n=m}^{\infty} n^{-p_m} \left(\sum_{k=1}^{m} |x_k| \right)^{p_m - 1} \sum_{k=1}^{m} |x_k| b_k^{-\frac{p_m p_k^*}{p_k}} \\ &= \sum_{k=1}^{m} |x_k| b_k^{-\frac{p_m p_k^*}{p_k}} \sum_{n=m}^{\infty} n^{-p_m} \left(\sum_{k=1}^{m} |x_k| \right)^{p_m - 1}. \end{aligned} \tag{3.6}$$

Since $\mathbf{x} \in ces(p_n)$ implies that

$$\frac{1}{n} \sum_{k=1}^{n} |x_k| \le K_1(p)$$

holds for all n; where and later on $K_i(p)$ is written instead of $K_i(\mathbf{p})$, thus, by (3.2),

$$\sum_{n=m}^{\infty} n^{-p_m}\left(\sum_{k=1}^{m}|x_k|\right)^{p_m-1} = \sum_{n=m}^{\infty} n^{-p_m}\left(\sum_{k=1}^{m}|x_k|\right)^{p_n-1} n^{p_n-p_m}\left(\sum_{k=1}^{m}|x_k|\right)^{p_m-p_n}$$

$$\leq K_2(p)\sum_{n=m}^{\infty} n^{-p_n}\left(\sum_{k=1}^{n}|x_k|\right)^{p_n-1} = K_2(p)b_m. \tag{3.7}$$

By (3.2) we also have that the sequence $\{b_k\}$ is bounded, whence

$$b_k^{\frac{p_k-p_m}{p_k-1}} \leq K_3(p)$$

also hold for any k and $m \geq k$. Hence we get that

$$\sum_{k=1}^{m}|x_k|b_k^{-\frac{p_m p_k^*}{p_k}} \equiv \sum_{k=1}^{m}|x_k|b_k^{-\frac{p_k^*}{p_k}} b^{\frac{p_k-p_m}{p_k-1}-1}$$

$$\leq K_3(p)b_m^{-1}\sum_{k=1}^{m} z_k^{p_k^*}. \tag{3.8}$$

Collecting the estimations (3.6), (3.7) and (3.8) we have that

$$\sum_{n=m}^{\infty} n^{-p_m}\left(\sum_{k=1}^{m} z_k^{p_k^*}\right)^{p_m-1} \leq K_4(p).$$

Hence an easy computation gives that

$$\sum_{k=1}^{m} z_k^{p_k^*} \leq K(p)^{\frac{1}{p_m-1}} m = K(p)^{p_m^*-1} m;$$

and herewith we have proved that $\mathbf{z} \in g(p_n^*)$.

This completes the proof of part (i).

(ii) The assumption $\mathbf{z} \in g(p_n^*)$ yields that

$$\sum_{k=1}^{n}|z_k|^{p_k^*} \leq K(p)^{\frac{1}{p_n-1}}\sum_{k=1}^{n} 1.$$

This and Lemma 2.2 with $w_k := k^{-1/2}$ imply

$$\sum_{k=1}^{n}|z_k|^{p_k^*}k^{-1/2} \leq K(p)^{\frac{1}{p_n-1}}\sum_{k=1}^{n} k^{-1/2}. \tag{3.9}$$

On the other hand, if $\mathbf{x} = \mathbf{y}\cdot\mathbf{z}$, applying Hölder's inequality, we have

$$\left(\sum_{k=1}^{n}|x_k|\right)^{p_n} = \left(\sum_{k=1}^{n}|y_k|k^{1/2p_n^*}k^{-1/2p_n^*}|z_k|\right)^{p_n}$$

$$\leq \left(\sum_{k=1}^{n}|y_k|^{p_n}k^{\frac{p_n-1}{2}}\right)\left(\sum_{k=1}^{n}k^{-1/2}|z_k|^{p_n^*}\right)^{p_n-1}.$$

This and (3.9) exhibit that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n} \le K_5(p)\sum_{n=1}^{\infty}n^{-p_n}\sum_{k=1}^{n}|y_k|^{p_n}k^{\frac{p_n-1}{2}}n^{\frac{p_n-1}{2}} \tag{3.10}$$

$$= K_5(p)\sum_{k=1}^{\infty}|y_k|^{p_n}\sum_{n=k}^{\infty}\left(\frac{k}{n}\right)^{\frac{p_n-1}{2}}\frac{1}{n} =: S_1, \quad \text{say.}$$

Since $\mathbf{y} \in l(p_n)$, the terms y_n are bounded, using this, furthermore the boundedness and the monotonicity of the sequence $\mathbf{p}$, we know that

$$S_1 \le K_6(p)\sum_{k=1}^{\infty}|y_k|^{p_k}\sum_{n=k}^{\infty}\left(\frac{k}{n}\right)^{\frac{p_k-1}{2}}\frac{1}{n}$$

$$\le K_6(p)\sum_{k=1}^{\infty}|y_k|^{p_k}k^{\frac{p_k-1}{2}}\sum_{n=k}^{\infty}n^{-1-\frac{1}{2}(p_k-1)}$$

$$\le K_7(p)\sum_{k=1}^{\infty}|y_k|^{p_k}. \tag{3.11}$$

The assumption $\mathbf{y} \in l(p_n)$, (3.10) and (3.11) plainly prove that $\mathbf{x} \in ces(p_n)$, as stated.

Herewith we have completed the proof of the theorem.

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