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COLLOCATION AND FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. We consider the problem of numerical inversion of Fredholm integral equations of the first kind via piecewise interpolation. One of the most important aspects of this technique is the choice of grid and collocation points. Theoretical results are developed which identify an optimal strategy for the distribution of collocation points for piecewise constant interpolation. The method, as outlined, can be readily extended to higher order schemes.

Key words and phrases: Collocation, Fredholm integral equations, weighted quadrature.

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1. Introduction

In this paper we will consider the problem of inverting Fredholm integral equations of the first kind, viz

$$g(\boldsymbol{y}) = \int_\Gamma K(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{x})\, d\Gamma(\boldsymbol{x}), \tag{1.1}$$

where g represents some known data at the point $\boldsymbol{y} \in \Gamma$ and K is some integrable kernel.

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The integral equation (1.1) is inherently ill-posed. That is, it can be shown that a small perturbation on g can give rise to an arbitrarily large perturbation in f. To explore this point, consider the singular integral

$$\int_0^1 \ln|x-y| n^\alpha e^{inx}\,dx = in^{\alpha-1}\left(\ln y - e^{in}\ln(1-y)\right) - \pi n^{\alpha-1} e^{iny} + O\left(n^{\alpha-2}\right). \tag{1.2}$$

For $0<\alpha<1$ and n large, then infinitely small changes for the integral correspond to infinitely large changes in the integrand. For this reason, numerical methods for solving such equations are often ill-fated and the simple illustration here shows this is often manifested in attempting to find the high frequency terms in the unknown. For example, a spectral expansion method would encounter problems as shown in (1.2) and this has been explored in [14].

Consider the one dimensional symmetric integral equation

$$g(y) = \int_a^b K|x-y| f(x)\,dx, \qquad a \le y \le b, \tag{1.3}$$

where both $K>0$ and g are known and f is the unknown function we wish to find. We assume that g is bounded but not necessarily analytic. To begin, define a grid

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{1.4}$$

and the interpolation scheme

$$f(x) = \begin{cases} f_{i-1}, & x \in [x_{i-1}, \xi_i) \\ f_i, & x \in [\xi_i, x_i) \end{cases}, \qquad \xi_i \in [x_{i-1}, x_i], \quad i = 1,2,\dots,n. \tag{1.5}$$

Thus, we may write (1.3) as

$$g(y) = f_0 \int_{x_0}^{\xi_1} K|x-y|\,dx + \sum_{i=1}^{n-1} f_j \int_{\xi_i}^{\xi_{i+1}} K|x-y|\,dx + f_n \int_{\xi_n}^{x_n} K|x-y|\,dx. \tag{1.6}$$

To obtain a solution we need to find the $n+1$ unknowns $f_0, f_1, f_2, \dots, f_n$. Thus we can formulate a linear system by evaluating (1.6) at $n+1$ collocation points.

To obtain a stable system, the distribution of collocation points must be considered as a function of both polynomial interpolation order and kernel singularity. Much work has been done where a convergence theory for piecewise constant and linear interpolants was developed [2, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22]. For an excellent review see [1]. Convergence of the numerical solution is guaranteed if one collocates evenly between the node points [1, 2, 9, 16, 19], though not necessarily to the solution [1, 4, pp. 260-262]. Recently, [6] extended this theory to include Hermite cubics.

In an effort to identify optimal collocation points, we will utilize a weighted Peano kernel theory as developed in [3, 7, 13, 15] to approximate the integral equation (1.3) and provide *a-priori* error bounds. The bounds are then minimized in order to produce an optimal grid as well as furnish the desired distribution of collocation points. The method is useful in that it can provide an abundance of error results in terms of desirable properties of f (monotonicity, p-norm, total bounded variation, Lipschitzian etc.)

2. MAIN RESULTS

We will assume $K(\cdot,y) : [a,b] \to (0,\infty)$ to be integrable and positive, that is $K(\cdot,y) \in L_1(a,b)$ and $K(x,y) \ge 0,\ \forall (x,y) \in [a,b]\times[a,b]$. In addition, we assume that $f : [a,b] \to \mathbb{R}$

has bounded first derivative and we approximate it using the constant functional

$$f(x) \approx \begin{cases} f(a), & a \le x \le \xi, \\ f(b), & \xi < x \le b. \end{cases} \tag{2.1}$$

We seek to write down an explicit formula for $f(a)$ and $f(b)$ in terms of g and K. The following theorem will be utilized.

Theorem 2.1. *[13, Theorem 7.21] Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Further, let $w : (a,b) \to [0,\infty)$ be an integrable function so that $\int_a^b w(t)\,dt < \infty$. Then for $x \in [a,b]$, the following inequality holds*

$$\left| \int_a^b w(t) f(t)\,dt - \left[m(a,x) f(a) + m(x,b) f(b) \right] \right| \le I(x) \|f'\|_\infty , \tag{2.2}$$

where

$$I(x) = \int_a^b p(x,t)\, w(t)\; dt, \tag{2.3}$$

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ b-t, & t \in (x,b] \end{cases}, \quad \textit{and} \quad m(a,b) = \int_a^b w(t)\,dt.$$

The bound $I(x)$ is minimized at the midpoint $x = (a+b)/2$.

Thus we can directly apply Theorem 2.1 to the integral equation (1.3) to establish that

$$g(y) = m\left(a, \frac{a+b}{2}; y\right) f(a) + m\left(\frac{a+b}{2}, b; y\right) f(b) + R(y), \tag{2.4}$$

where

$$|R(y)| \le \|f'\|_\infty \left(\int_a^{(a+b)/2} (x-a) K|x-y|\,dx + \int_{(a+b)/2}^b (b-x) K|x-y|\,dx \right), \tag{2.5}$$

and m has been redefined to

$$m(a,b;y) = \int_a^b K|x-y|\,dx. \tag{2.6}$$

Since (2.4) is linear in $f(a)$ and $f(b)$, we can collocate at the two points $a \le y_1 < y_2 \le b$ to obtain

$$f(a) = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \big(m_{22}(g_1 - R_1) - m_{12}(g_2 - R_2) \big) \tag{2.7}$$

and

$$f(b) = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \big(m_{11}(g_2 - R_2) - m_{21}(g_1 - R_1) \big), \tag{2.8}$$

where

$$m_{i1} = m\left(a, \frac{a+b}{2}; y_i\right), \; m_{i2} = m\left(\frac{a+b}{2}, b; y_i\right), \tag{2.9}$$
$$g_i = g(y_i), \text{ and } R_i = R(y_i), \text{ for } i = 1, 2.$$

We can now write down an approximation for both $f(a)$ and $f(b)$ and the associated error bound in terms of $\|f'\|_\infty$, y_1 and y_2. Optimal collocation points can then be identified by minimizing the error. This is established in the following theorem, where for simplicity we will assume that $y_2 = a + b - y_1$.

Theorem 2.2. *The integral equation (1.3) has an approximate solution (2.1) in which*

$$\left| f(a) - \left(\frac{M_1 g_1 - M_2 g_2}{M_1^2 - M_2^2} \right) \right| \le \|f'\|_\infty E(y) \quad \textit{and} \tag{2.10}$$
$$\left| f(b) - \left(\frac{M_1 g_2 - M_2 g_1}{M_1^2 - M_2^2} \right) \right| \le \|f'\|_\infty E(y)$$

where

$$M_1 = m_{11}, \quad M_2 = m_{12} \quad \textit{and} \tag{2.11}$$
$$E(y) = \frac{\left[\int_a^{\frac{a+b}{2}} (x-a) K|x-y| dx + \int_{\frac{a+b}{2}}^b (b-x) K|x-y| dx \right]}{\left| \int_a^{\frac{a+b}{2}} K|x-y| dx - \int_{\frac{a+b}{2}}^b K|x-y| dx \right|},$$

for $y = y_1 \in [a, (a+b)/2)$ *and* $y_2 = b + a - y_1$.

Proof. With the condition $y_2 = a + b - y_1$, it is a simple matter to show that

$$m_{11} = m_{22} \quad \text{and} \quad m_{12} = m_{21}.$$

Furthermore, we can also establish that

$$|R(y_1)| \le \|f'\|_\infty E(y) \quad \text{and} \quad |R(y_2)| \le \|f'\|_\infty E(y).$$

Hence, rearranging (2.7) and (2.8), using the above simplifications and the triangle inequality produces the result. □

Equation (2.10) provides explicit error bounds for functions f of bounded first derivative in terms of a collocation point $y \in [a, \frac{a+b}{2})$. Minimizing $E(y)$ should produce an optimal collocation strategy for this class. This is explored in the next section.

3. NUMERICAL EXPERIMENTS

In this section we apply the results of the previous section to the numerical solution of Symm's integral equation

$$g(y) = \int_0^1 \ln\left(\frac{1}{|x-y|} \right) f(x)\, dx, \qquad 0 \le y \le 1. \tag{3.1}$$

We choose an exact solution $f(x) = x^{3/2} + 1$, so that f' is bounded, but all higher derivatives are unbounded. All of the algebraic calculations of the previous section have been performed using Maple.

In this case, we have

$$g(y) = \frac{4}{15} y - \ln(y)\, y - \frac{7}{5} \ln(1-y) + \ln(1-y)\, y + \frac{4}{5} y^2 + \frac{29}{25} - \frac{4}{5} y^{5/2} \operatorname{Re}\left(\operatorname{arctanh} y^{-1/2} \right). \tag{3.2}$$

Using Maple, the approximation for $f(a)$ in equation (2.10) is

$$(3.3)\quad f^*(a) = \Bigg[\left(-\ln(y)\,y + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1-2\,y) - y\,\ln(2) + y\,\ln(1-2\,y) + \frac{1}{2}\right) \left(\frac{4}{15}\,y - \ln(y)\,y - \frac{7}{5}\ln(1-y) + \ln(1-y)\,y + \frac{4}{5}y^2 + \frac{29}{25} - \frac{4}{5}y^{5/2}\,\mathrm{Re}\left(\operatorname{arctanh}\,y^{-1/2}\right)\right) - \left(-\ln(1-y) + \ln(1-y)\,y - \frac{1}{2}\ln(2) + y\,\ln(2) + \frac{1}{2}\ln(1-2\,y) - y\,\ln(1-2\,y) + \frac{1}{2}\right)\left(\frac{107}{75} - \frac{4}{15}\,y - \ln(1-y)(1-y) - \frac{7}{5}\,\ln(y) + \ln(y)(1-y) + \frac{4}{5}(1-y)^2 - \frac{4}{5}(1-y)^{5/2}\,\mathrm{Re}\left(\operatorname{arctanh}(1-y)^{-1/2}\right)\right)\Bigg] \Bigg[\left(-\ln(y)\,y + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(1-2\,y) - y\,\ln(2) + y\,\ln(1-2\,y) + \frac{1}{2}\right)^2 - \left(-\ln(1-y) + \ln(1-y)\,y - \frac{1}{2}\ln(2) + y\,\ln(2) + \frac{1}{2}\ln(1-2\,y) - y\,\ln(1-2\,y) + \frac{1}{2}\right)^2\Bigg]^{-1}$$

and from equation (2.11), the bound for the theoretical error is

$$(3.4)\quad E(y) = \left[-\frac{1}{2}\ln(y)\,y^2 - y^2\ln(2) + \ln(1-2y)\left(y^2 + \frac{1}{4}\right) - \frac{1}{4}\ln(2) + \frac{3}{8} + y\ln(2) - y\ln(1-2y) + \ln(1-y)\left(y - \frac{1}{2} - \frac{1}{2}y^2\right)\right] \Big/ \Big[-\ln(y)\,y + (1-2y)\ln(2) - (1-2y)\ln(1-2y) + (1-y)\ln(1-y)\Big].$$

In Figure 3.1 we plot the theoretical error in $f(a)$. That is, a plot of $E(y)$ as a function of collocation point y. It is obvious that the error should increase as $y \to 1/2$ since at this point $y_1 = y_2$ and the linear system becomes singular.

In contrast to other results for interpolation of this order, the theoretical result shows that the optimal collocation point is not at the boundary $y = 0$ as would be expected but in the interior. For this particular kernel, the optimal point occurs near $y = 0.017$.

In Figure 3.2 we plot the numerical error in $f(a)$. That is, a plot of $|f(0) - f^*(0)|$ as a function of collocation point y. The optimal location of the collocation point is near $y = 0.019$. We can see that the theoretical error is qualitatively similar to the numerical error and that the optimal collocation point is close to that identified in the theoretical result.

4. Conclusion

The application of Peano kernel theory to first kind integral equations is a powerful technique. The theory can account for general properties of g, K and f. This contrasts with other methods where, for example, g is assumed analytic. In addition, there are a number weighted Peano kernel derived multi-point quadrature rules with error bounds in terms of f', f'' and $f^{(n)}$ [13] as

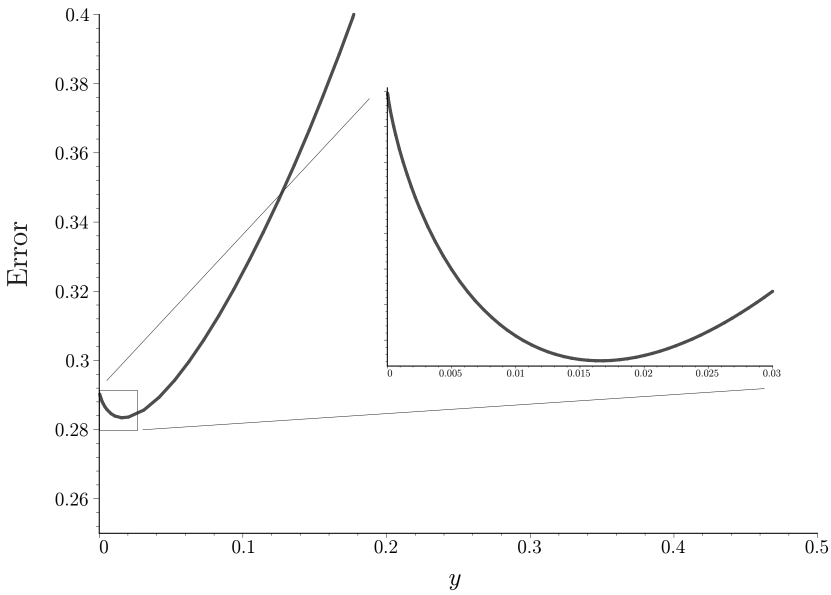

Figure 3.1: Theoretical error given by equation (3.4) as a function of collocation point y. The zoomed graph indicates an optimal collocation point near $y = 0.017$

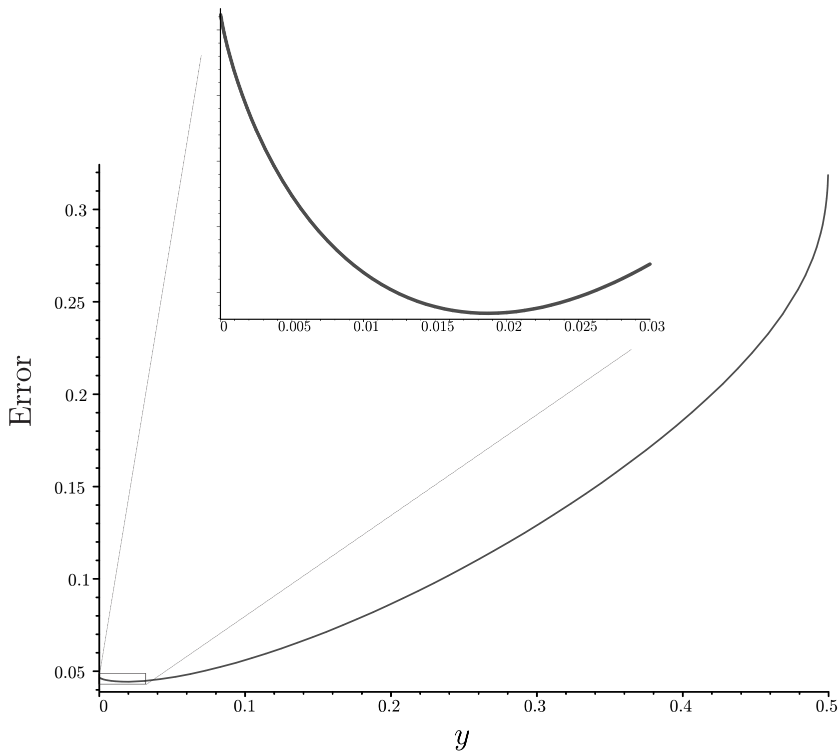

Figure 3.2: Numerical error, $|f(a) - f^*(a)|$, as a function of collocation point y. The zoomed graph indicates an optimal collocation point near $y = 0.019$

well as multiple dimensions [5]. The application of these may prove to be a fruitful source of results in the study of collocation points for integral equations.

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