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ON OPEN PROBLEMS OF F. QI

BENHARRAT BELAÏDI, ABDALLAH EL FARISSI, AND ZINELAÂBIDINE LATREUCH

DEPARTMENT OF MATHEMATICS
LABORATORY OF PURE AND APPLIED MATHEMATICS
UNIVERSITY OF MOSTAGANEM
B. P. 227 MOSTAGANEM, ALGERIA
belaidi@univ-mosta.dz

elfarissi.abdallah@yahoo.fr

z.latreuch@gmail.com

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ABSTRACT. In this paper, we give a complete answer to Problem 1 and a partial answer to Problem 2 posed by F. Qi in [2] and we propose an open problem.

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1. INTRODUCTION

Before, we state our results, for our own convenience, we introduce the following notations:

$$[0,\infty)^n \triangleq \underbrace{[0,\infty)\times[0,\infty)\times \ldots \times[0,\infty)}_{n\text{ times}} \tag{1.1}$$

and

$$(0,\infty)^n \triangleq \underbrace{(0,\infty)\times(0,\infty)\times \ldots \times(0,\infty)}_{n\text{ times}} \tag{1.2}$$

for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

In [2], F. Qi proved the following:

Theorem A. *For $(x_1, x_2, \ldots, x_n) \in [0,\infty)^n$ and $n \geqslant 2$, inequality*

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.3}$$

is valid. Equality in (1.3) holds if $x_i = 2$ for some given $1 \leqslant i \leqslant n$ and $x_j = 0$ for all $1 \leqslant j \leqslant n$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1.3) is the best possible.

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Theorem B. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then*

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \leqslant \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.4}$$

Equality in (1.4) holds if $x_i = 2$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1.4) is the best possible.

In the same paper, F. Qi posed the following two open problems:

Problem 1.1. *For $(x_1, x_2, ..., x_n) \in [0,\infty)^n$ and $n \geqslant 2$, determine the best possible constants $\alpha_n, \lambda_n \in \mathbb{R}$ and $\beta_n > 0$, $\mu_n < \infty$ such that*

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{1.5}$$

Problem 1.2. *What is the integral analogue of the double inequality (1.5)?*

Recently, Huan-Nan Shi gave a partial answer in [3] to Problem 1.1. The main purpose of this paper is to give a complete answer to this problem. Also, we give a partial answer to Problem 1.2. The method used in this paper will be quite different from that in the proofs of Theorem 1.1 of [2] and Theorem 1 of [3]. For some related results, we refer the reader to [1]. We will prove the following results.

Theorem 1.1. *Let $p \geqslant 1$ be a real number. For $(x_1, x_2, \ldots, x_n) \in [0,\infty)^n$ and $n \geqslant 2$, the inequality*

$$\frac{e^p}{p^p}\sum_{i=1}^{n} x_i^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.6}$$

is valid. Equality in (1.6) holds if $x_i = p$ for some given $1 \leqslant i \leqslant n$ and $x_j = 0$ for all $1 \leqslant j \leqslant n$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (1.6) is the best possible.

Theorem 1.2. *Let $0 < p \leqslant 1$ be a real number. For $(x_1, x_2, \ldots, x_n) \in [0,\infty)^n$ and $n \geqslant 2$, the inequality*

$$n^{p-1}\frac{e^p}{p^p}\sum_{i=1}^{n} x_i^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.7}$$

is valid. Equality in (1.7) holds if $x_i = \frac{p}{n}$ for all $1 \leqslant i \leqslant n$. Thus, the constant $n^{p-1}\frac{e^p}{p^p}$ in (1.7) is the best possible.

Theorem 1.3. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$ and $p \geqslant 1$ be a real number. Then*

$$\frac{e^p}{p^p}\sum_{i=1}^{\infty} x_i^p \leqslant \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.8}$$

Equality in (1.8) holds if $x_i = p$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (1.8) is the best possible.

Remark 1. In general, we cannot find $0 < \mu_n < \infty$ and $\lambda_n \in \mathbb{R}$ such that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}.$$

Proof. We suppose that there exists $0 < \mu_n < \infty$ and $\lambda_n \in \mathbb{R}$ such that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}.$$

Then for $(x_1, 1, ..., 1)$, we obtain as $x_1 \to +\infty$,

$$1 \leqslant e^{1-n}\mu_n \left(n - 1 + x_1^{\lambda_n}\right) e^{-x_1} \to 0.$$

This is a contradiction. □

Theorem 1.4. *Let $p > 0$ be a real number, $(x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ and $n \geqslant 2$ such that $0 < x_i \leqslant p$ for all $1 \leqslant i \leqslant n$. Then the inequality*

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p} \tag{1.9}$$

is valid. Equality in (1.9) holds if $x_i = p$ for all $1 \leqslant i \leqslant n$. Thus, the constant $\frac{p^p}{n}e^{np}$ is the best possible.

Remark 2. Let $p > 0$ be a real number, $(x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ and $n \geqslant 2$ such that $0 < x_i \leqslant p$ for all $1 \leqslant i \leqslant n$. Then

(i) *if $0 < p \leq 1$, we have*

$$n^{p-1}\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p}; \tag{1.10}$$

(ii) *if $p \geq 1$, we have*

$$\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p}. \tag{1.11}$$

Remark 3. Taking $p = 2$ in Theorems 1.1 and 1.3 easily leads to Theorems A and B respectively.

Remark 4. Inequality (1.6) can be rewritten as either

$$\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leqslant \prod_{i=1}^{n} e^{x_i} \tag{1.12}$$

or

$$\frac{e^p}{p^p} \left\|x\right\|_p^p \leqslant \exp \left\|x\right\|_1, \tag{1.13}$$

where $x = (x_1, x_2, ..., x_n)$ and $\left\|\cdot\right\|_p$ denotes the p-norm.

Remark 5. Inequality (1.8) can be rewritten as

$$\frac{e^p}{p^p} \sum_{i=1}^{\infty} x_i^p \leqslant \prod_{i=1}^{\infty} e^{x_i} \tag{1.14}$$

which is equivalent to inequality (1.12) for $x = (x_1, x_2, ...) \in [0, \infty)^{\infty}$.

Remark 6. Taking $x_i = \frac{1}{i}$ for $i \in \mathbb{N}$ in (1.6) and rearranging gives

$$p - p\ln p + \ln\left(\sum_{i=1}^{n} \frac{1}{i^p}\right) \leqslant \sum_{i=1}^{n} \frac{1}{i}. \tag{1.15}$$

Taking $x_i = \frac{1}{i^s}$ for $i \in \mathbb{N}$ and $s > 1$ in (1.8) and rearranging gives

$$p - p\ln p + \ln\left(\sum_{i=1}^{\infty} \frac{1}{i^{ps}}\right) = p - p\ln p + \ln\varsigma(ps) \leqslant \sum_{i=1}^{\infty} \frac{1}{i^s} = \varsigma(s), \tag{1.16}$$

where ς denotes the well-known Riemann Zêta function.

In the following, we give a partial answer to Problem 1.2.

Theorem 1.5. *Let $0 < p \leqslant 1$ be a real number, and let f be a continuous function on $[a, b]$. Then the inequality*

$$\frac{e^p}{p^p}(b-a)^{p-1}\int_a^b |f(x)|^p\,dx \leq \exp\left(\int_a^b |f(x)|\,dx\right) \tag{1.17}$$

is valid. Equality in (1.17) holds if $f(x) = \frac{p}{b-a}$. Thus, the constant $\frac{e^p}{p^p}(b-a)^{p-1}$ in (1.17) is the best possible.

Theorem 1.6. *Let $x > 0$. Then*

$$\Gamma(x) \leqslant \frac{2^{x+1}x^{x-1}}{e^x} \tag{1.18}$$

is valid, where Γ denotes the well-known Gamma function.

2. LEMMAS

Lemma 2.1. *For $x \in [0, \infty)$ and $p > 0$, the inequality*

$$\frac{e^p}{p^p}x^p \leqslant e^x \tag{2.1}$$

is valid. Equality in (2.1) holds if $x = p$. Thus, the constant $\frac{e^p}{p^p}$ in (2.1) is the best possible.

Proof. Letting $f(x) = p\ln x - x$ on the set $(0, \infty)$, it is easy to obtain that the function f has a maximal point at $x = p$ and the maximal value equals $f(p) = p\ln p - p$. Then, we obtain (2.1). It is clear that the inequality (2.1) also holds at $x = 0$. ☐

Lemma 2.2. *Let $p > 0$ be a real number. For $(x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ and $n \geqslant 2$, we have:*

(i) *If $p \geqslant 1$, then the inequality*

$$\sum_{i=1}^{n} x_i^p \leqslant \left(\sum_{i=1}^{n} x_i\right)^p \tag{2.2}$$

is valid.

(ii) *If $0 < p \leqslant 1$, then inequality*

$$n^{p-1}\sum_{i=1}^{n} x_i^p \leqslant \left(\sum_{i=1}^{n} x_i\right)^p \tag{2.3}$$

is valid.

Proof. (i) For the proof, we use mathematical induction. First, we prove (2.2) for $n = 2$. We have for any $(x_1, x_2) \neq (0, 0)$

$$\frac{x_1}{x_1 + x_2} \leq 1 \qquad \text{and} \qquad \frac{x_2}{x_1 + x_2} \leq 1. \tag{2.4}$$

Then, by $p \geqslant 1$ we get

$$\left(\frac{x_1}{x_1 + x_2}\right)^p \leqslant \frac{x_1}{x_1 + x_2} \qquad \text{and} \qquad \left(\frac{x_2}{x_1 + x_2}\right)^p \leqslant \frac{x_2}{x_1 + x_2}. \tag{2.5}$$

By addition from (2.5), we obtain

$$\left(\frac{x_1}{x_1 + x_2}\right)^p + \left(\frac{x_2}{x_1 + x_2}\right)^p \leqslant \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_1 + x_2}.$$

So,

$$x_1^p + x_2^p \leqslant (x_1 + x_2)^p\,. \tag{2.6}$$

It is clear that inequality (2.6) holds also at the point $(0, 0)$.

Now we suppose that

$$\sum_{i=1}^{n} x_i^p \leqslant \left(\sum_{i=1}^{n} x_i\right)^p \tag{2.7}$$

and we prove that

$$\sum_{i=1}^{n+1} x_i^p \leqslant \left(\sum_{i=1}^{n+1} x_i\right)^p. \tag{2.8}$$

We have by (2.6)

$$\left(\sum_{i=1}^{n+1} x_i\right)^p = \left(\sum_{i=1}^{n} x_i + x_{n+1}\right)^p \geqslant \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \tag{2.9}$$

and by (2.7) and (2.9), we obtain

$$\sum_{i=1}^{n+1} x_i^p = \sum_{i=1}^{n} x_i^p + x_{n+1}^p \leqslant \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \leqslant \left(\sum_{i=1}^{n+1} x_i\right)^p. \tag{2.10}$$

Then for all $n \geqslant 2$, (2.2) holds.

(ii) For $(x_1, x_2, \ldots, x_n) \in [0, \infty)^n$, $0 < p \leqslant 1$ and $n \geqslant 2$, we have

$$\left(\sum_{i=1}^{n} x_i\right)^p = \left(\sum_{i=1}^{n} n\frac{x_i}{n}\right)^p. \tag{2.11}$$

By using the concavity of the function $x \mapsto x^p$ $(x \geqslant 0,\ 0 < p \leqslant 1)$, we obtain from (2.11)

$$\left(\sum_{i=1}^{n} x_i\right)^p = \left(\sum_{i=1}^{n} n\frac{x_i}{n}\right)^p \geqslant \sum_{i=1}^{n} \frac{n^p x_i^p}{n} = n^{p-1}\sum_{i=1}^{n} x_i^p. \tag{2.12}$$

Hence (2.3) holds. □

3. PROOFS OF THE THEOREMS

We are now in a position to prove our theorems.

Proof of Theorem 1.1. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $p \geqslant 1$, we put $x = \sum_{i=1}^{n} x_i$. Then by (2.1), we have

$$\frac{e^p}{p^p}\left(\sum_{i=1}^{n} x_i\right)^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \tag{3.1}$$

and by (2.2) we obtain (1.6). □

Proof of Theorem 1.2. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $0 < p \leqslant 1$, we put $x = \sum_{i=1}^{n} x_i$. Then by (2.1), we have

$$\frac{e^p}{p^p}\left(\sum_{i=1}^{n} x_i\right)^p \leqslant \exp\left(\sum_{i=1}^{n} x_i\right) \tag{3.2}$$

and by (2.3) we obtain (1.7). □

Proof of Theorem 1.3. This can be concluded by letting $n \to +\infty$ in Theorem 1.1. □

Proof of Theorem 1.4. By the condition of Theorem 1.4, we have $0 < x_i \leqslant p$ for all $1 \leqslant i \leqslant n$. Then, $x_i^{-p} \geqslant p^{-p}$ for all $1 \leqslant i \leqslant n$. It follows that $\sum_{i=1}^{n} x_i^{-p} \geqslant np^{-p}$. Then we obtain

$$\sum_{i=1}^{n} x_i - \ln\left(\sum_{i=1}^{n} x_i^{-p}\right) \leqslant np - \ln\left(np^{-p}\right) = np + \ln\frac{1}{n} + p\ln p. \tag{3.3}$$

It follows that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leqslant \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p}.$$

The proof of Theorem 1.4 is completed. □

Proof of Theorem 1.5. Let $0 < p \leqslant 1$. By Hölder's inequality, we have

$$\int_a^b |f(x)|^p \, dx \leqslant \left(\int_a^b |f(x)| \, dx\right)^p (b-a)^{1-p}. \tag{3.4}$$

It follows that

$$(b-a)^{p-1} \int_a^b |f(x)|^p \, dx \leqslant \left(\int_a^b |f(x)| \, dx\right)^p. \tag{3.5}$$

On the other hand, by Lemma 2.1, we have

$$\frac{e^p}{p^p}\left(\int_a^b |f(x)| \, dx\right)^p \leq \exp\left(\int_a^b |f(x)| \, dx\right). \tag{3.6}$$

By (3.5) and (3.6), we get (1.17). □

Proof of Theorem 1.6. Let $x > 0$ and $t > 0$. Then by Lemma 2.1, we have

$$e^t \geqslant \frac{e^x}{x^x} t^x. \tag{3.7}$$

So,

$$e^{-t} \geqslant \frac{e^x}{x^x} t^x e^{-2t}. \tag{3.8}$$

It is clear that

$$1 \geqslant \frac{e^x}{x^x} \int_0^\infty t^x e^{-2t} dt = \frac{e^x}{2^{x+1} x^{x-1}} \Gamma(x). \tag{3.9}$$

The proof of Theorem 1.6 is completed. □

4. OPEN PROBLEM

Problem 4.1. *For $p \geq 1$ a real number, determine the best possible constant $\alpha \in \mathbb{R}$ such that*

$$\frac{e^p}{p^p} \alpha \int_a^b |f(x)|^p \, dx \leq \exp\left(\int_a^b |f(x)| \, dx \right).$$

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