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BOUNDARY VALUE PROBLEM FOR SECOND-ORDER DIFFERENTIAL OPERATORS WITH MIXED NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study a second order differential operator with mixed nonlocal boundary conditions combined weighting integral boundary condition with another two point boundary condition. Under certain conditions on the weighting functions and on the coefficients in the boundary conditions, called non regular boundary conditions, we prove that the resolvent decreases with respect to the spectral parameter in $L^p(0,1)$, but there is no maximal decreasing at infinity for $p \geq 1$. Furthermore, the studied operator generates in $L^p(0,1)$ an analytic semi group with singularities for $p \geq 1$. The obtained results are then used to show the correct solvability of a mixed problem for a parabolic partial differential equation with non regular boundary conditions.

Key words and phrases: Green's Function, Non Regular Boundary Conditions, Regular Boundary Conditions, Semi group with Singularities.

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1. INTRODUCTION

In the space $L^p(0,1)$ we consider the boundary value problem

$$\begin{cases} L(u) = u'' = f(x), \\ B_1(u) = a_0 u(0) + b_0 u'(0) + c_0 u(1) + d_0 u'(1) = 0, \\ B_2(u) = \int_0^1 R(t) u(t)\, dt + \int_0^1 S(t) u'(t)\, dt = 0, \end{cases} \tag{1.1}$$

where the functions $R, S \in C([0,1], \mathbb{C})$. We associate to problem (1.1) in space $L^p(0,1)$ the operator:

$$L_p(u) = u'',$$

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with domain $D(L_p) = \{u \in W^{2,p}(0,1) : B_i(u) = 0, i = 1,2\}$.

Many papers and books give the full spectral theory of Birkhoff regular differential operators with two point linearly independent boundary conditions, in terms of coefficients of boundary conditions. The reader should refer to [7, 10, 20, 21, 22, 28, 31, 33] and references therein. Few works have been devoted to the study of a non regular situation. The case of separated non regular boundary conditions was studied by W. Eberhard, J.W. Hopkins, D. Jakson, M.V. Keldysh, A.P. Khromov, G. Seifert, M.H. Stone, L.E. Ward (see S. Yakubov and Y. Yakubov [33] for exact references). A situation of non regular non-separated boundary conditions was considered by H. E. Benzinger [2], M. Denche [4], W. Eberhard and G. Freiling [8], M.G. Gasumov and A.M. Magerramov [12, 13], A.P. Khromov [18], Yu. A. Mamedov [19], A.A. Shkalikov [24], Yu. T. Silchenko [26], C. Tretter [29], A.I. Vagabov [30], S. Yakubov [32] and Y. Yakubov [34].

A mathematical model with integral boundary conditions was derived by [9, 23] in the context of optical physics. The importance of this kind of problem has been also pointed out by Samarskii [27].

In this paper, we study a problem for second order ordinary differential equations with mixed nonlocal boundary conditions combined with weighted integral boundary conditions and another two point boundary condition. Following the technique in [11, 20, 21, 22], we should bound the resolvent in the space $L^p(0,1)$ by means of a suitable formula for Green's function. Under certain conditions on the weighting functions and on the coefficients in the boundary conditions, called non regular boundary conditions, we prove that the resolvent decreases with respect to the spectral parameter in $L^p(0,1)$, but there is no maximal decreasing at infinity for $p \geq 1$. In contrast to the regular case this decreasing is maximal for $p = 1$ as shown in [11]. Furthermore, the studied operator generates in $L^p(0,1)$ an analytic semi group with singularities for $p \geq 1$. The obtained results are then used to show the correct solvability of a mixed problem for a parabolic partial differential equation with non regular non local boundary conditions.

2. GREEN'S FUNCTION

Let $\lambda \in \mathbb{C}$, $u_1(x) = u_1(x,\lambda)$ and $u_2(x) = u_2(x,\lambda)$ be a fundamental system of solutions to the equation

$$L(u) - \lambda u = 0.$$

Following [20], the Green's function of problem (1.1) is given by

$$G(x,s;\lambda) = \frac{N(x,s;\lambda)}{\Delta(\lambda)}, \tag{2.1}$$

where $\Delta(\lambda)$ it the characteristic determinant of the considered problem defined by

$$\Delta(\lambda) = \begin{vmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{vmatrix} \tag{2.2}$$

and

$$N(x,s;\lambda) = \begin{vmatrix} u_1(x) & u_2(x) & g(x,\ s,\ \lambda) \\ B_1(u_1) & B_1(u_2) & B_1(g)_x \\ B_2(u_1) & B_2(u_2) & B_2(g)_x \end{vmatrix} \tag{2.3}$$

for $x,\ s \in [0,1]$.The function $g(x,\ s,\ \lambda)$ is defined as follows

$$g(x,s;\lambda) = \pm\frac{1}{2}\frac{u_1(x)u_2(s) - u_1(s)u_2(x)}{u_1'(s)u_2(s) - u_1(s)u_2'(s)}, \tag{2.4}$$

where it takes the plus sign for $x > s$ and the minus sign for $x < s$.

Given an arbitrary $\delta \in \left(\frac{\pi}{2}, \pi\right)$, we consider the sector

$$\Sigma_\delta = \{\lambda \in \mathbb{C}, |\arg \lambda| \leq \delta, \lambda \neq 0\}.$$

For $\lambda \in \sum_\delta$, define ρ as the square root of λ with positive real part. For $\lambda \neq 0$, we can consider a fundamental system of solutions of the equation $u'' = \lambda u = \rho^2 u$ given by $u_1(t) = e^{-\rho t}$ and $u_2(t) = e^{\rho t}$.

In the following we are going to deduce an adequate formulae for $\Delta(\lambda)$ and $G(x, s; \lambda)$. First of all, for $j = 1, 2$, we have

$$B_1(u_j) = a_0 + (-1)^j b_0\rho + c_0 e^{(-1)^j \rho} + (-1)^j d_0 \rho e^{(-1)^j \rho},$$
$$B_2(u_j) = \int_0^1 R(t) e^{(-1)^j \rho t} dt + (-1)^j \rho \int_0^1 S(t) e^{(-1)^j \rho t} dt.$$

so we obtain from (2.2)

$$\Delta(\lambda) = \left(a_0 - b_0\rho + c_0 e^{-\rho} - d_0\rho e^{-\rho}\right)\left(\int_0^1 (R(t) + \rho S(t)) e^{\rho t}\, dt\right) - (a_0 + b_0\rho + c_0 e^{\rho} + d_0 \rho e^{\rho})\left(\int_0^1 (R(t) - \rho S(t)) e^{-\rho t}\, dt\right), \tag{2.5}$$

and $g(x, s; \lambda)$ has the form

$$g(x, s; \lambda) = \begin{cases} \dfrac{1}{4\rho}(e^{\rho(x-s)} - e^{\rho(s-x)}) & \text{if } x > s, \\ \\ \dfrac{1}{4\rho}(e^{\rho(s-x)} - e^{\rho(x-s)}) & \text{if } x < s. \end{cases}$$

Thus we have

$$B_1(g) = \left(a_0 - b_0\rho - c_0 e^{-\rho} + d_0\rho e^{-\rho}\right)\frac{e^{\rho s}}{4\rho} + (-a_0 - b_0\rho + c_0 e^{\rho} + d_0\rho e^{\rho})\frac{e^{-\rho s}}{4\rho},$$
$$B_2(g) = \frac{e^{\rho s}}{4\rho}\left(\int_0^s (R(t) - \rho S(t)) e^{-\rho t} dt + \int_s^1 (-R(t) + \rho S(t)) e^{-\rho t} dt\right) + \frac{e^{-\rho s}}{4\rho}\left(-\int_0^s (R(t) + \rho S(t)) e^{\rho t} dt + \int_s^1 (R(t) + \rho S(t)) e^{\rho t} dt\right).$$

After a long calculation, formula (2.3) can be written as

$$N(x, s; \lambda) = \varphi(x, s; \lambda) + \varphi_i(x, s; \lambda), \tag{2.6}$$

where

$$\varphi(x, s; \lambda) = \frac{1}{2\rho}\left[\left(\int_0^1 (a_0 + \rho b_0)(R(t) + \rho S(t)) e^{\rho t} dt - e^{\rho}(c_0 + \rho d_0)\int_0^s (R(t) + \rho S(t)) e^{\rho t} dt\right) e^{-\rho(x+s)} + \left(\int_s^1 (a_0 - \rho b_0)(R(t) - \rho S(t)) e^{-\rho t} dt - e^{\rho}(c_0 - \rho d_0)\int_0^s (R(t) - \rho S(t)) e^{-\rho t} dt\right) e^{\rho(x+s)}\right] \tag{2.7}$$

and the function $\varphi_i(x,s;\lambda)$ is given by

$$\varphi_i(x,s;\lambda) = \begin{cases} \varphi_1(x,s;\lambda) & \text{if } x > s, \\ \varphi_2(x,s;\lambda) & \text{if } x < s, \end{cases} \tag{2.8}$$

with

$$\begin{aligned} \varphi_1(x,s;\lambda) = \frac{e^{\rho(s-x)}}{2\rho} &\left(\int_0^s (a_0 + \rho b_0 + c_0 e^{\rho} + \rho e^{\rho} d_0)\left(R(t) - \rho S(t)\right) e^{-\rho t} dt \right. \\ &\left. - \int_s^1 (a_0 - \rho b_0)\left(R(t) + \rho S(t)\right) e^{\rho t} dt \right) \\ &+ \frac{e^{\rho(x-s)}}{2\rho} \left(\int_0^s \left(a_0 - \rho b_0 + c_0 e^{-\rho} - \rho e^{-\rho} d_0\right)\left(R(t) + \rho S(t)\right) e^{\rho t} dt \right. \\ &\left. - \int_0^1 (a_0 + \rho b_0)\left(R(t) - \rho S(t)\right) e^{-\rho t} dt \right), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \varphi_2(x,s;\lambda) = \frac{1}{2\rho} &\left[\left(- \int_s^1 (a_0 + \rho b_0 + c_0 e^{\rho} + \rho e^{\rho} d_0)\left(R(t) - \rho S(t)\right) e^{-\rho t} dt \right. \right. \\ &\left. + \int_0^1 (c_0 - \rho d_0) e^{-\rho} \left(R(t) + \rho S(t)\right) e^{\rho t} dt \right) e^{\rho(s-x)} \\ &- \left(\int_s^1 \left(a_0 - \rho b_0 + c_0 e^{-\rho} - \rho e^{-\rho} d_0\right)\left(R(t) + \rho S(t)\right) e^{\rho t} dt \right. \\ &\left. \left. - \int_0^1 (c_0 + \rho d_0) e^{\rho} \left(R(t) - \rho S(t)\right) e^{-\rho t} dt \right) e^{\rho(x-s)} \right]. \end{aligned} \tag{2.10}$$

3. BOUNDS ON THE RESOLVENT

Every $\lambda \in \mathbb{C}$ such that $\Delta(\lambda) \neq 0$ belongs to $\rho(L_p)$, and the associated resolvent operator $R(\lambda, L_p)$ can be expressed as a Hilbert Schmidt operator

$$(\lambda I - L_p)^{-1} f = R(\lambda, L_p) f = - \int_0^1 G(\cdot, s; \lambda) f(s) ds, \qquad f \in L^p(0,1). \tag{3.1}$$

Then, for every $f \in L^p(0,1)$ we estimate (3.1)

$$\|R(\lambda; L_p) f\|_{L^p(0,1)} \leq \left(\sup_{0 \leq s \leq 1} \int_0^1 |G(x,s;\lambda)|^p dx \right)^{\frac{1}{p}} \|f\|_{L^p(0,1)},$$

and so we need to bound

$$\left(\sup_{0 \leq s \leq 1} \int_0^1 |G(x,s;\lambda)|^p dx \right)^{\frac{1}{p}} = \frac{1}{|\Delta(\lambda)|} \left(\sup_{0 \leq s \leq 1} \int_0^1 |N(x,s;\lambda)|^p dx \right)^{\frac{1}{p}}.$$

3.1. **Estimation of $N(x,s;\lambda)$.** We will denote by $\|\cdot\|$ the supremum norm which is defined by $\|R\| = \sup_{0 \leq s \leq 1} |R(s)|$. Since

$$N(x,s;\lambda)=\begin{cases}\varphi(x,s;\lambda)+\varphi_1(x,s;\lambda) & \text{if } x>s\\ \varphi(x,s;\lambda)+\varphi_2(x,s;\lambda) & \text{if } x<s\end{cases};$$

then

$$\|N(x,s;\lambda)\|_{L^p}\le\|\varphi(x,s;\lambda)\|_{L^p}+\|\varphi_i(x,s;\lambda)\|_{L^p}, \tag{3.2}$$

from (2.8), we have

$$\|\varphi_i(x,s;\lambda)\|_{L^p}\le 2^{1/p}\left\{\left[\int_0^s|\varphi_2(x,s;\lambda)|^p\,dx\right]^{\frac{1}{p}}+\left[\int_s^1|\varphi_1(x,s;\lambda)|^p\,dx\right]^{\frac{1}{p}}\right\}. \tag{3.3}$$

From (2.7) we have

$$\begin{aligned}|\varphi(x,s;\lambda)|\le{}&\frac{e^{-(x+s)\operatorname{Re}\rho}}{2|\rho|}\Big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\int_s^1 e^{t\operatorname{Re}\rho}dt\\&+(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\,e^{\operatorname{Re}\rho}\int_0^s e^{t\operatorname{Re}\rho}dt\Big]\\&+\frac{e^{(x+s)\operatorname{Re}\rho}}{2|\rho|}\Big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\int_s^1 e^{-t\operatorname{Re}\rho}dt\\&+e^{-\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\int_0^s e^{-t\operatorname{Re}\rho}dt\Big]\end{aligned}$$

then

$$\begin{aligned}|\varphi(x,s;\lambda)|\le{}&\frac{e^{-(x+s)\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\left(e^{\operatorname{Re}\rho}-e^{s\operatorname{Re}\rho}\right)\\&+(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\left(e^{(s+1)\operatorname{Re}\rho}-e^{\operatorname{Re}\rho}\right)\big]\\&+\frac{e^{(x+s)\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\left(e^{-s\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)\\&+(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\left(e^{-\operatorname{Re}\rho}-e^{-(s+1)\operatorname{Re}\rho}\right)\big]\end{aligned}$$

and

$$\begin{aligned}&\int_0^1|\varphi(x,s;\lambda)|^p\,dx\\&\le\frac{2^p\int_0^1 e^{-px\operatorname{Re}\rho}dx}{(|\rho|\operatorname{Re}\rho)^p}\Big[(\|R\|+|\rho|\,\|S\|)^p(|a_0|+|\rho|\,|b_0|)^p\left(e^{(1-s)\operatorname{Re}\rho}-1\right)^p\\&\quad+(\|R\|+|\rho|\,\|S\|)^p\times(|c_0|+|\rho|\,|d_0|)^p\left(e^{\operatorname{Re}\rho}-e^{(1-s)\operatorname{Re}\rho}\right)^p\Big]\\&\quad+\frac{2^p\int_0^1 e^{px\operatorname{Re}\rho}dx}{(|\rho|\operatorname{Re}\rho)^p}\big[(\|R\|+|\rho|\,\|S\|)^p\times(|a_0|+|\rho|\,|b_0|)^p\left(1-e^{(s-1)\operatorname{Re}\rho}\right)^p\\&\quad+(\|R\|+|\rho|\,\|S\|)^p(|c_0|+|\rho|\,|d_0|)^p\left(e^{(s-1)\operatorname{Re}\rho}-e^{-1\operatorname{Re}\rho}\right)^p\Big],\end{aligned}$$

after calculation we obtain

$$\left(\int_0^1 |\varphi(x,s;\lambda)|^p\,dx\right)^{\frac{1}{p}} \le \frac{2^{1+\frac{2}{p}}e^{\operatorname{Re}\rho}}{|\rho|(\operatorname{Re}\rho)^{1+\frac{1}{p}}\,p^{1/p}}\Big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\Big[\left(e^{-s\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)$$
$$\times\left(1-e^{-p\operatorname{Re}\rho}\right)^{\frac{1}{p}}+\left(1-e^{-p\operatorname{Re}\rho}\right)^{\frac{1}{p}}\left(1-e^{(s-1)\operatorname{Re}\rho}\right)\Big]+(\|R\|+|\rho|\,\|S\|)$$
$$\times(|c_0|+|\rho|\,|d_0|)\Big[\left(1-e^{-p\operatorname{Re}\rho}\right)^{\frac{1}{p}}\left(1-e^{-s\operatorname{Re}\rho}\right)$$
$$+\left(1-e^{-p\operatorname{Re}\rho}\right)^{\frac{1}{p}}\left(e^{(s-1)\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)\Big]\Big]$$

as $\operatorname{Re}\rho>0$ so

$$\sup_{0\le s\le 1}\left(\int_0^1 |\varphi(x,s;\lambda)|^p\,dx\right)^{\frac{1}{p}} \le \frac{2^{1+\frac{2}{p}}e^{\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|c_0|)+|\rho|(|b_0|+|d_0|)\right]}{|\rho|(\operatorname{Re}\rho)^{1+\frac{1}{p}}\,p^{1/p}}. \tag{3.4}$$

From (2.9) we have

$$|\varphi_1(x,s;\lambda)| \le \frac{e^{(s-x)\operatorname{Re}\rho}}{2|\rho|}\Bigg[\left[(|a_0|+|\rho|\,|b_0|)+e^{\operatorname{Re}\rho}(|c_0|+|\rho|\,|d_0|)\right]$$
$$\times(\|R\|+|\rho|\,\|S\|)\int_0^s e^{-t\operatorname{Re}\rho}dt$$
$$+(|a_0|+|\rho|\,|b_0|)(\|R\|+|\rho|\,\|S\|)\int_0^1 e^{t\operatorname{Re}\rho}dt\Bigg]$$
$$+\frac{e^{(x-s)\operatorname{Re}\rho}}{2|\rho|}\Bigg[(\|R\|+|\rho|\,\|S\|)\Bigg[\left[(|a_0|+|\rho|\,|b_0|)\right.$$
$$\left.+e^{\operatorname{Re}\rho}(|c_0|+|\rho|\,|d_0|)\right]\int_0^s e^{t\operatorname{Re}\rho}dt$$
$$+(|a_0|+|\rho|\,|b_0|)(\|R\|+|\rho|\,\|S\|)\int_0^1 e^{-t\operatorname{Re}\rho}dt\Bigg]$$

then

$$|\varphi_1(x,s;\lambda)| \le \frac{e^{(s-x)\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\Big[(|a_0|+|\rho|\,|b_0|)(\|R\|+|\rho|\,\|S\|)\left(e^{\operatorname{Re}\rho}-e^{-s\operatorname{Re}\rho}\right)$$
$$+(|c_0|+|\rho|\,|d_0|)(\|R\|+|\rho|\,\|S\|)\left(e^{\operatorname{Re}\rho}-e^{(1-s)\operatorname{Re}\rho}\right)\Big]$$
$$+\frac{e^{(x-s)\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\Big[(\|R\|+|\rho|\,\|S\|)(|a_0|+|\rho|\,|b_0|)\left(e^{s\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)$$
$$+(|c_0|+|\rho|\,|d_0|)(\|R\|+|\rho|\,\|S\|)\left(e^{(s-1)\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)\Big]$$

and

$$\int_s^1 |\varphi_1(x,s;\lambda)|^p\,dx$$
$$\leq \frac{2^p\int_s^1 e^{-px\operatorname{Re}\rho}dx}{|\rho|^p(\operatorname{Re}\rho)^p}\Big[(|a_0|+|\rho|\,|b_0|)^p(\|R\|+|\rho|\,\|S\|)^p\left(e^{(1+s)\operatorname{Re}\rho}-1\right)^p$$
$$+(|c_0|+|\rho|\,|d_0|)^p(\|R\|+|\rho|\,\|S\|)^p\left(e^{(1+s)\operatorname{Re}\rho}-e^{\operatorname{Re}\rho}\right)^p\Big]$$
$$+\frac{2^p\int_s^1 e^{px\operatorname{Re}\rho}dx}{|\rho|^p(\operatorname{Re}\rho)^p}\Big[(|a_0|+|\rho|\,|b_0|)^p(\|R\|+|\rho|\,\|S\|)^p\left(1-e^{-(1+s)\operatorname{Re}\rho}\right)^p$$
$$+(|c_0|+|\rho|\,|d_0|)^p(\|R\|+|\rho|\,\|S\|)^p\left(e^{-\operatorname{Re}\rho}-e^{-(1+s)\operatorname{Re}\rho}\right)^p,$$

this yields

$$\int_s^1 |\varphi_1(x,s;\lambda)|^p\,dx \leq \frac{2^{p+1}e^{p\operatorname{Re}\rho}}{|\rho|^p(\operatorname{Re}\rho)^{p+1}}\big[(|a_0|+|\rho|\,|b_0|)^p(\|R\|+|\rho|\,\|S\|)^p$$
$$\times\left(1-e^{-(1+s)\operatorname{Re}\rho}\right)^p\left(1-e^{p(s-1)\operatorname{Re}\rho}\right)$$
$$+(|c_0|+|\rho|\,|d_0|)^p(\|R\|+|\rho|\,\|S\|)^p\left(1-e^{-s\operatorname{Re}\rho}\right)^p\left(1-e^{p(s-1)\operatorname{Re}\rho}\right)\big]$$

as $\operatorname{Re}\rho>0$ we obtain

$$\sup_{0\leq s\leq 1}\left(\int_s^1|\varphi_1(x,s;\lambda)|^p\,dx\right)^{\frac{1}{p}} \tag{3.5}$$
$$\leq \frac{2^{1+\frac{2}{p}}e^{\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|c_0|)+|\rho|(|b_0|+|d_0|)\right]}{|\rho|(\operatorname{Re}\rho)^{1+\frac{1}{p}}\,p^{1/p}}.$$

From (2.10) we have

$$|\varphi_2(x,s;\lambda)|\leq\frac{e^{(x-s)\operatorname{Re}\rho}}{2|\rho|}\Big[(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\int_0^1 e^{(1-t)\operatorname{Re}\rho}dt$$
$$+(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|\rho|\,|b_0|)+e^{-\operatorname{Re}\rho}(|c_0|+|\rho|\,|d_0|)\right]\int_0^1 e^{t\operatorname{Re}\rho}dt\Big]$$
$$+\frac{e^{(s-x)\operatorname{Re}\rho}}{2|\rho|}\left[(|c_0|+|\rho|\,|d_0|)\,(\|R\|+|\rho|\,\|S\|)\int_0^1 e^{(t-1)\operatorname{Re}\rho}dt\right.$$
$$\left.+\left[(|a_0|+|\rho|\,|b_0|)+e^{\operatorname{Re}\rho}(|c_0|+|\rho|\,|d_0|)\right](\|R\|+|\rho|\,\|S\|)\int_s^1 e^{-t\operatorname{Re}\rho}dt\right]$$

then

$$|\varphi_2(x,s;\lambda)|\leq\frac{e^{x\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\left[(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\left(e^{(1-s)\operatorname{Re}\rho}-e^{-\operatorname{Re}\rho}\right)\right.$$
$$\left.+(|a_0|+|\rho|\,|b_0|)\,(\|R\|+|\rho|\,\|S\|)\left(e^{(1-s)\operatorname{Re}\rho}-1\right)\right]$$
$$+\frac{e^{-x\operatorname{Re}\rho}}{2|\rho|\operatorname{Re}\rho}\left[(\|R\|+|\rho|\,\|S\|)(|c_0|+|\rho|\,|d_0|)\left(e^{\operatorname{Re}\rho}-e^{(s-1)\operatorname{Re}\rho}\right)\right.$$
$$\left.+(|a_0|+|\rho|\,|b_0|)(\|R\|+|\rho|\,\|S\|)\left(1-e^{(s-1)\operatorname{Re}\rho}\right)\right].$$

So

$$\int_0^s |\varphi_2(x,s;\lambda)|^p\,dx \le \frac{2^{p+1}e^{p\operatorname{Re}\rho}}{p\,|\rho|^p(\operatorname{Re}\rho)^{p+1}}\Big[(\|R\|+|\rho|\,\|S\|)^p(|c_0|+|\rho|\,|d_0|)^p\left(1-e^{(s-2)\operatorname{Re}\rho}\right)^p \times\left(1-e^{-ps\operatorname{Re}\rho}\right)+(|a_0|+|\rho|\,|b_0|)^p \times(\|R\|+|\rho|\,\|S\|)^p\left(1-e^{-ps\operatorname{Re}\rho}\right)\left(1-e^{(s-1)\operatorname{Re}\rho}\right)^p\Big]$$

as $\operatorname{Re}\rho>0$ we obtain

$$\sup_{0\le s\le 1}\left(\int_0^s|\varphi_2(x,s;\lambda)|^p\,dx\right)^{\frac{1}{p}} \le \frac{2^{1+\frac{2}{p}}e^{\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|c_0|)+|\rho|\,(|b_0|+|d_0|)\right]}{|\rho|\,(\operatorname{Re}\rho)^{1+\frac{1}{p}}\,p^{1/p}}. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$\|R(\lambda,L_p)\|\le\frac{2^{1+\frac{2}{p}}e^{\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|c_0|)+|\rho|\,(|b_0|+|d_0|)\right]}{|\triangle(\lambda)|\,|\rho|\,(\operatorname{Re}\rho)^{1+\frac{1}{p}}\,p^{1/p}}\left(2^{1+\frac{2}{p}}+1\right),$$

for $\rho\in\Sigma_{\frac{\delta}{2}}=\left\{\rho\in\mathbb{C}:|\arg\rho|\le\frac{\delta}{2},\rho\ne 0\right\}$, we have $(\operatorname{Re}\rho)^{-1}<\frac{1}{|\rho|\cos\left(\frac{\delta}{2}\right)}$, then

$$\|R(\lambda,L_p)\|\le\frac{2^{1+\frac{2}{p}}\left(2^{1+\frac{2}{p}}+1\right)e^{\operatorname{Re}\rho}(\|R\|+|\rho|\,\|S\|)\left[(|a_0|+|c_0|)+|\rho|\,(|b_0|+|d_0|)\right]}{|\triangle(\lambda)|\,|\rho|\left(|\rho|^{1+\frac{1}{p}}\right)\left(\cos\frac{\delta}{2}\right)^{1+\frac{1}{p}}p^{1/p}}.$$

Finally, we obtain for $\lambda=\rho^2\in\Sigma_\delta$

$$\|R(\lambda,L_p)\|_{L^p}\le\frac{c\,e^{\operatorname{Re}\rho}}{|\triangle(\lambda)|\,|\rho|^{1+\frac{1}{p}}}\left(\frac{\|R\|}{|\rho|}+\|S\|\right)\left[(|a_0|+|c_0|)+|\rho|\,(|b_0|+|d_0|)\right], \tag{3.7}$$

where

$$c=\frac{2^{1+\frac{2}{p}}\left(2^{1+\frac{2}{p}}+1\right)}{p^{1/p}\left(\cos\frac{\delta}{2}\right)^{1+\frac{1}{p}}}.$$

3.2. **Estimation of the Characteristic Determinant.** The next step is to determine the cases for which $|\Delta(\lambda)|$ remains bounded below. It will then be necessary to bound $|\Delta(\lambda)|$ appropriately. However, formula (2.5) is not useful for this purpose. It will be then necessary to make some additional regularity hypotheses on the functions R and S, and so we assume that the functions R and S are in $C^2([0,1],\mathbb{C})$.

Integrating twice by parts in (2.5), we obtain

$$\Delta(\lambda)=e^\rho\Big[\rho(d_0S(0)-b_0S(1))+(d_0S'(0)+b_0S'(1)+a_0S(1) -b_0R(1)-d_0R(0)+c_0S(0))+\frac{1}{\rho}(a_0R(1)-c_0R(0)+b_0R'(1) -a_0S'(1)-d_0R'(0)+c_0S'(0))+\frac{1}{\rho^2}(-a_0R'(1)-c_0R'(0))+\Phi(\rho)\Big], \tag{3.8}$$

where

$$\Phi(\rho) = \left[(a_0 - \rho b_0)e^{-\rho} + (c_0 - \rho d_0)e^{-2\rho}\right] \int_0^1 \left(\frac{R''(t)}{\rho^2} - \frac{S''(t)}{\rho}\right) e^{\rho t} dt$$
$$+ \left[(a_0 + \rho b_0)e^{-\rho} + (c_0 + \rho d_0)\right] \int_0^1 \left(\frac{R''(t)}{\rho^2} - \frac{S''(t)}{\rho}\right) e^{-\rho t} dt$$
$$+ e^{-\rho}\left[\frac{1}{\rho}\left(-b_0 R'(0) + d_0 R'(1) - c_0 S'(1) + a_0 S'(0) + c_0 R(1) - a_0 R(0)\right)\right.$$
$$\left. + \rho(b_0 S(0) - d_0 S(1))\right] + 2e^{-2\rho}\left[(a_0 + \rho b_0)(R(1) - \rho S(1))\right.$$
$$- (c_0 - \rho d_0)(R(0) + \rho S(0)) + \frac{1}{\rho^2}(a_0 + \rho b_0)(R'(1) - \rho S'(1))$$
$$\left. + \frac{1}{\rho^2}(c_0 - \rho d_0)(R'(0) + \rho S'(0))\right]. \tag{3.9}$$

After a straightforward calculation, we obtain the following inequality valid for $\rho \in \Sigma_{\frac{\delta}{2}}$,with $|\rho|$ sufficiently large,

$$|\Phi(\rho)| \le \left[(|a_0| + |\rho|\,|b_0|)\, e^{-\operatorname{Re}\rho} + (|c_0| + |\rho|\,|d_0|)\, e^{-2\operatorname{Re}\rho}\right] \left(\frac{\|R''\|}{|\rho|^2} + \frac{\|S''\|}{|\rho|}\right) \left(\frac{e^{\operatorname{Re}\rho} - 1}{\operatorname{Re}\rho}\right)$$
$$+ \left[(|a_0| + |\rho|\,|b_0|)\, e^{-\operatorname{Re}\rho} + (|c_0| + |\rho|\,|d_0|)\right] \left(\frac{\|R''\|}{|\rho|^2} + \frac{\|S''\|}{|\rho|}\right) \left(\frac{1 - e^{-\operatorname{Re}\rho}}{\operatorname{Re}\rho}\right)$$
$$+ 2e^{-\operatorname{Re}\rho}\left[\frac{1}{|\rho|}\left|-b_0 R'(0) + d_0 R'(1) - c_0 S'(1) + a_0 S'(0) + c_0 R(1) - a_0 R(0)\right|\right.$$
$$\left. + |\rho|\,|b_0 S(0) - d_0 S(1)|\right] + e^{-2\operatorname{Re}\rho}\left[\frac{1}{|\rho|}(|a_0| + |\rho|\,|b_0|)(\|R\| + |\rho|\,\|S\|)\right.$$
$$+ \frac{1}{|\rho|}(|c_0| + |\rho|\,|d_0|)(\|R\| + |\rho|\,\|S\|) + (|a_0| + |\rho|\,|b_0|)\left(\frac{\|R'\| + |\rho|\,\|S'\|}{|\rho|^2}\right)$$
$$\left. + (|c_0| + |\rho|\,|d_0|)\left(\frac{\|R'\| + |\rho|\,\|S'\|}{|\rho|^2}\right)\right].$$

Then

$$|\Phi(\rho)| \le \frac{2}{|\rho|\cos\frac{\delta}{2}}\left[\frac{(|a_0| + |c_0|)}{|\rho|} + (|b_0| + |d_0|)\left(\frac{\|R''\|}{|\rho|} + \|S''\|\right)\right]$$
$$+ \frac{1}{|\rho|\cos\frac{\delta}{2}}\left[\frac{1}{|\rho|^2}\left|-b_0 R'(0) + d_0 R'(1) - c_0 S'(1) + a_0 S'(0) + c_0 R(1) - a_0 R(0)\right|\right.$$
$$\left. + |b_0 S(0) - d_0 S(1)|\right] + \frac{1}{|\rho|\,(\cos\frac{\delta}{2})^2}\left[\left(\frac{|a_0|}{|\rho|} + |b_0|\right)\left(\frac{\|R\|}{|\rho|} + \|S\|\right)\right.$$
$$+ \left(\frac{|c_0|}{|\rho|} + |d_0|\right)\left(\frac{\|R\|}{|\rho|} + \|S\|\right) + \left(\frac{|a_0|}{|\rho|} + |b_0|\right)\left(\frac{\|R'\|}{|\rho|^2} + \frac{\|S'\|}{|\rho|}\right)$$
$$\left. + \left(\frac{|c_0|}{|\rho|} + |d_0|\right)\left(\frac{\|R'\|}{|\rho|^2} + \frac{\|S'\|}{|\rho|}\right)\right]. \tag{3.10}$$

Where we have used that $Re(\rho) \geq |\rho| \cos(\frac{\delta}{2})$, $(1 - e^{-\operatorname{Re}\rho}) < 1$,

$$e^{-\operatorname{Re}\rho} \leq \frac{2}{|\rho|^2 (\cos \frac{\delta}{2})^2} and e^{-2\operatorname{Re}\rho} \leq \frac{1}{2|\rho|^2 (\cos \frac{\delta}{2})^2}, e^{-\operatorname{Re}\rho} < 1.$$

There are several cases to analyze depending on the functions R and S.

- **Case 1.**

Suppose that $S \neq 0$, $d_0 \neq 0$, $b_0 \neq 0$, $d_0 S(0) - b_0 S(1) = 0$ and

$$k_1 = d_0 S'(0) + b_0 S'(1) + a_0 S(1) - b_0 R(1) - d_0 R(0) + c_0 S(0) \neq 0.$$

From (3.8), we have for $|\rho|$ sufficiently large

$$\begin{aligned} |\triangle(\lambda)| \geq e^{\operatorname{Re}\rho} \Big[&\left| d_0 S'(0) + b_0 S'(1) + a_0 S(1) - b_0 R(1) - d_0 R(0) + c_0 S(0) \right| \\ &- \frac{1}{|\rho|} \left| a_0 R(1) - c_0 R(0) + b_0 R'(1) - a_0 S'(1) - d_0 R'(0) + c_0 S'(0) \right| \\ &- \frac{1}{|\rho|^2} \left| a_0 R'(1) + c_0 R'(0) \right| - |\Phi(\rho)| \Big]. \end{aligned}$$

From (3.10) we deduce for $\rho \in \Sigma_{\frac{\delta}{2}}$, $|\rho| \geq r_0 > 0$.

$$|\Phi(\rho)| \leq \frac{c(r_0)}{|\rho|}.$$

Then, we have

$$\begin{aligned} |\triangle(\lambda)| \geq e^{\operatorname{Re}\rho} \Big[&\left| d_0 S'(0) + b_0 S'(1) + a_0 S(1) - b_0 R(1) - d_0 R(0) + c_0 S(0) \right| \\ &- \frac{1}{|\rho|} \left| a_0 R(1) - c_0 R(0) + b_0 R'(1) - a_0 S'(1) - d_0 R'(0) + c_0 S'(0) \right| \\ &- \frac{1}{|\rho|^2} \left| a_0 R'(1) + c_0 R'(0) \right| - \frac{c(r_0)}{|\rho|} \Big]. \end{aligned}$$

we can now choose $r_0 > 0$, such that

$$\begin{aligned} &\frac{1}{r_0} \left| a_0 R(1) - c_0 R(0) + b_0 R'(1) - a_0 S'(1) - d_0 R'(0) + c_0 S'(0) \right| \\ &\quad + \frac{1}{r_0^2} \left| a_0 R'(1) + c_0 R'(0) \right| + \frac{c(r_0)}{|\rho|} \\ &\qquad \leq \frac{1}{2} \left| d_0 S'(0) + b_0 S'(1) + a_0 S(1) - b_0 R(1) - d_0 R(0) + c_0 S(0) \right|, \end{aligned}$$

then, for $\rho \in \Sigma_{\frac{\delta}{2}}$, $|\rho| \geq r_0 > 0$, we get

$$|\triangle(\lambda)| \geq \frac{e^{\operatorname{Re}\rho}}{2} |k_1|.$$

From (3.7), we deduce the following bound, valid for every $|\arg \rho| \leq \frac{\delta}{2}$, $\rho \neq 0$

$$\|R(\lambda, L_p)\| \leq \frac{2c}{|\rho|^{\frac{1}{p}} |k_1|} \left(\left(\frac{|a_0| + |c_0|}{|\rho|} \right) + (|b_0| + |d_0|) \right) \left(\frac{\|R\|}{|\rho|} + \|S\| \right),$$

then

$$\|R(\lambda, L_p)\| \leq \frac{c_1}{|\lambda|^{\frac{1}{2p}}},$$

as $|\lambda| \longrightarrow +\infty$, where

$$c_1 = \frac{2c\ \|S\|\left(|b_0| + |d_0|\right)}{|k_1|}.$$

- **Case 2.**

If $b_0 = d_0 = 0$, $S \neq 0$ and $a_0 S\left(1\right) + c_0 S\left(0\right) = 0$, with

$$k_2 = a_0 R\left(1\right) - c_0 R\left(0\right) - a_0 S^{'}\left(1\right) + c_0 S^{'}\left(0\right) \neq 0,$$

we have the following bound, valid for $\lambda \in \Sigma_\delta$ and $|\lambda|$ big enough,

$$\|R\left(\lambda, L_p\right)\| \leq \frac{2c\left(|a_0| + |c_0|\right)}{|\rho|^{\frac{1}{p}}\,|k_2|}\left(\frac{\|R\|}{|\rho|} + \|S\|\right),$$

then

$$\|R\left(\lambda, L_p\right)\| \leq \frac{c_1}{|\lambda|^{\frac{1}{2p}}},$$

as $|\lambda| \longrightarrow +\infty$, where

$$c_1 = \frac{2c\ \left(|a_0| + |c_0|\right)\|S\|}{|k_2|}.$$

- **Case 3.**

If $S = 0$, $b_0 \neq 0$, $d_0 \neq 0$ and $b_0 R(1) + d_0 R\left(0\right) = 0$, with

$$k_3 = a_0 R\left(1\right) - c_0 R\left(0\right) + b_0 R^{'}\left(1\right) - d_0 R^{'}\left(0\right) \neq 0.$$

Similarly, we get

$$\|R\left(\lambda, L_p\right)\| \leq \frac{2c\,\|R\|}{|\rho|^{1+\frac{1}{p}}\,|k_3|}\left(\frac{\left(|a_0| + |c_0|\right)}{|\rho|} + \left(|b_0| + |d_0|\right)\right),$$

then, we have

$$\|R\left(\lambda, L_p\right)\| \leq \frac{c_1}{|\lambda|^{\frac{1}{2p}}},$$

as $|\lambda| \longrightarrow +\infty$, where

$$c_1 = \frac{2c\,\|R\|\left(|b_0| + |d_0|\right)}{|k_3|}.$$

- **Case 4.**

If $b_0 = d_0 = 0$, $S = 0$ and $a_0 R\left(1\right) - c_0 R\left(0\right) = 0$ with

$$k_4 = a_0 R^{'}\left(1\right) - c_0 R^{'}\left(0\right) \neq 0.$$

Again in this case, we have

$$\|R\left(\lambda, L_p\right)\| \leq \frac{2c\left(|a_0| + |c_0|\right)\|R\|}{|\rho|^{\frac{1}{p}}\,|k_4|},$$

then

$$\|R\left(\lambda, L_p\right)\| \leq \frac{c_1}{|\lambda|^{\frac{1}{2p}}},$$

as $|\lambda| \longrightarrow +\infty$, where

$$c_1 = \frac{c\ \|R\|\left(|a_0| + |c_0|\right)}{|k_4|}.$$

Definition 3.1. The boundary value conditions in (1.1) are called non regular if the functions $R, S \in C^2([0,1], \mathbb{C})$ and if and only if one of the following conditions holds
1-$d_0 S(0) - b_0(1) = 0$, $b_0 \neq 0, d_0 \neq 0$, $S \neq 0$ and

$$d_0 S^{'}(0) + b_0 S^{'}(1) + a_0 S(1) - b_0 R(1) - d_0 R(0) + c_0 S(0) \neq 0$$

2- $b_0 = d_0 = 0$, $\|S\| \neq 0$, $a_0 S(1) + c_0 S(0) = 0$ with

$$a_0 R(1) - c_0 R(0) - a_0 S^{'}(1) + c_0 S^{'}(0) \neq 0$$

3-$S = 0$, $b_0 \neq 0$, $d_0 \neq 0$, $b_0 R(1) +_0 R(0) \neq 0$ with

$$a_0 R(1) - c_0 R(0) + b_0 R^{'}(1) - d_0 R^{'}(0) \neq 0$$

4-$b_0 = d_0 = 0$, $S = 0$, $a_0 R(1) - c_0 R(0) = 0$ with

$$a_0 R^{'}(1) - c_0 R^{'}(0) \neq 0.$$

This proves the following theorem

Theorem 3.1. *If the boundary value conditions in (1.1) are non regular, then $\Sigma_\delta \subset \rho(L_p)$ for sufficiently large $|\lambda|$ and there exists $c > 0$ such that*

$$\|R(\lambda, L_p)\| \leq \frac{c}{|\lambda|^{\frac{1}{2p}}}.$$

Remark 3.2. From Theorem 3.1 it follows that the operator L_p, for $p \neq \infty$, generates an analytic semigroup with singularities [25] of type $A(2p-1,\ 4p-1)$.

3.3. **Application.** In the following, we apply the above obtained results to the study of a class of a mixed problem for a parabolic equation with an weighted integral boundary condition combined with another two point boundary condition of the form

$$\text{(3.11)} \qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} - a\dfrac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) \\ L_1(u) = a_0 u(0,t) + b_0 u^{'}(0,t) + c_0 u(1,t) + d_0 u^{'}(1,t) = 0, \\ L_2(u) = \int_0^1 R(\xi)\, u(t,\xi)\, d\xi + \int_0^1 S(\xi)\, u^{'}(t,\xi)\, d\xi = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where $(t,\xi) \in [0,T] \times [0,1]$. Boundary value problems for parabolic equations with integral boundary conditions are studied by [1, 3, 5, 6, 14, 15, 16, 17, 35] using various methods. For instance, the potential method in [3] and [17], Fourier method in [1, 14, 15, 16] and the energy inequalities method has been used in [5, 6, 35]. In our case, we apply the method of operator differential equation. The study of the problem is then reduced to a cauchy problem for a parabolic abstract differential equation, where the operator coefficients has been previously studied. For this purpose, let E, E_1, and E_2 be Banach spaces. Introduce two Banach spaces

$$C_\mu((0,T], E) = \left\{ f/f \in C((0,T], E),\ \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| < \infty \right\},\ \mu \geq 0,$$

$$C_\mu^\gamma((0,T], E) = \left\{ f/f \in C((0,T], E),\ \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| \right. \\ \left. + \sup_{0<t<t+h\leq T} \|f(t+h) - f(t)\| h^{-\gamma} t^\mu < \infty \right\},\ \mu \geq 0,\ \gamma \in (0,1],$$

and a linear space

$$C^1\left((0,T],E_1,\ E_2\right)=\left\{f/f\in C\left((0,T],E_1\right)\cap C^1\left((0,T],E_2\right)\right\},\ E_1\subset E_2,$$

where $C\left((0,T],E\right)$ and $C^1\left((0,T],E\right)$ are spaces of continuous and continuously differentiable, respectively, vector-function from $(0,T]$ into E. We denote, for a linear operator A in a Banach space E, by

$$E(A)=\left\{u/u\in D(A),\ \|u\|_{E(A)}=\left(\|Au\|^2+\|u\|^2\right)^{\frac{1}{2}}\right\},$$

and

$$C^1\left((0,T],E(A),E\right)=\left\{f/f\in C\left((0,T],E(A)\right),\ f'\in C\left((0,T],E\right)\right\}.$$

Let us derive a theorem which was proved by various methods in [25] and [31, 33]. Consider, in a Banach space E, the Cauchy problem

$$\begin{cases} u'(t)=Au(t)+f(t),\ t\in[0,T], \\ u(0)=u_0, \end{cases} \tag{3.12}$$

where A is, generally speaking, unbounded linear operator in E, u_0 is a given element of E, $f(t)$ is a given vector-function and $u(t)$ is an unknown vector-function in E.

Theorem 3.3. *Let the following conditions be satisfied:*

(1) *A is a closed linear operator in a Banach space E and for some $\beta\in(0,1]$, $\alpha>0$*

$$\|R(\lambda,A)\|\le C|\lambda|^{-\beta},\qquad |\arg\lambda|\le\frac{\pi}{2}+\alpha,\ |\lambda|\to\infty;$$

(2) *$f\in C_\mu^\gamma\left((0,T],E\right)$ for some $\gamma\in(1-\beta,1]$, $\mu\in[0,\beta)$;*

(3) *$u_0\in D(A)$.*

Then the Cauchy problem (3.12) has a unique solution

$$u\in C\left([0,T],E\right)\cap C^1\left((0,T],E(A),E\right)$$

and for the solution u the following estimates hold

$$\|u(t)\|\le C\left(\|Au_0\|+\|u_0\|+\|f\|_{C_\mu((0,t],E)}\right),\ t\in(0,T],$$

$$\left\|u'(t)\right\|+\|Au(t)\|\le C\left(t^{\beta-1}\left(\|Au_0\|+\|u_0\|\right)+t^{\beta-\mu-1}\|f\|_{C_\mu^\gamma((0,t],E)}\right),\ t\in(0,T].$$

As a result of this we get the following theorem

Theorem 3.4. *Let the following conditions be satisfied*

(1) $a\neq0$, $|\arg a|<\dfrac{\pi}{2}$,

(2) *the functions $R(t)$, $S(t)\in C^2\left([0,1],\mathbb{C}\right)$ and one of the following conditions is satisfied* $d_0S(0)-b_0S(1)=0$, $b_0\neq0$, $d_0\neq0$, *and* $S\neq0$

$$d_0S'(0)+b_0S'(1)+a_0S(1)-b_0R(1)-d_0R(0)+c_0S(0)\neq0,$$

or $b_0=0$, $d_0=0$, $S\neq0$, $a_0S(1)+c_0S(0)=0$ *with*

$$a_0R(1)-c_0R(0)-a_0S'(1)+c_0S'(0)\neq0,$$

or $b_0\neq0$, $d_0\neq0$, $S=0$, $b_0R(1)+d_0R(0)=0$ *with*

$$a_0R(1)-c_0R(0)+b_0R'(1)-d_0R'(0)\neq0,$$

or $b_0 = d_0 = 0$, $S = 0$, $a_0 R(1) - c_0 R(0) = 0$ *with*

$$a_0 R'(1) - c_0 R'(0) \neq 0.$$

(3) $f \in C^{\gamma}_{\mu}((0,T], L^q(0,1))$ *for some* $\gamma \in \left(1 - \frac{1}{2q}, 1\right]$ *and some* $\mu \in \left[0, \frac{1}{2q}\right)$,

(4) $u_0 \in W^2_q\left((0,1), L_i u = 0, i = \overline{1,2}\right)$.

Then the problem (3.11) has a unique solution

$$u \in C((0,T], L^q(0,1)) \cap C^1\left((0,T], W^2_q(0,1), L^q(0,1)\right)$$

and for this solution we have the estimates:

$$\|u(t,\cdot)\|_{L^q(0,1)} \leq c\left(\|u_0\|_{W^2_q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}\right), \quad t \in (0,T], \tag{3.13}$$

$$\begin{aligned} &\|u''(t,\cdot)\|_{L^q(0,1)} + \|u'(t,\cdot)\|_{L^q(0,1)} \\ &\qquad \leq c\left(t^{\frac{1}{2q}-1}\|u_0\|_{W^2_q(0,1)} + t^{\frac{1}{2q}-1-\mu}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}\right), \quad t \in (0,T]. \end{aligned} \tag{3.14}$$

Proof. We consider in the space $L^q(0,1)$ $1 \leq q < \infty$, the operator A defined by

$$A(u) = au''(x), D(A) = \left\{u \in W^2_q(0,1),\ L_i(u) = 0, i = \overline{1,2}\right\}.$$

Then problem (3.11) can be written as

$$\begin{cases} u'(t) = Au(t) + f(t), \\ u(0) = u_0, \end{cases}$$

where $u(t) = u(t,\cdot)$, $f(t) = f(t,\cdot)$, and $u_0 = u_0(\cdot)$ are functions with values in the Banach space $L^q(0,1)$. From Theorem 3.1 we conclude that $\|R(\lambda, A)\| \leq c|\lambda|^{-\frac{1}{2q}}$, for $|\arg\lambda| \leq \frac{\pi}{2} + \alpha$, as $|\lambda| \to \infty$.

Then, from Theorem 3.3 the problem (3.11) has a unique solution

$$u \in C((0,T], L^q(0,1)) \cap C^1\left((0,T], W^2_q(0,1), L^q(0,1)\right)$$

and we have the following estimates

$$\|u(t,.)\|_{L^q(0,1)} \leq c\left(\|Au_0\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}\right), \tag{3.15}$$

$$\begin{aligned} &\left\|u'(t)\right\|_{L^q(0,1)} + \|Au(t,\cdot)\|_{L^q(0,1)} \\ &\qquad \leq c\left(t^{\frac{1}{2q}-1}\left(\|Au_0\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)}\right) + t^{\frac{1}{2q}-1-\mu}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}\right), \end{aligned} \tag{3.16}$$

where $t \in [0,T]$, from (3.15) we get

$$\begin{aligned} \|u(t,\cdot)\|_{L_q(0,1)} &\leq c\left(\|u_0''\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}\right) \\ &\leq c\left(\|u_0\|_{W^2_q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}\right), \quad t \in [0,T]. \end{aligned}$$

and from (3.16) we get

$$\begin{aligned} &\left\|u'(t)\right\|_{L^q(0,1)} + \|u''(t,\cdot)\|_{L^q(0,1)} \\ &\quad \leq c\left(t^{\frac{1}{2q}-1}\left(\|Au_0\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)}\right) + t^{\frac{1}{2q}-1-\mu}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}\right) \\ &\quad \leq c\left(t^{\frac{1}{2q}-1}\|u_0\|_{W^2_q(0,1)} + t^{\frac{1}{2q}-1-\mu}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}\right), \quad t \in [0,T]. \end{aligned}$$

which gives the desired result. □

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