Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 4, Article 120, 2005

MULTIVARIATE VERSION OF A JENSEN-TYPE INEQUALITY

ROBERT A. AGNEW

DEERFIELD, IL 60015-3007, USA
raagnew@aol.com
URL: http://members.aol.com/raagnew

Received 12 May, 2005; accepted 22 September, 2005
Communicated by C.E.M. Pearce

ABSTRACT. A univariate Jensen-type inequality is generalized to a multivariate setting.

Key words and phrases: Convex functions, Tchebycheff methods, Jensen's inequality.

2000 *Mathematics Subject Classification.* Primary 26D15.

1. INTRODUCTION

The following theorem was proved in [1], using Tchebycheff methods [4], [5], to extend a result obtained in [2] for the Laplace transform. It was later reproved in [3], [6], [7] using Jensen's inequality.

Theorem 1.1. *Let X be a nonnegative random variable with $E(X) = \mu > 0$ and $E(X^2) = \lambda < \infty$. Suppose that $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$ and $g(x) = f(x)/x$ convex on $(0, \infty)$. Then, $E(f(X)) \geq \mu g(\lambda/\mu) = (\mu^2/\lambda)\ f(\lambda/\mu)$ and the bound is sharp.*

We next provide a natural multivariate generalization of Theorem 1.1, using the same approach as [1], followed by examples to illustrate its application.

2. MAIN RESULT

Let $S = (0, \infty)^n$ and let $g_1, ..., g_n$ be real-valued functions on S. For any column vector $x = (x_1, \dots, x_n)^T \in S$, let $f(x) = \sum_{i=1}^n x_i g_i(x)$ and let e_i denote the i^{th} unit column vector in $\mathbb{R}^n$.

Theorem 2.1. *Let $g_1, ..., g_n$ be convex on S , and let $X = (X_1, \dots, X_n)^T$ be a random column vector in S with $E(X) = \mu = (\mu_1, \dots, \mu_n)^T$ and $E(XX^T) = \Sigma + \mu\mu^T$ for covariance matrix Σ. Then,*

$$E(f(X)) \geq \sum_{i=1}^{n} \mu_i\, g_i\left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right) \tag{2.1}$$

ISSN (electronic): 1443-5756

147-05

and the bound is sharp.

Proof. By convexity, for any $\xi_i \in S$, there exists a $b_i(\xi_i) \in \mathbb{R}^n$ such that

$$g_i(x) \geq g_i(\xi_i) + b_i(\xi_i)^T (x - \xi_i) \tag{2.2}$$

for all $x \in S$, i.e., there exists a supporting hyperplane at ξ_i. Hence,

$$\begin{aligned} E(f(X)) &= \sum_{i=1}^{n} E(X_i g_i(X)) \\ &\geq \sum_{i=1}^{n} E\left(X_i \left(g_i(\xi_i) + b_i(\xi_i)^T (X - \xi_i)\right)\right) \\ &\geq \sum_{i=1}^{n} \mu_i \left(g_i(\xi_i) + b_i(\xi_i)^T \left(E\left(\frac{XX_i}{\mu_i}\right) - \xi_i\right)\right) \end{aligned} \tag{2.3}$$

But

$$E(XX_i) = E\left(XX^T e_i\right) = E\left(XX^T\right) e_i = \Sigma e_i + \mu\mu_i.$$

Then, (2.2) and (2.3) together imply that

$$\xi_i = E\left(\frac{XX_i}{\mu_i}\right) = \frac{\Sigma e_i}{\mu_i} + \mu$$

yields the maximum bound which is obviously attained when X is concentrated at μ. □

Theorem 2.1 is a true multivariate extension as the following examples illustrate. As indicated in [2] for the Laplace transform, certain extensions are only nominally multivariate and fall within the domain of Theorem 1.1 because the random variables are combined in a univariate linear combination.

3. EXAMPLES

Example 3.1. Let $g_i(x) = \alpha_i + \beta_i^T x$ be linear with $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^n$. Then

$$f(x) = \sum_{i=1}^{n} x_i\, g_i(x) = \sum_{i=1}^{n} x_i \left(\alpha_i + \beta_i^T x\right)$$

is a general quadratic function which can also be written as $f(x) = \alpha^T x + x^T B x$ where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $B = [\beta_1, \ldots, \beta_n]^T$. Then we have

$$\begin{aligned} E(f(X)) &= E\left(\sum_{i=1}^{n} X_i \left(\alpha_i + \beta_i^T X\right)\right) \\ &= \sum_{i=1}^{n} \left(\alpha_i\, \mu_i + \beta_i^T (\Sigma\, e_i + \mu\mu_i)\right) \\ &= \sum_{i=1}^{n} \mu_i \left(\alpha_i + \beta_i^T \left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right)\right) \\ &= \alpha^T \mu + \mu^T B\, \mu + tr\ (B\,\Sigma) \end{aligned}$$

so the Theorem 2.1 bound is, not surprisingly, exact in this general quadratic case.

Example 3.2. Let $g_i(x) = \rho_i \prod_{j=1}^{n} x_j^{-\gamma_{ij}}$ with $\rho_i > 0$ and $\gamma_{ij} > 0$. Here, the g_i might represent Cournot-type price functions (inverse demand functions) for quasi-substitutable products where x_i is the supply of product i and $g_i(x_1, \ldots, x_n)$ is the equilibrium price of product i, given its supply and the supplies of its alternates. Then, $x_i\, g_i(x)$ represents the revenue from product i and $f(x) = \sum_{i=1}^{n} x_i\, g_i(x)$ represents total market revenue for the ensemble of products. In this context, we would normally expect $\gamma_{ij} \in (0, 1)$ for viable products. Then, with probabilistic supplies, we have

$$E(f(X)) \geq \sum_{i=1}^{n} \mu_i\, g_i\left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right) = \sum_{i=1}^{n} \mu_i\, \rho_i \prod_{j=1}^{n} \left(\frac{\sigma_{ij}}{\mu_i} + \mu_j\right)^{-\gamma_{ij}}$$

where σ_{ij} is the ij^{th} element of Σ. This example demonstrates that Theorem 2.1 has an interesting application in economic oligopoly theory.

In Example 3.2, $g_i(x) = e^{h_i(x)}$ where

$$h_i(x) = ln\, \rho_i - \sum_{j=1}^{n} \gamma_{ij} \ln x_j$$

is convex on S . In general, if $k : \mathbb{R} \to \mathbb{R}$ is convex nondecreasing and $h : S \to \mathbb{R}$ is convex, then $g(x) = k(h(x))$ is convex on S since

$$\begin{aligned} k\left(h\left(\lambda\, x^{(1)} + (1-\lambda)\; x^{(2)}\right)\right) &\leq k\left(\lambda\, h\left(x^{(1)}\right) + (1-\lambda)\; h\left(x^{(2)}\right)\right) \\ &\leq \lambda\, k\left(h\left(x^{(1)}\right)\right) + (1-\lambda)\; k\left(h\left(x^{(2)}\right)\right) \end{aligned}$$

for any $x^{(1)}$, $x^{(2)} \in S$ and $\lambda \in [0, 1]$. Other examples satisfying Theorem 2.1 can be generated by composing the linear functions of Example 3.1 with convex nondecreasing functions like $k(u) = e^u$, $k(u) = u + \sqrt{u^2 + 1} = e^{\sinh^{-1} u}$, or $k(u) = \max(0, u)$.

REFERENCES

[1] R.A. AGNEW, Inequalities with application in economic risk analysis, *J. Appl. Prob.*, **9** (1972), 441–444.

[2] D. BROOK, Bounds for moment generating functions and for extinction probabilities, *J. Appl. Prob.*, **3** (1966), 171–178.

[3] B. GULJAŠ, C.E.M. PEARCE AND J. PEČARIĆ, Jensen's inequality for distributions possessing higher moments, with applications to sharp bounds for Laplace-Stieltjes transforms, *J. Austral. Math. Soc. Ser. B*, **40** (1998), 80–85.

[4] S. KARLIN AND W.J. STUDDEN, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Wiley Interscience, 1966.

[5] J.F.C. KINGMAN, On inequalities of the Tchebychev type, *Proc. Camb. Phil. Soc.*, **59** (1963), 135–146.

[6] C.E.M. PEARCE AND J.E. PEČARIĆ, An integral inequality for convex functions, with application to teletraffic congestion problems, *Math. of Opns. Res.*, **20** (1995), 526–528.

[7] A.O. PITTENGER, Sharp mean-variance bounds for Jensen-type inequalities, *Stat. & Prob. Letters*, **10** (1990), 91–94.