ON THE RATE OF CONVERGENCE OF SOME ORTHOGONAL POLYNOMIAL EXPANSIONS

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Abstract: In this paper we estimate the rate of pointwise convergence of certain orthogonal expansions for measurable and bounded functions.

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Contents

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1. Introduction

Let H_n be the class of all polynomials of degree not exceeding n and let w be a weight function defined on $I = [-1, 1]$, i.e. $w(t) \geq 0$ for all $t \in I$ and

$$\int_{-1}^{1} |t|^k w(t)dt < \infty \qquad \text{for} \qquad k = 0, 1, 2, \dots$$

Then there is a unique system $\{p_n\}$ of polynomials such that $p_n \in H_n$, $p_n \equiv p_n(w; x) = \gamma_n x^n$ + lower degree terms, where $\gamma_n > 0$ and

$$\int_{-1}^{1} p_n(t)p_m(t)w(t)dt = \delta_{n,m}$$

(see [9, Chap. II]). If fw is integrable on I, then by $S_n[f](w; x)$ we denote the n-th partial sum of the Fourier series of the function f with respect to the system $\{p_n\}$, i.e.

$$S_n[f](w; x) := \sum_{k=0}^{n-1} a_k p_k(x) = \int_{-1}^{1} f(t)K_n(x, t)w(t)dt,$$

where

$$a_k := \int_{-1}^{1} f(t)p_k(t)w(t)dt, \qquad k = 0, 1, 2, \dots$$

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)p_k(t), \qquad n = 1, 2, \dots \tag{1.1}$$

In 1985 (see [6, p. 485]) R. Bojanic proved the following

Orthogonal Polynomial Expansions
Małgorzata Powierska
vol. 8, iss. 3, art. 11, 2007

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Theorem 1.1. *Let w be a weight function and suppose that for all $x \in (-1,1)$ and $n = 1, 2, \dots$*

$$0 < w(x) \le K(1-x^2)^{-A}, \tag{1.2}$$

$$|p_n(x)| \le K(1-x^2)^{-B}, \tag{1.3}$$

$$\left| \int_{-1}^{x} w(t) p_n(t) dt \right| \le \frac{C}{n}, \tag{1.4}$$

where A, B, C, K are some non-negative constants. If f is a function of bounded variation in the Jordan sense on I, then

$$\left| S_n[f](w;x) - \frac{1}{2}\left(f(x+) + f(x-)\right) \right| \le \frac{\varphi(x)}{n} \sum_{k=1}^{n} V\left(g_x; x - \frac{1+x}{k}, x + \frac{1-x}{k}\right) + \frac{1}{2}|f(x-) - f(x+)|\,|S_n[\psi_x](w;x)|,$$

where $f(x+), f(x-)$ denote the one-sided limits of f at the point x, the function g_x is given by

$$g_x(t) := \begin{cases} f(t) - f(x-) & \textit{if } -1 \le t < x, \\ 0 & \textit{if } t = x, \\ f(t) - f(x+) & \textit{if } x < t \le 1 \end{cases} \tag{1.5}$$

and

$$\psi_x(t) := \operatorname{sgn}_x(t) = \begin{cases} 1 & \textit{if } t > x, \\ 0 & \textit{if } t = x, \\ -1 & \textit{if } t < x. \end{cases} \tag{1.6}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Moreover, $\varphi(x) > 0$ *for* $x \in (-1,1)$ *and* $V(g_x;a,b)$ *is the total variation of* g_x *on* $[a,b]$.

In this paper, we extend this Bojanic result to the case of measurable and bounded functions f on I (in symbols $f \in M(I)$). We will estimate the rate of convergence of $S_n[f](w;x)$ at those points $x \in I$ at which f possesses finite one-sided limits $f(x+), f(x-)$. In our main estimate we use the modulus of variation $v_n(g_x;a,b)$ of the function g_x on some intervals $[a,b] \subset I$. For positive integers n, the modulus of variation of a function g on $[a,b]$ is defined by

$$\nu_n(g;a,b) := \sup_{\pi_n} \sum_{k=0}^{n-1} |g(x_{2k+1}) - g(x_{2k})|,$$

where the supremum is taken over all systems π_n of n non-overlapping open intervals $(x_{2k}, x_{2k+1}) \subset (a,b)$, $k = 0,1,\dots,n-1$ (see [2]). In particular, we obtain estimates for the deviation $\left|S_n[f](w;x) - \frac{1}{2}\left(f(x+) + f(x-)\right)\right|$ for functions $f \in BV_\Phi(I)$. We will say that a function f, defined on the interval I belongs to the class $BV_\Phi(I)$, if

$$V_\Phi(f;I) := \sup_{\pi} \sum_{k} \Phi\left(|f(x_k) - f(t_k)|\right) < \infty,$$

where the supremum is taken over all finite systems π of non-overlapping intervals $(x_k, t_k) \subset I$. It will be assumed that Φ is a continuous, convex and strictly increasing function on the interval $[0,\infty)$, such that $\Phi(0) = 0$. The symbol $V_\Phi(f;a,b)$ will denote the total Φ-variation of f on the interval $[a,b] \subset I$. In the special case, if $\Phi(u) = u^p$ for $u \geq 0$ $(p \geq 1)$, we will write $BV_p(I)$ instead of $BV_\Phi(I)$, and $V_p(f;a,b)$ instead of $V_\Phi(f;a,b)$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Lemmas

In this section we first mention some results which are necessary for proving the main theorem.

Lemma 2.1. *Under the assumptions (1.2), (1.3) and (1.4), we have for* $n \geq 2$

$$\left|\int_{-1}^{s} K_n(x,t)w(t)dt\right| \leq \frac{4CK}{n-1}\,\frac{(1-x^2)^{-B}}{x-s} \qquad (-1 \leq s < x < 1), \tag{2.1}$$

$$\left|\int_{s}^{1} K_n(x,t)w(t)dt\right| \leq \frac{4CK}{n-1}\,\frac{(1-x^2)^{-B}}{s-x} \qquad (-1 < x < s \leq 1), \tag{2.2}$$

$$\int_{x-\frac{1+x}{n}}^{x} |K_n(x,t)w(t)|\,dt \leq 2^{A+B}K^3\frac{1+x}{(1-x^2)^{A+2B}} \qquad (-1 < x < 1), \tag{2.3}$$

$$\int_{x}^{x+\frac{1-x}{n}} |K_n(x,t)w(t)|\,dt \leq 2^{A+B}K^3\frac{1-x}{(1-x^2)^{A+2B}} \qquad (-1 < x < 1), \tag{2.4}$$

$$|K_n(x,t)w(t)| \leq \frac{2K^3}{|x-t|}\,\frac{1}{(1-x^2)^{B}(1-t^2)^{B+A}} \qquad \text{if} \quad x \neq t,\ -1 < x < 1,\ -1 < t < 1. \tag{2.5}$$

Proof. In order to prove (2.1), let us observe that by the Christoffel-Darboux formula ([3, p. 26] or [9, p. 42]) we have

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n}\,\frac{p_{n-1}(t)p_n(x) - p_{n-1}(x)p_n(t)}{x-t}. \tag{2.6}$$

vol. 8, iss. 3, art. 11, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Using the mean-value theorem and (1.3), we get for $-1 \leq s < x < 1$,

$$\left|\int_{-1}^{s} K_n(x,t)w(t)dt\right| \leq \frac{\gamma_{n-1}}{\gamma_n} \cdot \frac{K(1-x^2)^{-B}}{x-s} \left\{\left|\int_{\varepsilon}^{s} p_{n-1}(t)w(t)dt\right| + \left|\int_{\eta}^{s} p_n(t)w(t)dr\right|\right\},$$

where $\varepsilon, \eta \in [-1, s]$. From the inequality $\frac{\gamma_{n-1}}{\gamma_n} \leq 1$ (see [6, p. 488]) and from the assumption (1.4) our estimate (2.1) follows immediately.

The proof of (2.2) is similar.

In view of (1.1) and the assumptions (1.2), (1.3), we have

$$\begin{aligned}\int_{x-\frac{1+x}{n}}^{x} |K_n(x,t)w(t)|\, dt &\leq \frac{nK^3}{(1-x^2)^B} \int_{x-\frac{1+x}{n}}^{x} \frac{dt}{(1-t^2)^{A+B}} \\ &\leq 2^{A+B}K^3 \frac{1+x}{(1-x^2)^{A+2B}}.\end{aligned}$$

In the same way, we get (2.4).

Applying identity (2.6), assumptions (1.2) and (1.3), we can easily prove (2.5). □

Lemma 2.2. *Suppose that $g \in M(I)$ is equal to zero at a fixed point $x \in (-1,1)$ and that assumptions (1.2), (1.3), (1.4) are satisfied with A, B such that $A+B < 1$. Then for $n \geq 3$*

$$\left|\int_{x}^{1} g(t)K_n(x,t)w(t)dt\right| \leq \frac{c_1}{(1-x^2)^{A+2B} n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{\nu_j(g; t_{n-j}, 1)}{j^{1+A+B}} \tag{2.7}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$+\frac{c_2}{(1-x^2)^{1+B}}\left\{\sum_{j=1}^{n-1}\frac{\nu_j(g;x,t_j)}{j^2}+\frac{\nu_{n-1}(g;x,1)}{n-1}\right\},$$

where $t_j = x + j(1-x)/n$ $(j = 1, 2, \ldots, n)$, $c_1 = 8K^3/(1-A-B)$, $c_2 = 8K(3K^2 + 2C)$.

Proof. Observe that

$$\int_x^1 g(t)K_n(x,t)w(t)dt \tag{2.8}$$

$$= \int_x^{t_1} g(t)K_n(x,t)w(t)dt + \sum_{j=1}^{n-1} g(t_j)\int_{t_j}^{t_{j+1}} K_n(x,t)w(t)dt$$

$$+ \int_{t_{n-1}}^{1} (g(t)-g(t_{n-1}))K_n(x,t)w(t)dt$$

$$+ \sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} (g(t)-g(t_j))K_n(x,t)w(t)dt$$

$$= I_1 + I_2 + I_3 + I_4, \quad \text{say.}$$

In view of (2.4),

$$|I_1| \le \int_x^{t_1} |g(t)-g(x)|\,|K_n(x,t)w(t)|dt \le \frac{2K^3(1-x)}{(1-x^2)^{A+2B}}\nu_1(g;x,t_1). \tag{2.9}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Applying the Abel transformation we get

$$I_2 = g(t_1)\sum_{k=1}^{n-1}\int_{t_k}^{t_{k+1}} K_n(x,t)w(t)dt + \sum_{j=1}^{n-2}(g(t_{j+1})-g(t_j))\sum_{k=j+1}^{n-1}\int_{t_k}^{t_{k+1}} K_n(x,t)w(t)dt$$
$$= (g(t_1)-g(x))\int_{t_1}^{1} K_n(x,t)w(t)dt + \sum_{j=1}^{n-2}(g(t_{j+1})-g(t_j))\int_{t_{j+1}}^{1} K_n(x,t)w(t)dt.$$

Next, using the inequality (2.2) and once more the Abel transformation we obtain

$$|I_2| \le \frac{4CK}{(n-1)(1-x^2)^B}\left(\frac{|g(t_1)-g(x)|}{t_1-x} + \sum_{j=1}^{n-2}|g(t_{j+1})-g(t_j)|\frac{1}{(t_{j+1}-x)}\right)$$
$$\le \frac{4CKn}{(n-1)(1-x^2)^B(1-x)}\left\{|g(t_1)-g(x)| + \sum_{j=1}^{n-2}\frac{1}{(j+1)(j+2)}\sum_{k=1}^{j}|g(t_{k+1}-g(t_k)| + \frac{1}{n-1}\sum_{k=1}^{n-3}|g(t_{k+1})-g(t_k)|\right\}.$$

Hence, in view of the definition of the modulus of variation and its elementary properties,

$$|I_2| \le \frac{8CK}{(1-x)(1-x^2)^B}\left(\sum_{k=1}^{n-1}\frac{\nu_k(g;x,t_k)}{k^2} + \frac{\nu_{n-1}(g;x,1)}{n-1}\right) \tag{2.10}$$

(see the proof of Lemma 1 in [8]).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Next, by inequality (2.5),

$$|I_3| \le \frac{2K^3}{(1-x^2)^B}\nu_1(g;t_{n-1},1)\int_{t_{n-1}}^1 \frac{dt}{(t-x)(1-t^2)^{A+B}} \tag{2.11}$$
$$\le \frac{4K^3\nu_1(g;t_{n-1},1)}{(1-x^2)^B(1-x)(1+x)^{A+B}}\int_{t_{n-1}}^1 \frac{dt}{(1-t)^{A+B}}$$
$$= \frac{4K^3\nu_1(g;t_{n-1},1)}{(1-x^2)^{A+2B}n^{1-(A+B)}(1-(A+B))}$$

and

$$|I_4| \le \frac{2K^3}{(1-x^2)^B}\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \frac{|g(t)-g(t_j)|}{(t_j-x)(1-t_{j+1})^{A+B}(1+t_j)^{A+B}}dt$$
$$\le \frac{2K^3n^{1+A+B}}{(1-x^2)^{A+2B}(1-x)}\sum_{j=1}^{n-2}\int_{t_j}^{t_{j+1}} \frac{|g(t)-g(t_j)|}{j(n-j-1)^{A+B}}dt$$
$$= \frac{2K^3n^{1+A+B}}{(1-x^2)^{A+2B}(1-x)}\sum_{j=1}^{n-2}\int_0^h \frac{|g(s+t_j)-g(t_j)|}{j(n-j-1)^{A+B}}dt$$
$$= \frac{2K^3n^{1+A+B}}{(1-x^2)^{A+2B}(1-x)}\int_0^h \left\{\sum_{j=1}^{m}\frac{|g(s+t_j)-g(t_j)|}{j(n-j-1)^{A+B}} + \sum_{j=m+1}^{n-2}\frac{|g(s+t_j)-g(t_j)|}{j(n-j-1)^{A+B}}\right\}ds,$$

where $h=(1-x)/n$ and $m=[n/2]$. Next, arguing similarly to the proof of the

journal of inequalities in pure and applied mathematics
issn: 1443-5756

lemma in [7] (the estimate of I_4) we obtain

$$(2.12)\quad |I_4| \le \frac{2K^3}{(1-x^2)^{A+2B}} \left\{ 2\cdot 6^{A+B} \sum_{j=2}^{n-1} \frac{\nu_j(g;x,t_j)}{j^2} + \frac{6^{A+B}\nu_{n-1}(g;x,1)}{n-1} + \frac{4}{n^{1-(A+B)}} \sum_{j=2}^{n-1} \frac{\nu_j(g;t_{n-j},1)}{j^{1+A+B}} + 2\frac{\nu_{n-1}(g;x,1)}{n^{1-(A+B)}(n-1)^{A+B}} \right\}.$$

In view of (2.8), (2.9), (2.10), (2.11) and (2.12) we get the desired estimation. □

By symmetry, the analogous estimate for the integral $\int_{-1}^{x} g(t)K_n(x,t)w(t)dt$ can be deduced as well. Namely, we have

$$(2.13)\quad \left| \int_{-1}^{x} g(t)K_n(x,t)w(t)dt \right| \le \frac{c_1}{(1-x^2)^{A+2B} n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{\nu_j(g;-1,s_{n-j})}{j^{1+A+B}} + \frac{c_2}{(1-x^2)^{1+B}} \left\{ \sum_{j=1}^{n-1} \frac{\nu_j(g;s_j,x)}{j^2} + \frac{\nu_{n-1}(g;-1,x)}{n-1} \right\},$$

where $s_j = x - j(1+x)/n \ (j = 1,2,\dots,n),\ c_1, c_2$ are the same as in Lemma 2.2.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Results

Suppose that $f \in M(I)$ and that at a fixed point $x \in (-1,1)$ the one-sided limits $f(x+), f(x-)$ exist. As is easily seen

$$S_n[f](w;x) - \frac{1}{2}(f(x+)+f(x-)) = \int_{-1}^{1} g_x(t)K_n(x,t)w(t)dt + \frac{1}{2}(f(x+)-f(x-))S_n[\psi_x](w;x), \tag{3.1}$$

where g_x and ψ_x are defined by (1.5) and (1.6), respectively.

The first term on the right-hand side of identity (3.1) can be estimated via (2.7) and (2.13). Consequently, we get:

Theorem 3.1. *Let w be a weight function and let assumptions (1.2), (1.3), (1.4) be satisfied with $A+B<1$. If $f \in M(I)$ and if the limits $f(x+), f(x-)$ at a fixed $x \in (-1,1)$ exist, then for $n \geq 3$ we have*

$$\begin{aligned} &\left| S_n[f](w;x) - \frac{1}{2}(f(x+)+f(x-)) \right| \\ &\leq \frac{c_1}{(1-x^2)^{A+2B} n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{\nu_j(g_x; t_{n-j}, 1) + \nu_j(g_x; -1, s_{n-j})}{j^{1+A+B}} \\ &\quad + \frac{c_2}{(1-x^2)^{1+B}} \left\{ \sum_{j=1}^{n-1} \frac{\nu_j(g; x, t_j) + \nu_j(g_x; s_j, x)}{j^2} \right. \\ &\quad \left. + \frac{\nu_{n-1}(g_x; -1, x) + \nu_{n-1}(g_x; x, 1)}{n-1} \right\} + \frac{1}{2}(f(x+)-f(x-))S_n[\psi_x](w;x), \end{aligned} \tag{3.2}$$

where t_j, s_j, c_1, c_2 are defined above (in Section 2).

Orthogonal Polynomial Expansions
Małgorzata Powierska
vol. 8, iss. 3, art. 11, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Theorem 3.2. *Let $f \in BV_\Phi(I)$ and let assumptions (1.2), (1.3), (1.4) be satisfied with $A+B<1$. Then for every $x \in (-1,1)$, and all $n \geq 3$,*

$$\left| S_n[f])w;x) - \frac{1}{2}(f(x+)+f(x-)) \right| \tag{3.3}$$
$$\leq \frac{c_3}{(1-x^2)^{1+B}} \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n} V_\Phi\left(g_x; x, x+\frac{1-x}{k}\right) + \frac{k}{n} V_\Phi\left(g_x; x-\frac{1+x}{k}, x\right)\right)$$
$$+ \frac{c_4(x)}{(1-x^2)^{A+2B} n^{1-(A+B)}} \sum_{k=1}^{n-1} \frac{1}{k^{A+B}} \Phi^{-1}\left(\frac{1}{k}\right) + \frac{1}{2}|f(x+)-f(x-)|\,|S_n[\psi_x](w;x)|,$$

where $c_3 = 10c_2$, $c_4(x) = c_1(\max\{1, V_\Phi(g_x;x,1)\} + \max\{1, V_\Phi(g_x;-1,x)\})$ and Φ^{-1} denotes the inverse function of Φ.

Proof. It is known that, for every positive integer j and for every subinterval $[a,b]$ of $[-1,x]$ (or $[x,1]$),

$$\nu_j(g_x;a,b) \leq j\Phi^{-1}\left(\frac{1}{j} V_\Phi(g_x;a,b)\right)$$

(see [2, p. 537]). Consequently,

$$\frac{1}{n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{\nu_j(g_x, t_{n-j}, 1)}{j^{1+A+B}} \leq \frac{\max\{V_\Phi(g_x;x,1),1\}}{n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{1}{j^{A+B}} \Phi^{-1}\left(\frac{1}{j}\right).$$

Moreover

$$\sum_{j=1}^{n-1} \frac{\nu_j(g_x;x,t_j)}{j^2} \leq 8 \sum_{j=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n} V_\Phi\left(g_x;x,x+\frac{1-x}{k}\right)\right)$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

(see [7, Section 3]). Similarly,

$$\frac{\nu_{n-1}(g_x;x,1)}{n-1} \le 2\Phi^{-1}\left(\frac{V_\Phi(g_x;x,1)}{n}\right) \le 2\sum_{k=1}^{n-1}\frac{1}{k}\Phi^{-1}\left(\frac{k}{n}V_\Phi\left(g_x;x,x+\frac{1-x}{k}\right)\right).$$

Analogous estimates for the other terms in the inequality (3.2), corresponding to the interval $[-1,x]$, can be obtained as well. Theorem 3.1 and the above estimates give the desired result. □

Remark 1. Since the function g_x is continuous at the point x, we have $\lim_{t\to 0} V_\Phi(g_x;x,x+t)=0$. Consequently, under the additional assumption,

$$\sum_{k=1}^{\infty}\frac{1}{k}\Phi^{-1}\left(\frac{1}{k}\right)<\infty \tag{3.4}$$

and

$$\lim_{n\to\infty} S_n[\psi_x](w;x)=0, \tag{3.5}$$

the right-hand side of inequality (3.3) converges to zero as $n\to\infty$.

In particular, if $f\in BV_p(I)$ with $p\ge 1$, i.e. if $\Phi(u)=u^p$ for $u\ge 0$, then (3.4) holds true. Moreover, the function λ defined as $\lambda(t)=f(\cos t)$ is 2π-periodic and of bounded p-th power variation on $[-\pi,\pi]$. Hence, in view of the theorem of Marcinkiewicz ([5, p. 38]), its L^p-integral modulus of continuity

$$\omega(\lambda;\delta)_p := \sup_{|h|\le\delta}\left(\int_{-\pi}^{\pi}|\lambda(x+h)-\lambda(x)|^p dx\right)^{1/p}$$

satisfies the inequality

$$\omega(\lambda;\delta)_p \le \delta^{1/p}V_p(\lambda;0,3\pi) \quad \text{for} \quad 0\le\delta\le\pi.$$

Orthogonal Polynomial Expansions
Małgorzata Powierska
vol. 8, iss. 3, art. 11, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Consequently, if $1 \leq p \leq 2$, then

$$\omega(\lambda;\delta)_2 \leq \delta^{1/2} V_2(\lambda;0,3\pi) \leq \delta^{1/2}(V_p(\lambda;0,3\pi))^{2/p},$$

which means that $\lambda \in \operatorname{Lip}\left(\frac{1}{2},2\right)$. Applying now the Freud theorem ([3, V. Theorem 7.5]) we can easily state that in the case of $f \in BV_p(I)$ with $1 \leq p \leq 2$, condition (3.5) holds. So, from Theorem 3.2 we get:

Corollary 3.3. *Let w be a weight function satisfying $0 < w(x) \leq M(1-x^2)^{-1/2}$ for $x \in (-1,1)$ $(M = const.)$ and let (1.3), (1.4) be satisfied with $0 < B < 1/2$. If $f \in BV_p(I)$, where $1 \leq p \leq 2$, then $S_n[f](w;x)$ converges to $\frac{1}{2}(f(x+)+f(x-))$ at every $x \in (-1,1)$, where w is continuous.*

From our theorems we can also obtain some results concerning the rate of uniform convergence of $S_n[f](w;x)$. Namely, we have:

Corollary 3.4. *Let conditions (1.2), (1.3), (1.4) be satisfied with $A + B < 1$. If f is continuous on the interval I and if $-1 < a < b < 1$, then for all $x \in [a,b]$ and all integers $n \geq 3$*

$$|S_n[f](w;x) - f(x)| \leq c(a,b,A,B)\left\{\omega\left(f;\frac{1}{n}\right)\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=m+1}^{n}\frac{\nu_k(f;-1,1)}{k^2}\right\},$$

where $\omega(f;\delta)$ denotes the modulus of continuity of f on I, $c(a,b,A,B)$ is a positive constant depending only on a,b,A,B and m is an arbitrary integer, such that $m < n$.

Proof. It is known ([2, 8]) that, for every interval $[a,b] \subset [-1,1]$ and for every positive integer j,

$$\nu_j(f;a,b) \leq 2j\omega\left(f;\frac{b-a}{j}\right).$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Therefore,

$$\nu_j(g_x; s_j, x) \le 4j\omega\left(f; \frac{1}{n}\right), \quad \nu_j(g_x; x, t_j) \le 4j\omega\left(f; \frac{1}{n}\right)$$

and

$$\frac{1}{n^{1-(A+B)}} \sum_{j=1}^{n-1} \frac{\nu_j(g_x, t_{n-j}, 1) + \nu_j(g_x; -1, s_{n-j})}{j^{1+A+B}} \le \frac{8}{1-(A+B)}\omega\left(f; \frac{1}{n}\right).$$

Using the above estimation and inequality (3.2) we get the desired result. □

Clearly, Corollary 3.4 yields some criterions for the uniform convergence of orthogonal polynomial expansions on each compact interval contained in $(-1, 1)$ (cf. [2, 7]).

Finally, let us note that our results can be applied to the Jacobi orthonormal polynomials $\left\{p_n^{(\alpha,\beta)}\right\}$ determined via the Jacobi weight $w(x) := w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, where $\alpha > -1, \beta > -1$. In this case, the fulfillment of (1.2) and (1.3) with some A, B follows from the definition of the weight $w^{(\alpha,\beta)}(x)$ and from Theorem 8.1 in [3] (Chap. I). Estimate (1.4) can be verified via the known formula

$$\int_x^1 p_n^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t)dt = \left(\frac{n}{n+\alpha+\beta+1}\right)^{\frac{1}{2}} \frac{(1-x)^{(\alpha+1)}(1+x)^{(\beta+1)}}{n} p_n^{(\alpha+1,\beta+1)}(x)$$

(cf. [6, identity (51)]) and the inequality

$$\left|p_{n-1}^{(\alpha,\beta)}(x)\right| \le c(\alpha,\beta)\left((1-x)^{1/2} + \frac{1}{n}\right)^{-\alpha-1/2}\left((1+x)^{1/2} + \frac{1}{n}\right)^{-\beta-1/2}$$

(see e.g. [4, inequality (12)]). Moreover, it was stated by R. Bojanic that in the case of the Jacobi polynomials condition (3.5) is satisfied (see [6, estimate (12)]).

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In particular, our general estimations given in Theorems 3.1, 3.2 and in Corollary 3.3 remain valid for the Legendre polynomials (see [7]). The rate of pointwise convergence of the Legendre polynomial expansions for functions f of bounded variation in the Jordan sense on I (i.e. for $f \in BV_1(I)$ was first obtained in [1].

Orthogonal Polynomial Expansions
Małgorzata Powierska
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