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POWERS OF CLASS $wF(p, r, q)$ OPERATORS

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Abstract

This paper is to discuss powers of class $wF(p,r,q)$ operators for $1 \geq p > 0$, $1 \geq r > 0$ and $q \geq 1$; and an example is given on powers of class $wF(p,r,q)$ operators.

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Contents

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators in H, and a capital letter (such as T) denote an element of $B(H)$. An operator T is said to be k-hyponormal for $k > 0$ if $(T^*T)^k \geq (TT^*)^k$, where T^* is the adjoint operator of T. A k-hyponormal operator T is called hyponormal if $k = 1$; semi-hyponormal if $k = 1/2$. Hyponormal and semi-hyponormal operators have been studied by many authors, such as [1, 11, 16, 20, 21]. It is clear that every k-hyponormal operator is q-hyponormal for $0 < q \leq k$ by the Löwner-Heinz theorem ($A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $1 \geq \alpha \geq 0$). An invertible operator T is said to be $\log$-hyponormal if $\log T^*T \geq \log TT^*$, see [18, 19]. Every invertible k-hyponormal operator for $k > 0$ is $\log$-hyponormal since $\log t$ is an operator monotone function. $\log$-hyponormality is sometimes regarded as 0-hyponormal since $(X^k - 1)/k \to \log X$ as $k \to 0$ for $X > 0$.

As generalizations of k-hyponormal and $\log$-hyponormal operators, many authors introduced many classes of operators, see the following.

Definition A ([5, 6]).

(1) *For $p > 0$ and $r > 0$, an operator T belongs to class $A(p,r)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

(2) *For $p > 0, r \geq 0$ and $q \geq 1$, an operator T belongs to class $F(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}.$$

For each $p > 0$ and $r > 0$, class $A(p,r)$ contains all p-hyponormal and log-hyponormal operators. An operator T is a class $A(k)$ operator ([9]) if and only if T is a class $A(k,1)$ operator, T is a class $A(1)$ operator if and only if T is a class A operator ([9]), and T is a class $A(p,r)$ operator if and only if T is a class $F\left(p,r,\frac{p+r}{r}\right)$ operator.

Aluthge-Wang [3] introduced w-hyponormal operators defined by $\left|\widetilde{T}\right| \geq |T| \geq \left|\widetilde{T}^*\right|$ where the polar decomposition of T is $T = U|T|$ and $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation of T. As a generalization of w-hyponormality, Ito [12] and Yang-Yuan [25, 26] introduced the classes $wA(p,r)$ and $wF(p,r,q)$ respectively.

Definition B.

(1) *For $p > 0, r > 0$, an operator T belongs to class $wA(p,r)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r} \quad \text{and} \quad |T|^{2p} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p}{p+r}}.$$

(2) *For $p > 0, r \geq 0$, and $q \geq 1$, an operator T belongs to class $wF(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}} \quad \text{and} \quad |T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}},$$

denoting $(1-q^{-1})^{-1}$ by q^* (when $q > 1$) because q and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents.

An operator T is a w-hyponormal operator if and only if T is a class $wA(\frac{1}{2},\frac{1}{2})$ operator, T is a class $wA(p,r)$ operator if and only if T is a class $wF(p,r,\frac{p+r}{r})$ operator.

Ito [15] showed that the class $A(p,r)$ coincides with the class $wA(p,r)$ for each $p > 0$ and $r > 0$, class A coincides with class $wA(1,1)$. For each $p > 0$, $r \geq 0$ and $q \geq 1$ such that $rq \leq p+r$, [25] showed that class $wF(p,r,q)$ coincides with class $F(p,r,q)$.

Halmos ([11, Problem 209]) gave an example of a hyponormal operator T whose square T^2 is not hyponormal. This problem has been studied by many authors, see [2, 10, 14, 22, 27]. Aluthge-Wang [2] showed that the operator T^n is (k/n)-hyponormal for any positive integer n if T is k-hyponormal.

In this paper, we firstly discuss powers of class $wF(p,r,q)$ operators for $1 \geq p > 0$, $1 \geq r > 0$ and $q \geq 1$. Secondly, we shall give an example on powers of class $wF(p,r,q)$ operators.

2. Result and Proof

The following assertions are well-known.

Theorem A ([15]). *Let* $1 \geq p > 0$, $1 \geq r > 0$. *Then* T^n *is a class* $wA(\frac{p}{n}, \frac{r}{n})$ *operator.*

Theorem B ([13]). *Let* $1 \geq p > 0, 1 \geq r \geq 0$, $q \geq 1$ *and* $rq \leq p + r$. *If* T *is an invertible class* $F(p, r, q)$ *operator, then* T^n *is a* $F(\frac{p}{n}, \frac{r}{n}, q)$ *operator.*

Theorem C ([25]). *Let* $1 \geq p > 0$, $1 \geq r \geq 0$; $q \geq 1$ *when* $r = 0$ *and* $\frac{p+r}{r} \geq q \geq 1$ *when* $r > 0$. *If* T *is a class* $wF(p, r, q)$ *operator, then* T^n *is a class* $wF(\frac{p}{n}, \frac{r}{n}, q)$ *operator.*

Here we generalize them to the following.

Theorem 2.1. *Let* $1 \geq p > 0$, $1 \geq r > 0$; $q > \frac{p+r}{r}$. *If* T *is a class* $wF(p, r, q)$ *operator such that* $N(T) \subset N(T^*)$, *then* T^n *is a class* $wF(\frac{p}{n}, \frac{r}{n}, q)$ *operator.*

In order to prove the theorem, we require the following assertions.

Lemma A ([8]). *Let* $\alpha \in \mathbb{R}$ *and* X *be invertible. Then* $(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X$ *holds, especially in the case* $\alpha \geq 1$, *Lemma A holds without invertibility of* X.

Theorem D ([15]). *Let* $A, B \geq 0$. *Then for each* $p, r \geq 0$, *the following assertions hold:*

(1) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r \Rightarrow \left(A^{\frac{p}{2}} B^r A^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \leq A^p$.

(2) $\left(A^{\frac{p}{2}} B^r A^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \leq A^p$ *and* $N(A) \subset N(B) \Rightarrow \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r$.

Theorem E ([24]). *Let T be a class wA operator. Then $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$ and $|T^*|^2 \geq |(T^2)^*| \geq \cdots \geq |(T^n)^*|^{\frac{2}{n}}$ hold.*

Theorem F ([25]). *Let T be a class $wF(p_0, r_0, q_0)$ operator for $p_0 > 0, r_0 \geq 0$ and $q_0 \geq 1$. Then the following assertions hold.*

(1) *If $q \geq q_0$ and $r_0 q \leq p_0 + r_0$, then T is a class $wF(p_0, r_0, q)$ operator.*

(2) *If $q^* \geq q_0^*$, $p_0 q^* \leq p_0 + r_0$ and $N(T) \subset N(T^*)$, then T is a class $wF(p_0, r_0, q)$ operator.*

(3) *If $rq \leq p + r$, then class $wF(p, r, q)$ coincides with class $F(p, r, q)$.*

Theorem G ([25]). *Let T be a class $wF\left(p_0, r_0, \frac{p_0+r_0}{\delta_0+r_0}\right)$ operator for $p_0 > 0$, $r_0 \geq 0$ and $-r_0 < \delta_0 \leq p_0$. Then T is a class $wF\left(p, r, \frac{p+r}{\delta_0+r}\right)$ operator for $p \geq p_0$ and $r \geq r_0$.*

Proposition A ([25]). *Let $A, B \geq 0$; $1 \geq p > 0$, $1 \geq r > 0$; $\frac{p+r}{r} \geq q \geq 1$. Then the following assertions hold.*

(1) *If $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$ and $B \geq C$, then $\left(C^{\frac{r}{2}} A^p C^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq C^{\frac{p+r}{q}}$.*

(2) *If $B^{\frac{p+r}{q}} \geq \left(B^{\frac{r}{2}} C^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$, $A \geq B$ and the condition*

$$(*) \qquad \text{if } \lim_{n\to\infty} B^{\frac{1}{2}} x_n = 0 \text{ and } \lim_{n\to\infty} A^{\frac{1}{2}} x_n \text{ exists, then } \lim_{n\to\infty} A^{\frac{1}{2}} x_n = 0$$

holds for any sequence of vectors $\{x_n\}$, then $A^{\frac{p+r}{q}} \geq \left(A^{\frac{r}{2}} C^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$.

Proof of Theorem 2.1. Put $\delta = \frac{p+r}{q} - r$, then $-r < \delta < 0$ by the hypothesis. Moreover, if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r+\delta}{p+r}} \geq |T^*|^{2(r+\delta)} \quad \text{and} \quad |T|^{2(p-\delta)} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p-\delta}{p+r}},$$

then T is a class wA operator by Theorem G and Theorem D, so that the following hold by taking $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |(T^n)^*|^{\frac{2}{n}}$ in Theorem E

$$A_n \geq \cdots \geq A_2 \geq A_1 \quad \text{and} \quad B_1 \geq B_2 \geq \cdots \geq B_n. \tag{2.1}$$

Meanwhile, A_n and A_1 satisfy the following for any sequence of vectors $\{x_m\}$ (see [24])

$$\text{if } \lim_{m\to\infty} A_1^{\frac{1}{2}}x_m = 0 \text{ and } \lim_{m\to\infty} A_n^{\frac{1}{2}}x_m \text{ exists, then } \lim_{m\to\infty} A_n^{\frac{1}{2}}x_m = 0.$$

Then the following holds by Proposition A

$$(A_n)^{\frac{p+r}{q^*}} \geq \left((A_n)^{\frac{p}{2}}(B_1)^r(A_n)^{\frac{p}{2}}\right)^{\frac{1}{q^*}} \geq \left((A_n)^{\frac{p}{2}}(B_n)^r(A_n)^{\frac{p}{2}}\right)^{\frac{1}{q^*}},$$

and it follows that

$$|T^n|^{\frac{2(p+r)}{nq^*}} \geq \left(|T^n|^{\frac{p}{n}}|(T^n)^*|^{\frac{2r}{n}}|T^n|^{\frac{p}{n}}\right)^{\frac{1}{q^*}}.$$

We assert that $N(T) \subset N(T^*)$ implies $N(T^n) \subset N((T^n)^*)$.

In fact,

$$\begin{aligned} x \in N(T^n) &\Rightarrow T^{n-1}x \in N(T) \subseteq N(T^*) \\ &\Rightarrow T^{n-2}x \in N(T^*T) = N(T) \subseteq N(T^*) \\ &\dots \\ &\Rightarrow x \in N(T) \subseteq N(T^*) \\ &\Rightarrow x \in N(T^*) \subseteq N((T^n)^*), \end{aligned}$$

thus

$$\left(|(T^n)^*|^{\frac{r}{n}}|T^n|^{\frac{2p}{n}}|(T^n)^*|^{\frac{r}{n}}\right)^{\frac{1}{q}} \geq |(T^n)^*|^{\frac{2(p+r)}{nq}}$$

holds by Theorem D and the Löwner-Heinz theorem, so that T^n is a class $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator. □

3. An Example

In this section we give an example on powers of class $wF(p,r,q)$ operators.

Theorem 3.1. *Let A and B be positive operators on H, U and D be operators on $\bigoplus_{k=-\infty}^{\infty} H_k$, where $H_k \cong H$, as follows*

$$U = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 1 & (0) & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

$$D = \begin{pmatrix} \ddots & & & & & & \\ & B^{\frac{1}{2}} & & & & & \\ & & B^{\frac{1}{2}} & & & & \\ & & & (A^{\frac{1}{2}}) & & & \\ & & & & A^{\frac{1}{2}} & & \\ & & & & & A^{\frac{1}{2}} & \\ & & & & & & \ddots \end{pmatrix},$$

where $(\cdot)$ shows the place of the $(0,0)$ matrix element, and $T = UD$. Then the following assertions hold.

(1) *If T is a class $wF(p,r,q)$ operator for $1 \geq p > 0$, $1 \geq r \geq 0$, $q \geq 1$ and $rq \leq p+r$, then T^n is a $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator.*

(2) *If T is a class $wF(p,r,q)$ operator such that $N(T) \subset N(T^*)$, $1 \geq p > 0$, $1 \geq r \geq 0$, $q \geq 1$ and $rq > p+r$, then T^n is a $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator.*

Remark 1. *Noting that Theorem 3.1 holds without the invertibility of A and B, this example is a modification of ([4], Theorem 2) and ([23], Lemma 1).*

We need the following well-known result to give the proof.

Theorem H (Furuta inequality [7], in brief FI). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

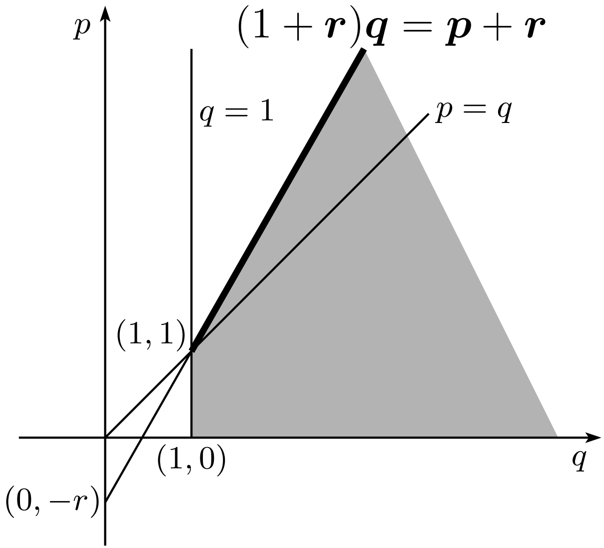
(i)
$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii)
$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Theorem H yields the Löwner-Heinz inequality by putting $r = 0$ in (i) or (ii) of FI. It was shown by Tanahashi [17] that the domain drawn for p, q and r in the Figure is the best possible for Theorem H.



Proof of Theorem 3.1. By simple calculations, we have

$$|T|^2 = \begin{pmatrix} \ddots & & & & & & \\ & B & & & & & \\ & & B & & & & \\ & & & (A) & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix},$$

$$|T^*|^2 = \begin{pmatrix} \ddots & & & & & & \\ & B & & & & & \\ & & B & & & & \\ & & & (B) & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix},$$

therefore

$$|T^*|^r|T|^{2p}|T^*|^r = \begin{pmatrix} \ddots & & & & & & \\ & B^{p+r} & & & & & \\ & & B^{p+r} & & & & \\ & & & (B^{\frac{r}{2}}A^pB^{\frac{r}{2}}) & & & \\ & & & & A^{p+r} & & \\ & & & & & A^{p+r} & \\ & & & & & & \ddots \end{pmatrix}$$

and

$$|T|^p|T^*|^{2r}|T|^p = \begin{pmatrix} \ddots & & & & & & \\ & B^{p+r} & & & & & \\ & & B^{p+r} & & & & \\ & & & (A^{\frac{p}{2}}B^rA^{\frac{p}{2}}) & & & \\ & & & & A^{p+r} & & \\ & & & & & A^{p+r} & \\ & & & & & & \ddots \end{pmatrix},$$

thus the following hold for $n \geq 2$

$$
\begin{aligned}
&T^{n^*}T^n \\
&= \begin{pmatrix}
\ddots &&&&&&&&&& \\
& B^n &&&&&&&&& \\
&& B^n &&&&&&&& \\
&&& B^{\frac{n-1}{2}}AB^{\frac{n-1}{2}} &&&&&&& \\
&&&& \ddots &&&&&& \\
&&&&& B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}} &&&&& \\
&&&&&& \ddots &&&& \\
&&&&&&& B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}} &&& \\
&&&&&&&& (A^n) && \\
&&&&&&&&& A^n & \\
&&&&&&&&&& \ddots
\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&T^nT^{n^*} \\
&= \begin{pmatrix}
\ddots &&&&&&&&&& \\
& B^n &&&&&&&&& \\
&& (B^n) &&&&&&&& \\
&&& A^{\frac{1}{2}}B^{n-1}A^{\frac{1}{2}} &&&&&&& \\
&&&& \ddots &&&&&& \\
&&&&& A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}} &&&&& \\
&&&&&& \ddots &&&& \\
&&&&&&& A^{\frac{n-1}{2}}BA^{\frac{n-1}{2}} &&& \\
&&&&&&&& A^n && \\
&&&&&&&&& A^n & \\
&&&&&&&&&& \ddots
\end{pmatrix}
\end{aligned}
$$

Proof of (1). T is a class $wF(p,r,q)$ operator is equivalent to the following

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \quad \text{and} \quad A^{\frac{p+r}{q^*}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{1}{q^*}},$$

T^n belongs to class $wF(\frac{p}{n},\frac{r}{n},q)$ is equivalent to the following (3.1) and (3.2).

$$(3.1)\quad \begin{cases} (B^{\frac{r}{2}}(B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}})^{\frac{p}{n}}B^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \\ (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \\ \left(\left(A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}}\right)^{\frac{r}{2n}}A^p\left(A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}}\right)^{\frac{r}{2n}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}}\right)^{\frac{p+r}{nq}} \\ \qquad\qquad\qquad\qquad \text{where } j=1,2,\ldots,n-1. \end{cases}$$

$$(3.2)\quad \begin{cases} \left(\left(B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}}\right)^{\frac{p}{2n}}B^r\left(B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}}\right)\right)^{\frac{1}{q^*}} \geq \left(B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}}\right)^{\frac{p+r}{nq^*}} \\ A^{\frac{p+r}{q^*}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{1}{q^*}} \\ A^{\frac{p+r}{q^*}} \geq \left(A^{\frac{p}{2}}\left(A^{\frac{j}{2}}B^{n-j}A^{\frac{j}{2}}\right)^{\frac{r}{n}}A^{\frac{p}{2}}\right)^{\frac{1}{q^*}} \\ \qquad\qquad\qquad\qquad \text{where } j=1,2,\ldots,n-1. \end{cases}$$

We only prove (3.1) because of Theorem D.

Step 1. To show

$$\left(B^{\frac{r}{2}}\left(B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}}\right)^{\frac{p}{n}}B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$$

for $j = 1, 2, ..., n-1$.

In fact, T is a class $wF(p, r, q)$ operator for $1 \geq p > 0$, $1 \geq r \geq 0$, $q \geq 1$ and $rq \leq p + r$ implies T belongs to class $wF\left(j, n-j, \frac{n}{\delta+j}\right)$, where $\delta = \frac{p+r}{q} - r$ by Theorem G and Theorem D, thus

$$\left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)^{\frac{\delta+j}{n}} \geq B^{\delta+j} \quad \text{and} \quad A^{n-j-\delta} \geq \left(A^{\frac{n-j}{2}} B^{j} A^{\frac{n-j}{2}}\right)^{\frac{n-j-\delta}{n}}$$

Therefore the assertion holds by applying (i) of Theorem H to $\left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)^{\frac{\delta+j}{n}}$ and $B^{\delta+j}$ for $\left(1 + \frac{r}{\delta+j}\right) q \geq \frac{p}{\delta+j} + \frac{r}{\delta+j}$.

Step 2. To show

$$\left(\left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{r}{2n}} A^{p} \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{r}{2n}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{p+r}{nq}}$$

for $j = 1, 2, ..., n-1$.

In fact, similar to Step 1, the following hold

$$\left(B^{\frac{n-j}{2}} A^{j} B^{\frac{n-j}{2}}\right)^{\frac{\delta+n-j}{n}} \geq B^{\delta+n-j} \quad \text{and} \quad A^{j-\delta} \geq \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{j-\delta}{n}},$$

this implies that $A^{j} \geq \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{j}{n}}$ by Theorem D. Therefore the assertion holds by applying (i) of Theorem H to A^{j} and $\left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{j}{n}}$ for $(1 + \frac{r}{j})q \geq \frac{p}{j} + \frac{r}{j}$.

Proof of (2). This part is similar to Proof of (1), so we omit it here. □

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