

ON THE ITERATED GREEN FUNCTIONS ON A BOUNDED DOMAIN AND THEIR RELATED KATO CLASS OF POTENTIALS

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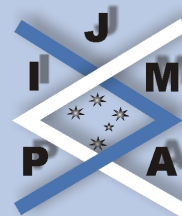
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Abstract: We use the results of Zhang [15, 16] and Davies [7] on the behavior of the heat kernel $p(t, x, y)$ on a bounded $C^{1,1}$ domain D to find again the result of Grunau-Sweers [9] concerning the estimates of the iterated Greens functions $G_{m,n}(D)$. Next, we use these estimates to characterize, by means of $p(t, x, y)$, the Kato class $K_{m,n}(D)$ and we give new examples of functions belonging to this class.



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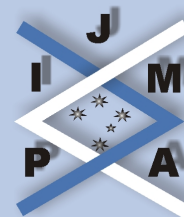
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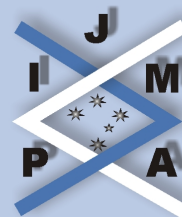
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1. Introduction

Let D be a bounded $C^{1,1}$ domain in \mathbb{R}^n , $n \geq 3$ and $p(t, x, y)$ be the density of the Gauss semigroup on D . Combining the results of Zhang [15], [16] and those of Davies or Davies-Simon [7], [8] a qualitatively sharp understanding of the boundary behaviour of $p(t, x, y)$ is given as follows: There exist positive constants c_1, c_2 and λ_0 depending only on D such that for all $t > 0$ and $x, y \in D$,

$$(1.1) \quad \left(\frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left(\frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) \frac{c_1 e^{-\lambda_0 t - c_2 \frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}} \\ \leq p(t, x, y) \leq \left(\frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left(\frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) \frac{e^{-\lambda_0 t - \frac{|x-y|^2}{c_2 t}}}{c_1 t^{\frac{n}{2}}},$$

where $\delta(x)$ denotes the Euclidean distance from x to the boundary of D .

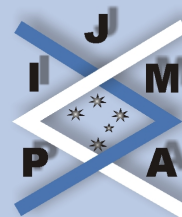
Let $G(x, y)$ be the Green's function of the laplacien Δ in D with a Dirichlet condition on ∂D . Then G is given by

$$(1.2) \quad G(x, y) = \int_0^\infty p(t, x, y) dt, \quad \text{for } x, y \in D.$$

For a positive integer m , we denote by $G_{m,n}$ the Green's function of the operator $u \mapsto (-\Delta)^m u$ on D with Navier boundary conditions $\Delta^j u = 0$ on ∂D for $0 \leq j \leq m - 1$. Then $G_{1,n} = G$ and $G_{m,n}$ satisfies for $m \geq 2$

$$G_{m,n}(x, y) = \int_D \int_D G(x, z) G_{m-1,n}(z, y) dz.$$

Using the Fubini theorem and the Chapman-Kolmogorov identity, we show by in-



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duction that for each $m \geq 1$ and $x, y \in D$ we have

$$(1.3) \quad G_{m,n}(x, y) = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} p(t, x, y) dt.$$

In this paper we will use (1.1) and (1.3) to find again the result of Grunau and Sweers in [9] concerning the sharp estimates of $G_{m,n}$. More precisely we will give another proof for the case $n \geq 3$ of the following theorem.

Theorem 1.1 (see [9]). *On D^2 we have*

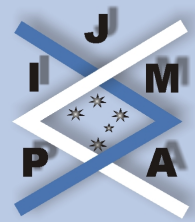
$$G_{m,n}(x, y) \sim H_{m,n}(x, y) = \begin{cases} \frac{1}{|x-y|^{n-2m}} \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \text{if } n > 2m, \\ \text{Log}\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \text{if } n = 2m, \\ \sqrt{\delta(x)\delta(y)} \min\left(1, \frac{\sqrt{\delta(x)\delta(y)}}{|x-y|}\right) & \text{if } n = 2m - 1, \\ \delta(x)\delta(y) \text{Log}\left(2 + \frac{1}{|x-y|^2 + \delta(x)\delta(y)}\right) & \text{if } n = 2m - 2, \\ \delta(x)\delta(y) & \text{if } n < 2m - 2, \end{cases}$$

where the symbol \sim is defined in the notations below.

As a second step we will also use (1.1) and (1.3) to give new contributions in the case $n > 2m$ to the study of the Kato class $K_{m,n}(D)$ defined in [11] for $m = 1$ and in [2] for $m \geq 2$ as follows.

Definition 1.1. A Borel measurable function q in D belongs to the Kato class $K_{m,n}(D)$ if q satisfies the following condition

$$(1.4) \quad \lim_{\alpha \rightarrow 0} \sup_{x \in D} \int_{D \cap \{|x-y| \leq \alpha\}} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) |q(y)| dy = 0.$$



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We note that in the case $m = 1$, the class $K_{1,n}(D)$ properly contains the classical Kato class $K_n(D)$ introduced in [1] as the natural class of singular functions which replaces the L^p -Lebesgue spaces in order that the weak solutions of the Schrödinger equation are continuous and satisfy a Harnack principle. More precisely, it is shown in [11] that the function $\rho_\alpha(y) = \frac{1}{\delta^\alpha(y)}$ belongs to $K_{1,n}(D)$ if and only if $\alpha < 2$ but for $1 \leq \alpha < 2$, $\rho_\alpha \notin K_n(D)$.

Our second contribution here is to exploit estimates of Theorem 1.1 on the one hand, to give new examples of functions belonging to the class $K_{m,n}(D)$ and to characterize this class by means of the density of the Gauss semigroup in D on the other hand. In particular we will prove the following results for the unit ball.

Proposition 1.2. For $\lambda, \mu \in \mathbb{R}$ and $y \in B(0, 1)$ we put

$$\rho_{\lambda, \mu}(y) = \frac{1}{(1 - |y|)^\lambda \left[\text{Log}\left(\frac{2}{1-|y|}\right) \right]^\mu}.$$

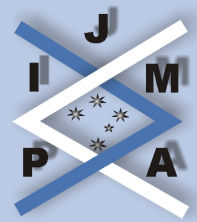
For $m \geq 2$ we have

$\rho_{\lambda, \mu} \in K_{m,n}(B(0, 1))$ if and only if $\lambda < 3$ or $(\lambda = 3$ and $\mu > 1)$.

Theorem 1.3. Let $n > 2m$ and q be a Borel measurable function in D . Then the following assertions are equivalent:

- 1) $q \in K_{m,n}(B(0, 1))$
- 2) $\lim_{t \rightarrow 0} \left(\sup_{x \in B} \int_0^t \int_B \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) |q(y)| dy ds \right) = 0$

We also note that in the case $m = 1$, similar characterizations have been obtained by Aizenman and Simon in [1] for the Kato class $K_n(\mathbb{R}^n)$ and by Bachar and Mâagli in [4] for the half space \mathbb{R}_+^n , where they introduce a new Kato class that properly



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contains the classical one. This was extended for $m \geq 2$ by Mâagli and Zribi [12] to the class $K_{m,n}(\mathbb{R}^n)$ and by Bachar [3] to the class $K_{m,n}(\mathbb{R}_+^n)$. The density of the Gauss semigroup in the case of \mathbb{R}^n and \mathbb{R}_+^n are explicitly known, but this is not the case for a bounded $C^{1,1}$ domain even if D is an open ball.

In order to simplify our statements, we define some convenient notations.

Notations.

- i) For $x, y \in D$, we denote by $\delta(x)$ the Euclidean distance from x to the boundary of D , $[x, y]^2 = |x - y|^2 + \delta(x)\delta(y)$ and d is the diameter of D .
- ii) For $a, b \in \mathbb{R}$, we denote by $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.
- iii) Let f and g be two nonnegative functions on a set S .
We say that $f \preceq g$, if there exists $c > 0$ such that

$$f(x) \leq c g(x) \quad \text{for all } x \in S.$$

We say that $f \sim g$, if there exists $C > 0$ such that

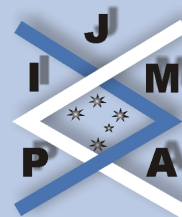
$$\frac{1}{C}g(x) \leq f(x) \leq Cg(x) \quad \text{for all } x \in S.$$

The following properties will be used several times

- iv) For $a, b \geq 0$, we have

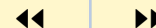
$$(1.5) \quad \frac{ab}{a+b} \leq \min(a, b) \leq 2 \frac{ab}{a+b}$$

$$(1.6) \quad (a+b)^p \sim a^p + b^p \quad \text{for } p \in \mathbb{R}^+.$$



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$$(1.7) \quad \min(1, a) \min(1, b) \leq \min(1, ab) \leq \min(1, a) \max(1, b)$$

$$(1.8) \quad \frac{a}{1+a} \leq \text{Log}(1+a)$$

v) Let $\eta, \nu > 0$ and $0 < \gamma \leq 1$. Then we have

$$(1.9) \quad \text{Log}(1+t) \preceq t^\gamma, \quad \text{for } t \geq 0.$$

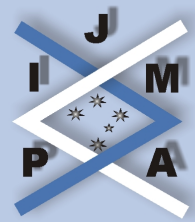
$$(1.10) \quad \text{Log}(1+\eta t) \sim \text{Log}(1+\nu t), \quad \text{for } t \geq 0.$$

Finally we note that since for each $a \geq b \geq 0$ and $c > 0$ we have

$$\begin{aligned} \frac{(a+1)(b+1)}{1+ab} e^{-c(b-a)^2} &= \left(1 + \frac{a+b}{1+ab}\right) e^{-c(b-a)^2} \\ &= \left(1 + \frac{2a+\xi}{1+a(a+\xi)}\right) e^{-c\xi^2} \\ &\leq (2+\xi)e^{-c\xi^2} \leq C. \end{aligned}$$

Then, using (1.5) we deduce that for each $x, y \in D$ and $0 < t \leq 1$ we have

$$\begin{aligned} \min\left(\frac{\delta(x)\delta(y)}{t}, 1\right) &\leq C \min\left(\frac{\delta(x)}{\sqrt{t}}, 1\right) \min\left(\frac{\delta(y)}{\sqrt{t}}, 1\right) e^{c \frac{|\delta(x)-\delta(y)|^2}{t}} \\ &\leq C \min\left(\frac{\delta(x)}{\sqrt{t}}, 1\right) \min\left(\frac{\delta(y)}{\sqrt{t}}, 1\right) e^{c \frac{|x-y|^2}{t}}. \end{aligned}$$



So, using this fact, (1.7) and the fact that D is bounded we deduce that estimates (1.1) can be written as follows:

There exist positive constants c, C and λ such that

$$(1.11) \quad \frac{1}{C} h_{\frac{1}{c}, \lambda}(t, x, y) \leq p(t, x, y) \leq C h_{c, \lambda}(t, x, y),$$

where

$$(1.12) \quad h_{c, \lambda}(t, x, y) := \begin{cases} \min\left(\frac{\delta(x)\delta(y)}{t}, 1\right) \frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}}, & \text{if } 0 < t \leq 1 \\ \delta(x)\delta(y)e^{-\lambda t}, & \text{if } t > 1. \end{cases}$$

Throughout the paper, the letter C will denote a generic positive constant which may vary from line to line.

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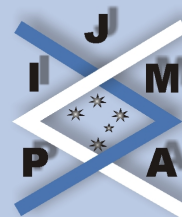


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2. Proof of Theorem 1.1

First we need the following lemma.

Lemma 2.1. For each $x, y \in D$ we have

a) For $n \geq 2m$

$$\delta(x)\delta(y) \leq \min \left(1, \frac{\delta(x)\delta(y)}{|x-y|^2} \right) \frac{d^{n-2m+2}}{|x-y|^{n-2m}}.$$

b)

$$\delta(x)\delta(y) \leq d^2 \min \left(1, \frac{\delta(x)\delta(y)}{|x-y|^2} \right) \leq 2d^2 \operatorname{Log} \left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2} \right).$$

Now we will give the proof of Theorem 1.1. More precisely, using (1.3) and (1.11) we will prove that for each $c > 0$, we have

$$\int_0^\infty t^{m-1} h_{c,\lambda}(t, x, y) dt \sim H_{m,n}(x, y).$$

Without loss of generality we will assume that $\lambda = 1$, $c = 1$ and denote by $h_{1,1}(t, x, y) = h(t, x, y)$. Hence, using a change of variable, we obtain

$$\begin{aligned} & \int_0^\infty t^{m-1} h(t, x, y) dt \\ &= C \delta(x)\delta(y) + \int_0^1 t^{m-1} \min \left(\frac{\delta(x)\delta(y)}{t}, 1 \right) \frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}}} dt \\ &= C \delta(x)\delta(y) + |x-y|^{2m-n} \int_{|x-y|^2}^\infty r^{\frac{n}{2}-m-1} \min \left(\frac{\delta(x)\delta(y)}{|x-y|^2} r, 1 \right) e^{-r} dr. \end{aligned}$$

Since we will sometimes omit e^{-r} and we need to integrate the functions $r \rightarrow r^{\frac{n}{2}-m-1}$ and $r \rightarrow r^{\frac{n}{2}-m}$ near zero or near ∞ , we will discuss the following cases

Case 1. $n > 2m$. Using (1.7) we obtain

$$\begin{aligned} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}, 1\right) \min(r, 1) &\leq \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2} r, 1\right) \\ &\leq \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}, 1\right) \max(r, 1). \end{aligned}$$

Hence the lower bound follows from the fact that

$$\int_{|x-y|^2}^{\infty} \min(1, r) r^{\frac{n}{2}-m-1} e^{-r} dr \geq \int_{d^2}^{\infty} \min(1, r) r^{\frac{n}{2}-m-1} e^{-r} dr = C$$

and the upper bound follows from Lemma 2.1.

Case 2. $n = 2m$. In this case

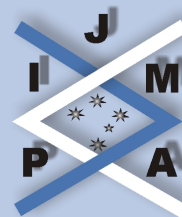
$$\int_0^{\infty} t^{m-1} h(t, x, y) dt = C \delta(x)\delta(y) + \int_{|x-y|^2}^{\infty} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2}, \frac{1}{r}\right) e^{-r} dr.$$

So using (1.5) and the fact that

$$\frac{|x-y|^2 + (2d^2 + 1)\delta(x)\delta(y)}{1 + \delta(x)\delta(y)} \geq |x-y|^2 + \delta(x)\delta(y),$$

we obtain

$$\begin{aligned} \int_0^{\infty} t^{m-1} h(t, x, y) dt &\geq \int_{|x-y|^2}^{2d^2+1} \frac{\delta(x)\delta(y)}{|x-y|^2 + r\delta(x)\delta(y)} dr \\ &= C \operatorname{Log} \left(\frac{|x-y|^2 + (2d^2 + 1)\delta(x)\delta(y)}{|x-y|^2(1 + \delta(x)\delta(y))} \right) \end{aligned}$$



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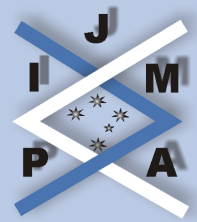


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$$\geq C \operatorname{Log} \left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2} \right).$$

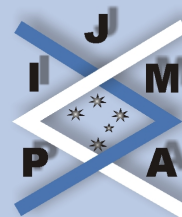
To prove the upper inequality we use (1.5), (1.11) and (1.10) to obtain

$$\begin{aligned} & \int_{|x-y|^2}^{\infty} \min \left(\frac{\delta(x)\delta(y)}{|x-y|^2}, \frac{1}{r} \right) e^{-r} dr \\ & \leq C \int_{|x-y|^2}^{\infty} \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} e^{-r} dr \\ & \leq C \int_{|x-y|^2}^{d^2+1} \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} dr + C \frac{\delta(x)\delta(y)}{[x, y]^2} \int_{1+d^2}^{\infty} e^{-r} dr \\ & = C \operatorname{Log} \left(\frac{|x-y|^2 + (d^2+1)\delta(x)\delta(y)}{|x-y|^2(1+\delta(x)\delta(y))} \right) + C \frac{\delta(x)\delta(y)}{[x, y]^2} \\ & \leq C \operatorname{Log} \left(1 + \frac{(d^2+1)\delta(x)\delta(y)}{|x-y|^2(1+\delta(x)\delta(y))} \right) + C \frac{\delta(x)\delta(y)}{|x, y|^2 + \delta(x)\delta(y)} \\ & \leq C \operatorname{Log} \left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2} \right). \end{aligned}$$

Hence the result follows from Lemma 2.1.

Case 3. $n = 2m - 1$. In this case

$$\begin{aligned} & \int_0^{\infty} t^{m-1} h(t, x, y) dt \\ & = C\delta(x)\delta(y) + |x-y| \int_{|x-y|^2}^{\infty} r^{-\frac{1}{2}} \min \left(\frac{\delta(x)\delta(y)}{|x-y|^2}, \frac{1}{r} \right) e^{-r} dr \\ & \leq C \frac{\delta(x)\delta(y)}{|x-y|} \left(d + \int_0^{\infty} r^{-\frac{1}{2}} e^{-r} dr \right) = C \frac{\delta(x)\delta(y)}{|x-y|}. \end{aligned}$$



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On the other hand, an integration by parts shows that

$$\begin{aligned}
 & |x - y| \int_{|x-y|^2}^{\infty} r^{-\frac{1}{2}} \min \left(\frac{\delta(x)\delta(y)}{|x-y|^2}, \frac{1}{r} \right) e^{-r} dr \\
 & \leq C \frac{\delta(x)\delta(y)}{|x-y|} \int_0^{d^2 \frac{|x-y|^2}{\delta(x)\delta(y)}} r^{-\frac{1}{2}} dr + |x-y| \int_{d^2 \frac{|x-y|^2}{\delta(x)\delta(y)}}^{\infty} r^{-\frac{3}{2}} e^{-r} dr \\
 & \leq C \sqrt{\delta(x)\delta(y)} + |x-y| \left[-2r^{-\frac{1}{2}} e^{-r} \right]_{d^2 \frac{|x-y|^2}{\delta(x)\delta(y)}}^{\infty} \\
 & \leq C \sqrt{\delta(x)\delta(y)}.
 \end{aligned}$$

Hence

$$\int_0^{\infty} t^{m-1} h(t, x, y) dt \leq C \min \left(\sqrt{\delta(x)\delta(y)}, \frac{\delta(x)\delta(y)}{|x-y|} \right).$$

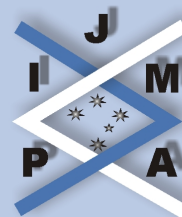
For the lower inequality we discuss two subcases

- If $\delta(x)\delta(y) \leq |x-y|^2$. Then from (1.7) we have

$$\begin{aligned}
 \int_0^{\infty} t^{m-1} h(t, x, y) dt & \geq |x-y| \min \left(1, \frac{\delta(x)\delta(y)}{|x-y|^2} \right) \int_{1+d^2}^{\infty} r^{-\frac{3}{2}} e^{-r} dr \\
 & = C \frac{\delta(x)\delta(y)}{|x-y|}.
 \end{aligned}$$

- If $|x-y|^2 \leq \delta(x)\delta(y)$. Then

$$\begin{aligned}
 \int_0^{\infty} t^{m-1} h(t, x, y) dt & \geq |x-y| \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge \frac{1}{r} \right) r^{-\frac{1}{2}} e^{-r} dr \\
 & \geq C \frac{\delta(x)\delta(y)}{|x-y|} \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} r^{-\frac{1}{2}} e^{-r} dr
 \end{aligned}$$



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$$\begin{aligned} &\geq C \frac{\delta(x)\delta(y)}{|x-y|} \int_{|x-y|^2}^{\frac{4d^2|x-y|^2}{(\delta(x)\delta(y))}} r^{-\frac{1}{2}} dr \\ &\geq C \delta(x)\delta(y) \left[\frac{2d}{\sqrt{\delta(x)\delta(y)}} - 1 \right] \\ &\geq C \sqrt{\delta(x)\delta(y)} \left[2d - \sqrt{\delta(x)\delta(y)} \right] \\ &\geq C \sqrt{\delta(x)\delta(y)}. \end{aligned}$$

Case 4. $n = 2m - 2$. In this case, we use (1.5) to deduce that

$$\begin{aligned} &\int_0^\infty t^{m-1} h(t, x, y) dt \\ &= C \delta(x)\delta(y) + |x-y|^2 \int_{|x-y|^2}^\infty \left(\frac{\delta(x)\delta(y)}{r|x-y|^2} \wedge \frac{1}{r^2} \right) e^{-r} dr. \\ &\sim \delta(x)\delta(y) + \delta(x)\delta(y) \int_{|x-y|^2}^\infty \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr. \end{aligned}$$

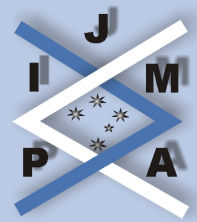
To prove the upper estimates we remark first that

$$\delta(x)\delta(y) \leq C \delta(x)\delta(y) \operatorname{Log} \left(2 + \frac{1}{[x, y]^2} \right)$$

and we discuss the following subcases

- If $\frac{1}{2} \leq \delta(x)\delta(y) \left(1 + \frac{1}{[x, y]^2} \right)$. Then $1 + \frac{1}{\delta(x)\delta(y)} \leq 4 + \frac{2}{[x, y]^2}$. So

$$\int_{|x-y|^2}^\infty \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr$$



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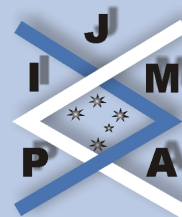
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$$\begin{aligned} &\leq \int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} \right) dr \\ &= \text{Log} \left(1 + \frac{1}{\delta(x)\delta(y)} \right) \\ &\leq \text{Log} 2 + \text{Log} \left(2 + \frac{1}{[x, y]^2} \right) \\ &\leq C \text{Log} \left(2 + \frac{1}{[x, y]^2} \right). \end{aligned}$$

- If $\delta(x)\delta(y) \left(1 + \frac{1}{[x, y]^2} \right) \leq \frac{1}{2}$. Then $\delta(x)\delta(y) ([x, y]^2 + 1) \leq \frac{1}{2} [x, y]^2$, which implies that $\delta(x)\delta(y) \leq |x - y|^2$ and consequently $[x, y]^2 \leq 2|x - y|^2$. Hence

$$\begin{aligned} &\int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr \\ &\leq C \text{Log} (1 + \delta(x)\delta(y)) e^{-|x-y|^2} + C \int_{|x-y|^2}^{\infty} \text{Log} \left(\frac{r}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr \\ &\leq C \text{Log} (1 + d^2) e^{-|x-y|^2} + C \int_{|x-y|^2}^{\infty} \text{Log} \left(\frac{r}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr \\ &\leq C \int_{|x-y|^2}^{\infty} \text{Log} \left(\frac{(1 + d^2)r}{\delta(x)\delta(y)r + |x-y|^2} \right) e^{-r} dr \\ &\leq C \int_{|x-y|^2}^{\infty} \text{Log} \left(\frac{(1 + d^2)r}{|x-y|^2(1 + \delta(x)\delta(y))} \right) e^{-r} dr \\ &\leq C \text{Log} \left(\frac{1 + d^2}{|x-y|^2} \right) + C \int_{|x-y|^2}^{\infty} \text{Log} \left(\frac{1}{1 + \delta(x)\delta(y)} r \right) e^{-r} dr \end{aligned}$$



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$$\begin{aligned} &\leq C \operatorname{Log} \left(\frac{1+d^2}{|x-y|^2} \right) + C \int_{|x-y|^2}^{\infty} \operatorname{Log}(r) e^{-r} dr \\ &\leq C \operatorname{Log} \left(\frac{1+d^2}{|x-y|^2} \right) + C \int_1^{\infty} \operatorname{Log}(r) e^{-r} dr \\ &\leq C + C \operatorname{Log} \left(\frac{1+d^2}{|x-y|^2} \right) \leq C \operatorname{Log} \left(2 + \frac{1}{[x, y]^2} \right). \end{aligned}$$

Hence

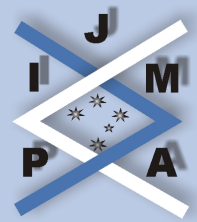
$$\int_0^{\infty} t^{m-1} h(t, x, y) dt \leq C \delta(x) \delta(y) \operatorname{Log} \left(2 + \frac{1}{[x, y]^2} \right).$$

Next we prove the lower estimates.

$$\begin{aligned} &\int_0^{\infty} t^{m-1} h(t, x, y) dt \\ &\sim \delta(x) \delta(y) + \delta(x) \delta(y) \int_{|x-y|^2}^{\infty} \left(\frac{1}{r} - \frac{\delta(x) \delta(y)}{\delta(x) \delta(y) r + |x-y|^2} \right) e^{-r} dr \\ &\geq C \delta(x) \delta(y) + C \delta(x) \delta(y) \int_{|x-y|^2}^{2d^2} \left(\frac{1}{r} - \frac{\delta(x) \delta(y)}{\delta(x) \delta(y) r + |x-y|^2} \right) dr \\ &= C \delta(x) \delta(y) + C \delta(x) \delta(y) \operatorname{Log} \left(\frac{2d^2(1 + \delta(x) \delta(y))}{|x-y|^2 + 2d^2 \delta(x) \delta(y)} \right). \end{aligned}$$

Let $\alpha > 1$ such that $\alpha \frac{2d^2}{2d^2+1} > 2[x, y]^2 + 1; \forall x, y \in D$. Then we have

$$\frac{2\alpha d^2(1 + \delta(x) \delta(y))}{|x-y|^2 + 2d^2 \delta(x) \delta(y)} \geq \frac{2\alpha d^2}{(1 + 2d^2)[x, y]^2} \geq 2 + \frac{1}{[x, y]^2}.$$



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Hence

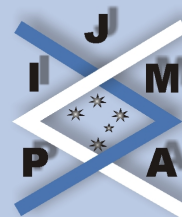
$$\begin{aligned}
 & \int_0^\infty t^{m-1} h(t, x, y) dt \\
 & \geq C \delta(x) \delta(y) \left[\text{Log } \alpha + \text{Log} \left(\frac{2d^2(1 + \delta(x)\delta(y))}{|x - y|^2 + 2d^2\delta(x)\delta(y)} \right) \right] \\
 & \geq C \delta(x) \delta(y) \text{Log} \left(\frac{2\alpha d^2(1 + \delta(x)\delta(y))}{|x - y|^2 + 2d^2\delta(x)\delta(y)} \right) \\
 & \geq C \delta(x) \delta(y) \text{Log} \left(2 + \frac{1}{[x, y]^2} \right).
 \end{aligned}$$

Case 5. $n < 2m - 2$. In this case we need only to prove the upper inequality.

$$\begin{aligned}
 & |x - y|^{2m-n} \int_{|x-y|^2}^\infty r^{\frac{n}{2}-m} \min \left(\frac{\delta(x)\delta(y)}{|x - y|^2}, \frac{1}{r} \right) e^{-r} dr \\
 & \leq \delta(x)\delta(y) |x - y|^{2m-n-2} \int_{|x-y|^2}^{d^2 \frac{|x-y|^2}{\delta(x)\delta(y)}} r^{\frac{n}{2}-m} dr + |x - y|^{2m-n} \int_{d^2 \frac{|x-y|^2}{\delta(x)\delta(y)}}^\infty r^{\frac{n}{2}-m-1} dr \\
 & \leq \frac{2}{2m - n - 2} \delta(x)\delta(y) \left[1 - \left(\frac{\delta(x)\delta(y)}{d^2} \right)^{m-1-\frac{n}{2}} \right] + \frac{2}{2m - n} \left(\frac{\delta(x)\delta(y)}{d^2} \right)^{m-\frac{n}{2}} \\
 & \leq \frac{2}{2m - n - 2} \delta(x)\delta(y) + \frac{2}{d^2(2m - n)} \left(\frac{\delta(x)\delta(y)}{d^2} \right)^{m-1-\frac{n}{2}} \delta(x)\delta(y) \\
 & \leq C \delta(x)\delta(y).
 \end{aligned}$$

This completes the proof of the theorem. \square

Now using estimates of Theorem 1.1 and similar arguments as in the proof of Corollary 2.5 in [5], we obtain the following.



Corollary 2.2. Let $r_0 > 0$. For each $x, y \in D$ such that $|x - y| \geq r_0$, we have

$$(2.1) \quad G_{m,n}(x, y) \preceq \frac{\delta(x)\delta(y)}{r_0^{n+2-2m}}.$$

Moreover, on D^2 the following estimates hold

$$\delta(x)\delta(y) \preceq G_{m,n}(x, y) \preceq \begin{cases} \frac{\delta(x)\delta(y)}{|x-y|^{n+1-2m}}, & \text{for } n \geq 2m \\ \delta(x) \wedge \delta(y), & \text{for } n \leq 2m - 1. \end{cases}$$

Iterated Green Functions

Habib Mâagli and Nouredine Zeddini

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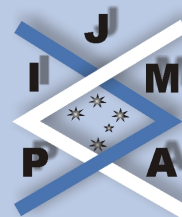
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3. The Kato Class $K_{m,n}(D)$

To give new examples of functions belonging to this class we need the following lemma

Lemma 3.1. For $\lambda, \mu \in \mathbb{R}$ and $x \in D$, let $\rho_{\lambda, \mu}(x) = \frac{1}{\delta^\lambda(x) [\text{Log}(\frac{2d}{\delta(x)})]^\mu}$. Then

$$\rho_{\lambda, \mu} \in L^1(D) \text{ if and only if } \lambda < 1 \text{ or } (\lambda = 1 \text{ and } \mu > 1).$$

Proof. Since for $\lambda < 0$ the function $\rho_{\lambda, \mu}$ is continuous and bounded in D we need only to prove the result for $\lambda \geq 0$.

Since D is a bounded $C^{1,1}$ domain and the function $t \mapsto \frac{1}{t^\lambda [\text{Log}(\frac{2d}{t})]^\mu}$ is decreasing near 0 for $\lambda > 0$, then the proof of the lemma on page 726 in [10] can be adapted. ■

Proposition 3.2. Let $m \geq 2$ and $p \in [1, \infty]$. Then $\rho_{\lambda, \mu}(\cdot) L^p(D) \subset K_{m,n}(D)$, provided that:

i) For $n \geq 2m - 1$, we have $\lambda < 2 + \frac{2(m-1)}{n} - \frac{1}{p}$ and $\frac{n}{2(m-1)} < p$.

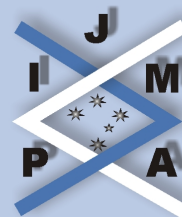
ii) For $n = 2m - 2$, we have $\lambda < 2 + \frac{n-1}{n} - \frac{1}{p}$ and $\frac{n}{n-1} < p$.

iii) For $n < 2m - 2$, we have $\lambda < 3 - \frac{1}{p}$.

Proof. Let $h \in L^p(D)$ and $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in D$ and $\alpha \in (0, 1)$, we put

$$I = I(x, \alpha) := \int_{B(x, \alpha) \cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) \rho_{\lambda, \mu}(y) h(y) dy.$$

Taking account of Theorem 1.1, we will discuss the following cases:



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Case 1. $n \geq 2m - 1$. In this case we have

$$I \preceq \int_{B(x,\alpha) \cap D} \frac{h(y)}{|x-y|^{n-2(m-1)}} \frac{dy}{\delta(y)^{\lambda-2} \left[\text{Log} \left(\frac{2d}{\delta(y)} \right) \right]^\mu}.$$

It follows from the Hölder inequality that

$$I \preceq \|h\|_p \left[\int_{B(x,\alpha) \cap D} \frac{1}{|x-y|^{(n-2(m-1))q}} \frac{dy}{\delta(y)^{(\lambda-2)q} \left[\text{Log} \left(\frac{2d}{\delta(y)} \right) \right]^{q\mu}} \right]^{\frac{1}{q}}.$$

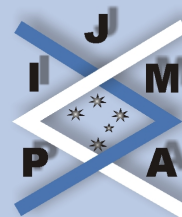
Since $\lambda < 2 + \frac{2(m-1)}{n} - \frac{1}{p}$ and $\frac{n}{2(m-1)} < p$, then $\lambda - 2 < \frac{1}{q} - \frac{n-2(m-1)}{n}$ and $q < \frac{n}{n-2(m-1)}$. Hence we can choose $q' > \max \left(1, \frac{1}{1-(\lambda-2)q} \right)$ so that $q q' < \frac{n}{n-2(m-1)}$ and $(\lambda - 2)q < 1 - \frac{1}{q'} := \frac{1}{r}$.

We apply the Hölder inequality again and Lemma 3.1 to deduce that

$$I \preceq \|h\|_p \left[\int_D \frac{dy}{\delta(y)^{(\lambda-2)qr} \left[\text{Log} \left(\frac{2d}{\delta(y)} \right) \right]^{qr\mu}} \right]^{\frac{1}{qr}} \alpha^{n-(n-2m+2)qq'}.$$

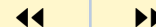
Hence $\sup_{x \in D} I(x, \alpha) \rightarrow 0$ as $\alpha \rightarrow 0$.

Case 2. $n = 2m - 2$. Assume that $\lambda < 2 + \frac{n-1}{n} - \frac{1}{p}$ and $\frac{n}{n-1} < p$, then $\lambda - 2 < \frac{1}{q} - \frac{1}{n}$ and $q < n$.



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Using (1.9), (1.6) and the Hölder inequality we obtain

$$\begin{aligned}
 I &\leq \int_{B(x,\alpha)\cap D} \left(1 + \frac{1}{[x,y]}\right) \frac{h(y)}{\delta(y)^{\lambda-2} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^\mu} dy \\
 &\leq \int_{B(x,\alpha)\cap D} \frac{1}{|x-y|} \frac{h(y)}{\delta(y)^{\lambda-2} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^\mu} dy \\
 &\leq \|h\|_p \left[\int_{B(x,\alpha)\cap D} \frac{1}{|x-y|^q} \frac{1}{\delta(y)^{(\lambda-2)q} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{q\mu}} dy \right]^{\frac{1}{q}}.
 \end{aligned}$$

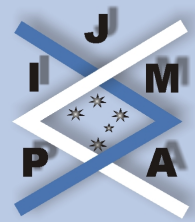
Let us choose $q' > 1$ and $r = \frac{q'}{q'-1}$ such that $qq' < n$ and $(\lambda - 2)qr < 1$. Then, using the Hölder inequality again and Lemma 3.1 we obtain

$$I \leq \|h\|_p \left[\int_D \frac{dy}{\delta(y)^{(\lambda-2)qr} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{qr\mu}} \right]^{\frac{1}{qr}} \alpha^{n-qq'}.$$

Hence $\sup_{x \in D} I(x, \alpha) \rightarrow 0$ as $\alpha \rightarrow 0$.

Case 3. $n < 2m - 2$. Using Theorem 1.1 and the Hölder inequality we obtain

$$\begin{aligned}
 I &\leq \int_{B(x,\alpha)\cap D} \frac{h(y)}{\delta(y)^{\lambda-2} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^\mu} dy \\
 &\leq \|h\|_p \left[\int_{B(x,\alpha)\cap D} \frac{1}{\delta(y)^{(\lambda-2)q} \left[\text{Log}\left(\frac{2d}{\delta(y)}\right)\right]^{q\mu}} dy \right]^{\frac{1}{q}}.
 \end{aligned}$$



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As in the preceding cases we choose $q' > 1$ so that $(\lambda - 2)qq' < 1$ to deduce from the Hölder inequality and Lemma 3.1 that $\sup_{x \in D} I(x, \alpha) \rightarrow 0$ as $\alpha \rightarrow 0$.

This completes the proof of the proposition. ■

Next, we will prove Proposition 1.2. So we need the following results

Lemma 3.3 (see [5]). *Let $x, y \in D$. Then the following properties are satisfied:*

1) *If $\delta(x)\delta(y) \leq |x - y|^2$ then*

$$\max(\delta(x), \delta(y)) \leq \frac{1 + \sqrt{5}}{2} |x - y|.$$

2) *If $|x - y|^2 \leq \delta(x)\delta(y)$ then*

$$\frac{(3 - \sqrt{5})}{2} \delta(x) \leq \delta(y) \leq \frac{(3 + \sqrt{5})}{2} \delta(x).$$

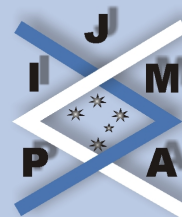
Lemma 3.4. *Let $q \in K_{m,n}(D)$. Then the function $: x \rightarrow \delta^2(x)q(x)$ is in $L^1(D)$.*

Proof. Let $q \in K_{m,n}(D)$. Then by (1.4), there exists $\alpha > 0$ such that for all $x \in D$ we have

$$\int_{(|x-y| \leq \alpha) \cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) |q(y)| dy \leq 1.$$

Let $x_1, x_2, \dots, x_p \in D$ such that $D \subset \bigcup_{i=1}^p B(x_i, \alpha)$. Then by Corollary 2.2, there exists $C > 0$ such that $\forall y \in B(x_i, \alpha) \cap D$ we have

$$\delta^2(y) \leq C \frac{\delta(y)}{\delta(x)} G_{m,n}(x_i, y).$$



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Hence

$$\begin{aligned} \int_D \delta^2(y)|q(y)|dy &\leq C \sum_{i=1}^p \int_{B(x_i,\alpha)\cap D} \frac{\delta(y)}{\delta(x_i)} G_{m,n}(x_i, y)|q(y)| dy \\ &\leq C p < \infty. \end{aligned}$$

■

Proof of Proposition 1.2. It follows from Lemmas 3.1 and 3.4 that a necessary condition for $\rho_{\lambda,\mu}$ to belong to $K_{m,n}(B)$ is that $\lambda < 3$ or ($\lambda = 3$ and $\mu > 1$). Let us prove that this condition is sufficient.

For $\lambda \leq 2$ the results follow from Proposition 3.2 by taking $p = \infty$. Hence we need only to prove the results for $2 < \lambda < 3$ or ($\lambda = 3$ and $\mu > 1$).

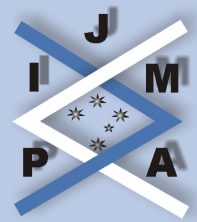
For $x \in D$ and $\alpha \in (0, 4e^{-\frac{\mu}{\lambda}})$, we put

$$\begin{aligned} I = I(x, \alpha) &:= \int_{B(x,\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) \rho_{\lambda,\mu}(y) dy \\ &= \int_{B(x,\alpha)\cap D} \frac{G_{m,n}(x, y)}{\delta(x)\delta(y)^{\lambda-1} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy. \end{aligned}$$

Taking account of Theorem 1.1 we distinguish the following cases.

Case 1. $n \geq 2m - 1$. Then we have

$$\begin{aligned} I &\preceq \int_{B(x,\alpha)\cap D_1} \frac{1}{|x-y|^{n-2(m-1)}} \frac{1}{(\delta(y))^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \\ &\quad + \int_{B(x,\alpha)\cap D_2} \frac{1}{|x-y|^{n-2(m-1)}} \frac{1}{(\delta(y))^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \end{aligned}$$



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$$= I_1 + I_2,$$

where

$$D_1 = \{x \in D : |x - y|^2 \leq \delta(x)\delta(y)\} \quad \text{and} \quad D_2 = \{x \in D : \delta(x)\delta(y) \leq |x - y|^2\}.$$

- If $y \in D_1$, then from Lemma 3.3, we have $\delta(x) \sim \delta(y)$ and so $|x - y| \preceq \delta(y)$.
Hence

$$\begin{aligned} I_1 &\preceq \int_{B(x,\alpha)} \frac{1}{|x - y|^{n-2m+\lambda} \left[\text{Log} \left(\frac{C}{|x-y|} \right) \right]^\mu} dy \\ &\preceq \int_0^\alpha \frac{r^{2m-(\lambda+1)}}{\left[\text{Log} \left(\frac{C}{r} \right) \right]^\mu} dr, \end{aligned}$$

which tends to zero as $\alpha \rightarrow 0$.

- If $y \in D_2$, then using Lemma 3.3, we have $\max(\delta(x), \delta(y)) \leq \frac{1+\sqrt{5}}{2}|x - y|$.
Hence,

$$I_2 \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^1 \frac{t^{n-1}}{(1-t)^{\lambda-2} \left[\text{Log} \left(\frac{4}{1-t} \right) \right]^\mu} \left(\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x - t\omega|^{n-2(m-1)}} \right) dt,$$

where σ is the normalized measure on the unit sphere S^{n-1} of \mathbb{R}^n .

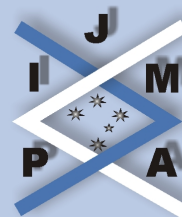
Now by elementary calculus, we have

$$\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x - t\omega|^{n-2(m-1)}} \preceq \frac{1}{(|x| \vee t)^{n-2(m-1)}} \preceq t^{2(m-1)-n}.$$

So

$$I_2 \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^1 \frac{t^{2m-3}}{(1-t)^{\lambda-2} \left[\text{Log} \left(\frac{4}{1-t} \right) \right]^\mu} dt,$$

which tends to zero as α tends to zero.



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Case 2. $n = 2m - 2$. In this case we have

$$\begin{aligned}
 I &\preceq \int_{B(x,\alpha) \cap D_1} \text{Log} \left(2 + \frac{1}{[x, y]^2} \right) \frac{1}{\delta(y)^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \\
 &\quad + \int_{B(x,\alpha) \cap D_2} \text{Log} \left(2 + \frac{1}{[x, y]^2} \right) \frac{1}{\delta(y)^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \\
 &= I_1 + I_2.
 \end{aligned}$$

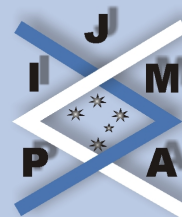
- If $y \in D_1$, it follows from the fact that $\text{Log}(2+t) \leq \sqrt{t}$ for $t \geq 2$ that

$$\begin{aligned}
 I_1 &\preceq \int_{B(x,\alpha) \cap D_1} \frac{1}{|x-y|(\delta(y))^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \\
 &\preceq \int_{B(x,\alpha) \cap D_1} \frac{1}{|x-y|^{\lambda-1} \left[\text{Log} \left(\frac{4}{|x-y|} \right) \right]^\mu} dy \\
 &\preceq \int_0^\alpha \frac{r^{n-\lambda}}{\left[\text{Log} \left(\frac{4}{r} \right) \right]^\mu} dr,
 \end{aligned}$$

which tends to zero as α tends to zero.

- If $y \in D_2$, then

$$\begin{aligned}
 I_2 &\preceq \int_{B(x,\alpha) \cap D_2} \text{Log} \left(2 + \frac{1}{|x-y|^2} \right) \frac{1}{\delta(y)^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy \\
 &\preceq \int_{B(x,\alpha) \cap D_2} \frac{1}{|x-y|^2 \delta(y)^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy
 \end{aligned}$$



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$$\begin{aligned} & \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^1 \frac{t^{n-1}}{(1-t)^{\lambda-2} \left[\text{Log} \left(\frac{4}{1-t} \right) \right]^\mu} \left(\int_{S^{n-1}} \frac{d\sigma(\omega)}{|x-t\omega|^2} \right) dt \\ & \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^1 \frac{t^{n-1}}{(1-t)^{\lambda-2} \left[\text{Log} \left(\frac{4}{1-t} \right) \right]^\mu} \frac{1}{(|x| \vee t)^2} dt \\ & \preceq \int_{1-\alpha(\frac{1+\sqrt{5}}{2})}^1 \frac{t^{n-3}}{(1-t)^{\lambda-2} \left[\text{Log} \left(\frac{4}{1-t} \right) \right]^\mu} dt, \end{aligned}$$

which tends to zero as α tends to zero.

Case 3. $n < 2m - 2$. In this case

$$I \preceq \int_{B(x,\alpha) \cap D} \frac{1}{(\delta(y))^{\lambda-2} \left[\text{Log} \left(\frac{4}{\delta(y)} \right) \right]^\mu} dy.$$

Hence the result follows from Lemma 3.3, using similar arguments as in the above cases.

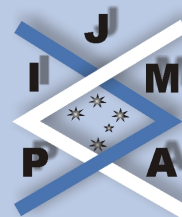
■

In the sequel we aim at proving Theorem 1.3. Below we present some preliminary results which we will need later.

Proposition 3.5.

a) For each $t > 0$ and all $x, y \in D$, we have

$$\int_0^t s^{m-1} p(s, x, y) ds \preceq G_{m,n}(x, y).$$



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b) Let $0 < t \leq 1$ and $x, y \in D$. Then

$$G_{m,n}(x, y) \preceq \int_0^t s^{m-1} p(s, x, y) ds,$$

provided that

- i) $n > 2m$ and $|x - y| \leq \sqrt{t}$; or
- ii) $n = 2m$ and $[x, y]^2 \leq t$; or
- iii) $n = 2m - 1$ and $|x - y|^2 + 2\delta(x)\delta(y) \leq t$.

Proof.

a) Follows from (1.3).

b) We deduce from (1.11) and (1.12) that

$$\int_0^t s^{m-1} p(s, x, y) ds \sim |x - y|^{2m-n} \int_{\frac{|x-y|^2}{t}}^{\infty} \min\left(\frac{\delta(x)\delta(y)}{|x-y|^2} r, 1\right) r^{\frac{n}{2}-m-1} e^{-r} dr.$$

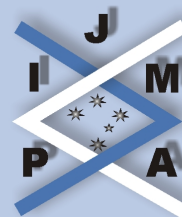
Next, we distinguish the following cases

i) $n > 2m$. In this case the result follows from (1.7) and Theorem 1.1.

ii) $n = 2m$. Using (1.5) we have

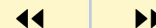
$$\begin{aligned} \int_0^t s^{m-1} p(s, x, y) ds &\geq C \int_{\frac{|x-y|^2}{t}}^2 \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)r + |x-y|^2} dr \\ &\geq C \operatorname{Log} \left(\frac{[x, y]^2 + \delta(x)\delta(y)}{\delta(x)\delta(y) + t} \cdot \frac{t}{|x-y|^2} \right). \end{aligned}$$

Now since $[x, y]^2 \leq t$ and the function $t \mapsto \frac{t}{\delta(x)\delta(y)+t}$ is nondecreasing, then the result follows from Theorem 1.1.



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iii) $n = 2m - 1$. As in the proof of Theorem 1.1 we distinguish two cases

- If $\delta(x)\delta(y) \leq |x - y|^2$. In this case the result follows from (1.7).
- If $\delta(x)\delta(y) > |x - y|^2$. Then

$$\int_0^t s^{m-1} p(s, x, y) ds \geq C \frac{\delta(x)\delta(y)}{|x - y|} \int_{\frac{|x-y|^2}{t}}^{\frac{|x-y|^2}{\delta(x)\delta(y)}} r^{-\frac{1}{2}} dr.$$

Since $|x - y|^2 + 2\delta(x)\delta(y) \leq t$, then

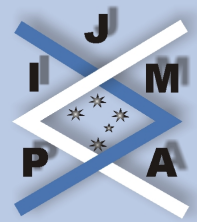
$$\left(\left(1 - \frac{1}{\sqrt{2}} \right) \frac{|x - y|}{\sqrt{\delta(x)\delta(y)}} + \frac{|x - y|}{\sqrt{t}} \right)^2 \leq \frac{|x - y|^2}{\delta(x)\delta(y)}.$$

Hence

$$\begin{aligned} \int_0^t s^{m-1} p(s, x, y) ds &\geq C \frac{\delta(x)\delta(y)}{|x - y|} \frac{|x - y|}{\sqrt{\delta(x)\delta(y)}} \\ &= C \sqrt{\delta(x)\delta(y)} \\ &\geq C G_{m,n}(x, y). \end{aligned}$$

Proposition 3.6. Let $q \in K_{m,n}(D)$. Then for each fixed $\alpha > 0$, we have

$$(3.1) \quad \sup_{t \leq 1} \left(\sup_{x \in D} \int_{(|x-y|>\alpha) \cap D} \frac{\delta(y)}{\delta(x)} p(t, x, y) |q(y)| dy \right) := M(\alpha) < \infty.$$



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Proof. Let $0 < t < 1$, $q \in K_{m,n}(D)$ and $0 < \alpha < 1$. Then using (1.11) and (1.12) we have

$$\begin{aligned} \int_{(|x-y|>\alpha)\cap D} \frac{\delta(y)}{\delta(x)} p(t, x, y) |q(y)| dy &\leq \frac{1}{t^{\frac{n}{2}+1}} \int_{(|x-y|>\alpha)\cap D} \delta^2(y) e^{-\frac{|x-y|^2}{t}} |q(y)| dy \\ &\leq \frac{e^{-\frac{\alpha^2}{t}}}{t^{\frac{n}{2}+1}} \int_D \delta^2(y) |q(y)| dy. \end{aligned}$$

Hence the result follows from Lemma 3.4. ■

Proof of Theorem 1.3. 2) \Rightarrow 1) Assume that

$$\lim_{t \rightarrow 0} \left(\sup_{x \in D} \int_D \int_0^t \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) |q(y)| ds dy \right) = 0.$$

Then by Proposition 3.5, there exists $C > 0$ such that for $\alpha > 0$ we have

$$\int_{(|x-y|\leq\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) |q(y)| dy \leq C \int_D \int_0^{\alpha^2} \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) ds dy,$$

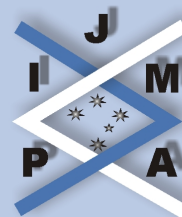
which shows that q satisfies (1.4).

1) \Rightarrow 2) Suppose that $q \in K_{m,n}(D)$ and let $\varepsilon > 0$. Then there exists $0 < \alpha < 1$ such that

$$\sup_{x \in D} \int_{(|x-y|\leq\alpha)\cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) |q(y)| dy \leq \varepsilon.$$

On the other hand, using Proposition 3.5 and (3.1), we have for $0 < t < 1$

$$\int_D \int_0^t \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) |q(y)| ds dy$$



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$$\begin{aligned} & \leq \int_{(|x-y| \leq \alpha) \cap D} \int_0^t \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) |q(y)| ds dy \\ & \quad + \int_{(|x-y| > \alpha) \cap D} \int_0^t \frac{\delta(y)}{\delta(x)} s^{m-1} p(s, x, y) |q(y)| ds dy \\ & \leq \int_{(|x-y| \leq \alpha) \cap D} \frac{\delta(y)}{\delta(x)} G_{m,n}(x, y) |q(y)| dy \\ & \quad + \int_0^t \int_{(|x-y| > \alpha) \cap D} \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| ds dy \\ & \leq \varepsilon + t M(\alpha). \end{aligned}$$

This achieves the proof. ■

Next we assume that $m = 1$ and we will give another characterization of the class $K_{1,n}(D)$.

Corollary 3.7. *Let $n \geq 3$ and q be a measurable function. For $\alpha > 0$, put*

$$G_\alpha q(x) = \int_D \int_0^\infty e^{-\alpha s} \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| ds dy, \text{ for } x \in D$$

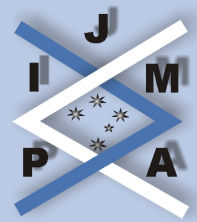
and

$$a(\alpha) = \sup_{x \in D} \int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| dy ds.$$

Then there exists $C > 0$ such that

$$\frac{1}{e} a(\alpha) \leq \|G_\alpha q\|_\infty \leq C a(\alpha),$$

where $\|G_\alpha q\|_\infty = \sup_{x \in D} |G_\alpha q(x)|$.



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In particular, we have

$$q \in K_{1,n}(D) \iff \lim_{\alpha \rightarrow \infty} \|G_{\alpha}q\|_{\infty} = 0.$$

Proof. Let $\alpha > 0$. Then using the Fubini theorem, we obtain for $x \in D$

$$\begin{aligned} G_{\alpha}q(x) &= \int_0^{\infty} \alpha e^{-\alpha t} \left[\int_0^t \int_D \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| dy ds \right] dt \\ &= \int_0^{\infty} e^{-t} \left[\int_0^{\frac{t}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| dy ds \right] dt. \end{aligned}$$

Hence, $\frac{1}{e} a(\alpha) \leq \|G_{\alpha}q\|_{\infty}$.

On the other hand if we denote by $[t]$ the integer part of t , then we have

$$\begin{aligned} G_{\alpha}q(x) &\leq \int_0^{\infty} e^{-t} \left[\sum_{k=0}^{[t]} \int_{\frac{k}{\alpha}}^{\frac{k+1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p(s, x, y) |q(y)| dy ds \right] dt \\ &\leq \int_0^{\infty} e^{-t} \left[\sum_{k=0}^{[t]} \int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p\left(s + \frac{k}{\alpha}, x, y\right) |q(y)| dy ds \right] dt. \end{aligned}$$

Now, using the Chapman-Kolmogorov identity and the Fubini theorem we obtain

$$\begin{aligned} &\int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(x)} p\left(s + \frac{k}{\alpha}, x, y\right) |q(y)| dy ds \\ &= \int_D \left(\int_0^{\frac{1}{\alpha}} \int_D \frac{\delta(y)}{\delta(z)} p(s, z, y) |q(y)| dy ds \right) \frac{\delta(z)}{\delta(x)} p\left(\frac{k}{\alpha}, x, z\right) dz \\ &\leq a(\alpha) \int_D \frac{\delta(z)}{\delta(x)} p\left(\frac{k}{\alpha}, x, z\right) dz. \end{aligned}$$

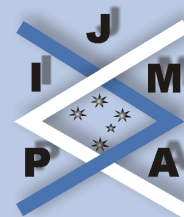
Since the first eigenfunction φ_1 associated to $-\Delta$ satisfies $\varphi_1(x) \sim \delta(x)$ and

$$\int_D p(t, x, z)\varphi_1(z)dz = e^{-\lambda_1 t}\varphi_1(x) \leq \varphi_1(x),$$

then

$$\|G_\alpha q\|_\infty \leq Ca(\alpha).$$

So, the last assertion follows from Theorem 1.3. ■



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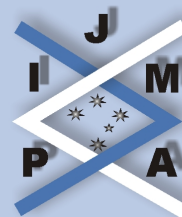
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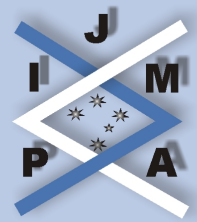


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