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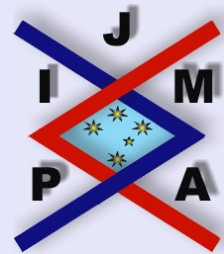
ON THE HOMOGENEOUS FUNCTIONS WITH TWO PARAMETERS AND ITS MONOTONICITY

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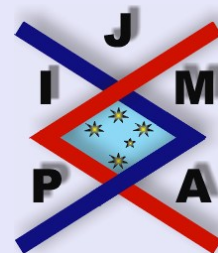


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Abstract

Suppose $f(x, y)$ is a positive homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$,

call $H_f(a, b; p, q) = \left[\frac{f(a^p, b^p)}{f(a^q, b^q)} \right]^{\frac{1}{p-q}}$ homogeneous function with two parameters.

If $f(x, y)$ is 2nd differentiable, then the monotonicity in parameters p and q of $H_f(a, b; p, q)$ depend on the signs of $I_1 = (\ln f)_{xy}$, for variable a and b depend on the sign of $I_{2a} = [(\ln f)_x \ln(y/x)]_y$ and $I_{2b} = [(\ln f)_y \ln(x/y)]_x$ respectively.

As applications of these results, a serial of inequalities for arithmetic mean, geometric mean, exponential mean, logarithmic mean, power-Exponential mean and exponential-geometric mean are deduced.

2000 Mathematics Subject Classification: Primary 26B35, 26E60; Secondary 26A48, 26D07

Key words: Homogeneous function with two parameters, f -mean with two-parameter, Monotonicity, Estimate for lower and upper bounds.

Thanks for Mr. Zhang Zhihua

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1. Introduction

The so-called two-parameter mean or extended mean values between two unequal positive numbers a and b were defined first by K.B. Stolarsky in [10] as

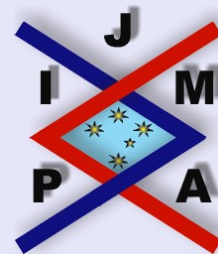
$$(1.1) \quad E(a, b; p, q) = \begin{cases} \left(\frac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ \left(\frac{a^p - b^p}{p(\ln a - \ln b)} \right)^{\frac{1}{p}} & p \neq 0, q = 0 \\ \left(\frac{a^q - b^q}{q(\ln a - \ln b)} \right)^{\frac{1}{q}} & p = 0, q \neq 0 \\ \exp\left(\frac{a^p \ln a - b^p \ln b}{a^p - b^p} - \frac{1}{p}\right) & p = q \neq 0 \\ \sqrt{ab} & p = q = 0 \end{cases} .$$

The monotonicity of $E(a, b; p, q)$ has been researched by E. B. Leach and M. C. Sholander in [4], and others also in [9, 8, 7, 6, 5, 11, 14, 15, 17] using different ideas and simpler methods.

As the generalized power-mean, C. Gini obtained a similar two-parameter type mean in [1]. That is:

$$(1.2) \quad G(a, b; p, q) = \begin{cases} \left(\frac{a^p + b^p}{a^q + b^q} \right)^{\frac{1}{p-q}} & p \neq q \\ \exp\left(\frac{a^p \ln a + b^p \ln b}{a^p + b^p}\right) & p = q \neq 0 \\ \sqrt{ab} & p = q = 0 \end{cases} .$$

Recently, the sufficient and necessary conditions comparing the two-parameter



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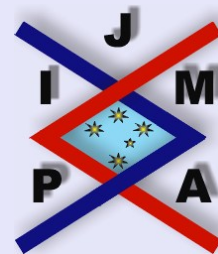
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mean with the Gini mean were put forward by using the so-called concept of “strong inequalities” ([3]).

From the above two-parameter type means, we find that their forms are both $\left(\frac{f(a^p, b^p)}{f(a^q, b^q)}\right)^{\frac{1}{p-q}}$, where $f(x, y)$ is a homogeneous function of x and y .

The main aim of this paper is to establish the concept of “two-parameter homogeneous functions”, and study the monotonicity of functions in the form of $\left(\frac{f(a^p, b^p)}{f(a^q, b^q)}\right)^{\frac{1}{p-q}}$. As applications of the main results, we will deduce three inequality chains which contain the arithmetic, geometric, exponential, logarithmic, power-exponential and exponential-geometric means, prove an upper bound for the Stolarsky mean in [12], and present two estimated expressions for the exponential mean.



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2. Basic Concepts and Main Results

Definition 2.1. Assume that $f : \mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathbb{R}_+$ is a homogeneous function of variable x and y , and is continuous and exists first order partial derivative, $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$, $(p, q) \in \mathbb{R} \times \mathbb{R}$. If $(1, 1) \notin \mathbb{U}$, then we define

$$(2.1) \quad \mathcal{H}_f(a, b; p, q) = \left[\frac{f(a^p, b^p)}{f(a^q, b^q)} \right]^{\frac{1}{p-q}} \quad (p \neq q, pq \neq 0),$$

$$(2.2) \quad \mathcal{H}_f(a, b; p, p) = \lim_{q \rightarrow p} \mathcal{H}_f(a, b; p, q) = G_{f,p}(a, b) \quad (p = q \neq 0),$$

where

$$(2.3) \quad G_{f,p}(a, b) = G_f^{\frac{1}{p}}(a^p, b^p), G_f(x, y) = \exp \left[\frac{x f_x(x, y) \ln x + y f_y(x, y) \ln y}{f(x, y)} \right],$$

$f_x(x, y)$ and $f_y(x, y)$ denote partial derivative to 1st and 2nd variable of $f(x, y)$ respectively.

If $(1, 1) \in \mathbb{U}$, then define further

$$(2.4) \quad \mathcal{H}_f(a, b; p, 0) = \left[\frac{f(a^p, b^p)}{f(1, 1)} \right]^{\frac{1}{p}} \quad (p \neq 0, q = 0),$$

$$(2.5) \quad \mathcal{H}_f(a, b; 0, q) = \left[\frac{f(a^q, b^q)}{f(1, 1)} \right]^{\frac{1}{q}} \quad (p = 0, q \neq 0),$$

$$(2.6) \quad \mathcal{H}_f(a, b; 0, 0) = \lim_{p \rightarrow 0} \mathcal{H}_f(a, b; p, 0) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} \quad (p = q = 0).$$



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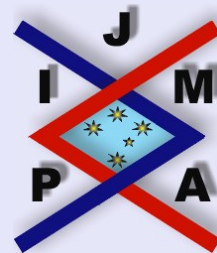


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From Lemma 3.1, $\mathcal{H}_f(a, b; p, q)$ is still a homogeneous function of positive numbers a and b . We call it a homogeneous function for positive numbers a and b with two parameters p and q , and call it a two-parameter homogeneous function for short. To avoid confusion, we also denote it by $\mathcal{H}_f(p, q)$ or $\mathcal{H}_f(a, b)$ or \mathcal{H}_f .

If $f(x, y)$ is a positive 1-order homogeneous mean function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, then call $\mathcal{H}_f(a, b; p, q)$ the two-parameter f -mean of positive numbers a and b .

Remark 1. If $f(x, y)$ is a positive 1-order homogeneous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, and is continuous and exists 1st order partial derivative, and satisfies $f(x, y) = f(y, x)$, then

$$G_{f,0}(a, b) = \mathcal{H}_f(a, b; 0, 0) = \sqrt{ab}.$$

In fact, by (2.3), we have

$$G_{f,0}(a, b) = \exp \left[\frac{f_x(1, 1) \ln a + f_y(1, 1) \ln b}{f(1, 1)} \right] = \mathcal{H}_f(a, b; 0, 0).$$

Since $f(x, y)$ is a positive 1-order homogeneous function, from (3.1) of Lemma 3.2, we obtain

$$(2.7) \quad \frac{1 \cdot f_x(1, 1)}{f(1, 1)} + \frac{1 \cdot f_y(1, 1)}{f(1, 1)} = 1.$$

If $f(x, y) = f(y, x)$, then $f_x(x, y) = f_y(y, x)$, so we have

$$(2.8) \quad f_x(1, 1) = f_y(1, 1).$$

By (2.7) and (2.8), we get

$$\frac{f_x(1, 1)}{f(1, 1)} = \frac{f_y(1, 1)}{f(1, 1)} = \frac{1}{2},$$

thereby $G_{f,0} = \sqrt{ab}$.

Thus it can be seen that despite the form of $f(x, y)$ we always have $\mathcal{H}_f(a, b; 0, 0) = G_{f,0}(a, b) = \sqrt{ab}$, so long as $f(x, y)$ is a positive 1-order homogeneous symmetric function defined on $\mathbb{R}_+ \times \mathbb{R}_+$.

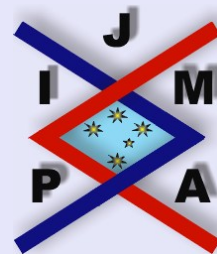
Example 2.1. In Definition 2.1, let $f(x, y) = L(x, y) = \frac{x-y}{\ln x - \ln y}$ ($x, y > 0, x \neq y$), we get (1.1), i.e.

$$(2.9) \quad \mathcal{H}_L(a, b; p, q) = \begin{cases} \left(\frac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ L^{\frac{1}{p}}(a^p, b^p) & p \neq 0, q = 0 \\ L^{\frac{1}{q}}(a^q, b^q) & p = 0, q \neq 0 \\ G_{L,p}(a, b) & p = q \neq 0 \\ G(a, b) & p = q = 0 \end{cases},$$

where

$$G_{L,p}(a, b) = E_p(a, b) = E_p^{\frac{1}{p}}(a^p, b^p) = E_p,$$

$$E(a, b) = e^{-1} \left(\frac{a^a}{b^b} \right)^{\frac{1}{a-b}}, \quad G(a, b) = \sqrt{ab}.$$



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Remark 2. *That*

$$E(a, b) = e^{-1} \left(\frac{a^a}{b^b} \right)^{\frac{1}{a-b}} \quad (a, b > 0 \text{ with } a \neq b)$$

is called the exponential mean of unequal positive numbers a and b , and is also called the identical mean and denoted by $I(a, b)$. To avoid confusion, we adopt our terms and notations in what follows.

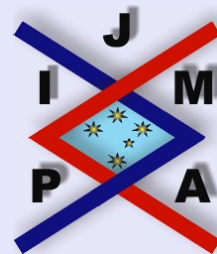
Example 2.2. In Definition 2.1, let $f(x, y) = A(x, y) = \frac{x+y}{2}$ ($x, y > 0, x \neq y$), we get (1.2), i.e.

$$(2.10) \quad \mathcal{H}_A(a, b; p, q) = \begin{cases} \left(\frac{a^p + b^p}{a^q + b^q} \right)^{\frac{1}{p-q}} & p \neq q \\ G_{A,p}(a, b) & p = q \neq 0 \\ G(a, b) & p = q = 0 \end{cases},$$

where $G_{A,p}(a, b) = Z_p(a, b) = Z_p^{\frac{1}{p}}(a^p, b^p) = Z_p$. $Z(a, b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$ is called the power-exponential mean between positive numbers a and b .

Example 2.3. In Definition 2.1, let $f(x, y) = E(x, y) = e^{-1} \left(\frac{x^x}{y^y} \right)^{\frac{1}{x-y}}$ ($x, y > 0, x \neq y$), then

$$(2.11) \quad \mathcal{H}_E(a, b; p, q) = \begin{cases} \left(\frac{E(a^p, b^p)}{E(a^q, b^q)} \right)^{\frac{1}{p-q}} & p \neq q \\ G_{E,p}(a, b) & p = q \neq 0 \\ G(a, b) & p = q = 0 \end{cases},$$



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where $G_{E,p}(a, b) = Y_p(a, b) = Y^{\frac{1}{p}}(a^p, b^p) = Y_p$. $Y(a, b) = Ee^{1-\frac{G^2}{L^2}}$ is called the exponential-geometric mean between positive numbers a and b , where $E = E(a, b)$, $L = L(a, b)$, $G = G(a, b)$.

Example 2.4. In Definition 2.1, let $f(x, y) = D(x, y) = |x - y|$ ($x, y > 0, x \neq y$), then

$$(2.12) \quad \mathcal{H}_D(a, b; p, q) = \begin{cases} \left| \frac{a^p - b^p}{a^q - b^q} \right|^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ G_{D,p}(a, b) & p = q \neq 0 \end{cases},$$

where $G_{D,p}(a, b) = G_{D,p} = e^{\frac{1}{p}} E^{\frac{1}{p}}(a^p, b^p) = e^{\frac{1}{p}} E_p$.

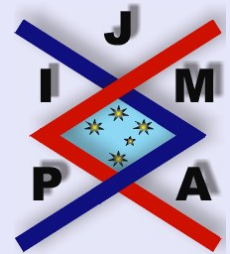
In order to avoid confusion, we rename $\mathcal{H}_L(a, b; p, q)$ (or $E(a, b; p, q)$) and $\mathcal{H}_A(a, b; p, q)$ (or $G(a, b; p, q)$) as the two-parameter logarithmic mean and two-parameter arithmetic mean respectively. In the same way, we call $\mathcal{H}_E(a, b; p, q)$ in Example 2.3 the two-parameter exponential mean.

In Example 2.4, since $D(x, y) = |x - y|$ is not a certain mean between positive numbers x and y , but one absolute value function of difference of two positive numbers, we call $\mathcal{H}_D(a, b; p, q)$ a two-parameter homogeneous function of difference.

It is obvious that the conception of two-parameter homogeneous functions has greatly developed the extension of the concept of two-parameter means.

For monotonicity of two-parameter homogeneous functions $\mathcal{H}_f(a, b; p, q)$, we have the following main results.

Theorem 2.1. Let $f(x, y)$ be a positive n -order homogenous function defined on $U(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable. If $I_1 = (\ln f)_{xy} > (<)0$,



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then $\mathcal{H}_f(p, q)$ is strictly increasing (decreasing) in both p and q on $(-\infty, 0) \cup (0, +\infty)$.

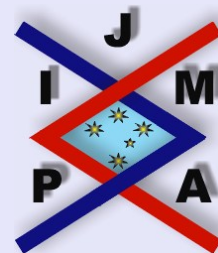
Corollary 2.2.

1. $\mathcal{H}_L(p, q), \mathcal{H}_A(p, q), \mathcal{H}_E(p, q)$ are strictly increasing both p and q on $(-\infty, +\infty)$,
2. $\mathcal{H}_D(p, q)$ is strictly decreasing both p and q on $(-\infty, 0) \cup (0, +\infty)$.

Theorem 2.3. Let $f(x, y)$ be a positive 1-order homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable.

1. If $I_{2a} = [(\ln f)_x \ln(y/x)]_y > (<)0$, then $\mathcal{H}_f(a, b)$ is strictly increasing (decreasing) in a .
2. If $I_{2b} = [(\ln f)_y \ln(x/y)]_x > (<)0$, then $\mathcal{H}_f(a, b)$ is strictly increasing (decreasing) in b .

Corollary 2.4. $\mathcal{H}_L(a, b), \mathcal{H}_D(a, b)$ is strictly increasing in both a and b .



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3. Lemmas and Proofs of the Main Results

For proving the main results in this article, we need some properties of homogeneous functions in [16]. For convenience, we quote them as follows.

Lemma 3.1. *Let $f(x, y)$, $g(x, y)$ be n, m -order homogenous functions over Ω respectively, then $f \cdot g$, f/g ($g \neq 0$) are $n + m, n - m$ -order homogenous functions over Ω respectively.*

If for a certain p with $(x^p, y^p) \in \Omega$, and $f^p(x, y)$ exists, then $f(x^p, y^p)$, $f^p(x, y)$ are both np -order homogeneous functions over Ω .

Lemma 3.2. *Let $f(x, y)$ be a n -order homogeneous function over Ω , and f_x, f_y both exist, then f_x, f_y are both $(n - 1)$ -order homogeneous functions over Ω , furthermore we have*

$$(3.1) \quad xf_x + yf_y = nf.$$

In particular, when $n = 1$ and $f(x, y)$ is second order differentiable over Ω , then

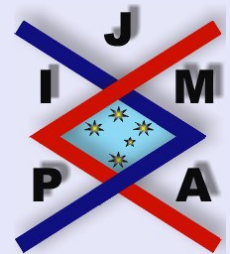
$$(3.2) \quad xf_x + yf_y = f,$$

$$(3.3) \quad xf_{xx} + yf_{xy} = 0,$$

$$(3.4) \quad xf_{xy} + yf_{yy} = 0.$$

Lemma 3.3. *Let $f(x, y)$ be a positive n -order homogenous function defined on $U(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable. Set*

$$T(t) = \ln f(a^t, b^t), \text{ where } x = a^t, y = b^t, a, b > 0,$$



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then

$$T''(t) = -xyI_1(\ln b - \ln a)^2, \text{ where } I_1 = \frac{\partial^2 \ln f(x, y)}{\partial x \partial y} = (\ln f)_{xy}.$$

Proof. Since $f(x, y)$ is a positive n -order homogeneous function, from (3.1), we can obtain $x(\ln f)_x + y(\ln f)_y = n$ or $x(\ln f)_x = n - y(\ln f)_y$, $y(\ln f)_y = n - x(\ln f)_x$, so

$$(3.5) \quad T'(t) = \frac{a^t f_x(a^t, b^t) \ln a + b^t f_y(a^t, b^t) \ln b}{f(a^t, b^t)}$$

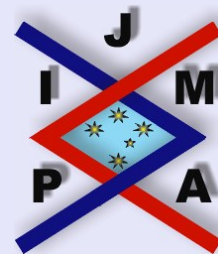
$$(3.6) \quad = \frac{x f_x(x, y) \ln a + y f_y(x, y) \ln b}{f(x, y)}$$

$$(3.7) \quad = x(\ln f)_x \ln a + y(\ln f)_y \ln b.$$

Hence

$$\begin{aligned} T''(t) &= \frac{\partial T'(t)}{\partial x} \frac{dx}{dt} + \frac{\partial T'(t)}{\partial y} \frac{dy}{dt} \\ &= [y(\ln f)_y(\ln b - \ln a) + n \ln a]_x a^t \ln a \\ &\quad + [x(\ln f)_x(\ln a - \ln b) + n \ln b]_y b^t \ln b \\ &= y(\ln f)_{yx}(\ln b - \ln a)x \ln a + x(\ln f)_{xy}(\ln a - \ln b)y \ln b \\ &= -xy(\ln f)_{xy}(\ln b - \ln a)^2 \\ &= -xyI_1(\ln b - \ln a)^2. \end{aligned}$$

□



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Lemma 3.4. Let $f(x, y)$ be a positive 1-order homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, and be second order differentiable. Set

$$S(t) = \frac{txf_x(x, y)}{f(x, y)}, \text{ where } x = a^t, y = b^t, a, b > 0,$$

then

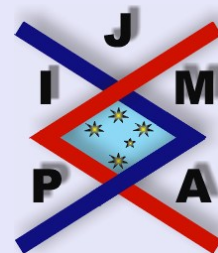
$$S'(t) = xyI_{2a}, \text{ where } I_{2a} = [(\ln f)_x \ln(y/x)]_y.$$

Proof.

$$\begin{aligned} S'(t) &= \frac{xf_x(x, y)}{f(x, y)} + t \frac{d}{dt} \left[\frac{xf_x(x, y)}{f(x, y)} \right] \\ &= x(\ln f)_x + t \left[\frac{\partial(x(\ln f)_x)}{\partial x} \frac{dx}{dt} + \frac{\partial(x(\ln f)_x)}{\partial y} \frac{dy}{dt} \right] \\ &= x(\ln f)_x + t \left[\frac{\partial(x(\ln f)_x)}{\partial x} a^t \ln a + \frac{\partial(x(\ln f)_x)}{\partial y} b^t \ln b \right] \\ &= x(\ln f)_x + t [x(x(\ln f)_x)_x \ln a + y(x(\ln f)_x)_y \ln b]. \end{aligned}$$

By Lemma 3.1, that $x(\ln f)_x = \frac{xf_x(x, y)}{f(x, y)}$ is a 0-order homogeneous function, from (3.1) of Lemma 3.2, we obtain

$$x [x(\ln f)_x]_x + y [x(\ln f)_x]_y = 0 \quad \text{or} \quad x [x(\ln f)_x]_x = -y [x(\ln f)_x]_y,$$



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hence

$$\begin{aligned}
 S'(t) &= x(\ln f)_x + ty [x(\ln f)_x]_y (\ln b - \ln a) \\
 &= x(\ln f)_x + txy(\ln f)_{xy} (\ln b - \ln a) \\
 &= x(\ln f)_x + xy(\ln f)_{xy} (\ln b^t - \ln a^t) \\
 &= x(\ln f)_x + xy(\ln f)_{xy} (\ln y - \ln x) \\
 &= xy [y^{-1}(\ln f)_x + (\ln f)_{xy} \ln(y/x)] \\
 &= xy [(\ln f)_x \ln(y/x)]_y = xyI_{2a}.
 \end{aligned}$$

□

Based on the above lemmas, then next we will go on proving the main results in this paper.

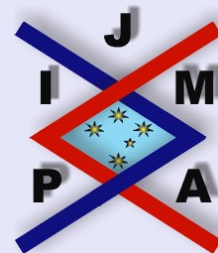
Proof of Theorem 2.1. Since $\mathcal{H}_f(p, q)$ is symmetric with respect to p and q , we only need to prove the monotonicity for p of $\ln \mathcal{H}_f$.

1) When $p \neq q$,

$$\begin{aligned}
 \ln \mathcal{H}_f &= \frac{1}{p-q} \ln \frac{f(a^p, b^p)}{f(a^q, b^q)} = \frac{T(p) - T(q)}{p-q}, \\
 \frac{\partial \ln \mathcal{H}_f}{\partial p} &= \frac{(p-q)T'(p) - T(p) + T(q)}{(p-q)^2}.
 \end{aligned}$$

Set $g(p) = (p-q)T'(p) - T(p) + T(q)$, then $g(q) = 0$, $g'(p) = (p-q)T''(p)$, and then exist $\xi = q + \theta(p-q)$ with $\theta \in (0, 1)$ by Mean-value Theorem, such that

$$\frac{\partial \ln \mathcal{H}_f}{\partial p} = \frac{g(p) - g(q)}{(p-q)^2} = \frac{g'(\xi)}{p-q} = \frac{(\xi-q)T''(\xi)}{p-q} = (1-\theta)T''(\xi).$$



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By Lemma 3.3, $T''(\xi) = -xyI_1(\ln b - \ln a)^2$, $x = a^\xi$, $y = b^\xi$. Obviously, when $I_1 < (>)0$, we get $\frac{\partial \ln \mathcal{H}_f}{\partial p} > (<)0$.

2) When $p = q$, from (2.2) and (3.6),

$$\ln \mathcal{H}_f = \ln G_f^{\frac{1}{p}}(a^p, b^p) = \frac{xf_x(x, y) \ln x + yf_y(x, y) \ln y}{f(x, y)} = T'(p),$$

$$\frac{\partial \ln \mathcal{H}_f}{\partial p} = T''(p) = -xyI_1(\ln b - \ln a)^2.$$

when $I_1 < (>)0$, we get $\frac{\partial \ln \mathcal{H}_f}{\partial p} > (<)0$.

Combining 1) with 2), the proof is completed. \square

Proof of Corollary 2.2. It follows from Theorem 2.1 that the monotonicity of $\mathcal{H}_f(p, q)$ depends on the sign of $I_1 = (\ln f)_{xy}$.

1) For $f(x, y) = L(x, y)$,

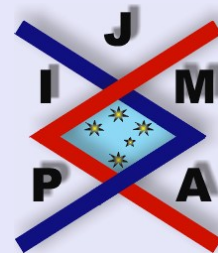
$$I_1 = (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2}$$

$$= \frac{1}{xy(x-y)^2} ((\sqrt{xy})^2 - L^2(x, y)).$$

By the well-known inequality $L(x, y) > \sqrt{xy}$ ([13]), we have $I_1 < 0$.

2) For $f(x, y) = A(x, y)$,

$$I_1 = (\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0.$$



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3) For $f(x, y) = E(x, y)$,

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^3} [2(x-y) - (x+y)(\ln x - \ln y)] \\ &= \frac{2(\ln x - \ln y)}{(x-y)^3} \left[L(x, y) - \frac{x+y}{2} \right]. \end{aligned}$$

By the well-known inequality $L(x, y) < \frac{x+y}{2}$ ([13]), we have $I_1 < 0$.

4) For $f(x, y) = D(x, y)$,

$$I_1 = (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0.$$

Applying mechanically Theorem 2.1, we immediately obtain Corollary 2.2. \square

Proof of Theorem 2.3.

1) Since

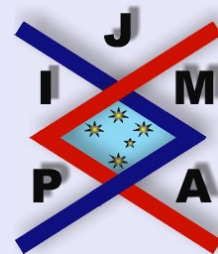
$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{1}{p-q} \left[\frac{pa^{p-1}f_x(a^p, b^p)}{f(a^p, b^p)} - \frac{qa^{q-1}f_x(a^q, b^q)}{f(a^q, b^q)} \right] = \frac{S(p) - S(q)}{a(p-q)},$$

by the Mean-value Theorem, there exists $\xi = q + \theta(p-q)$ with $\theta \in (0, 1)$, such that

$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{S(p) - S(q)}{a(p-q)} = a^{-1}S'(\xi).$$

From Lemma 3.4, $S'(\xi) = xyI_{2a}$, where $x = a^\xi, y = b^\xi$. Obviously, if $I_{2a} > 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} > 0$, so $\mathcal{H}_f(a, b)$ is strictly increasing in a ; If $I_{2a} < 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} < 0$, so $\mathcal{H}_f(a, b)$ is strictly decreasing in a .

2) It can be proved in the same way. \square



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Proof of Corollary 2.4.

1) For $f(x, y) = L(x, y)$,

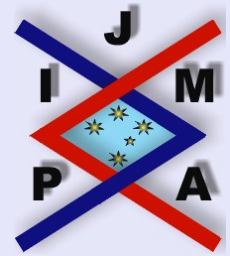
$$I_{2a} = [(\ln f)_x \ln(y/x)]_y = \frac{x/y - 1 - \ln(x/y)}{(x - y)^2}.$$

By the well-known inequality $\ln x < x - 1$ ($x > 0, x \neq 1$), we have $I_{2a} > 0$.

2) For $f(x, y) = D(x, y)$,

$$I_{2a} = [(\ln f)_x \ln(y/x)]_y = \frac{x/y - 1 - \ln(x/y)}{(x - y)^2} > 0.$$

Since $\mathcal{H}_L(a, b)$, $\mathcal{H}_D(a, b)$ are both symmetric with respect to a and b , applying mechanically Theorem 2.3, we immediately obtain Corollary 2.4. \square



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4. Some Applications

As direct applications of theorems and lemmas in this paper, we will present several examples as follows.

Example 4.1 (a G-A inequality chain). *By 1) of Corollary 2.2, for $f(x, y) = A(x, y), L(x, y)$ and $E(x, y)$, $\mathcal{H}_f(p, q)$ are strictly increasing in both p and q . So there are*

$$(4.1) \quad \mathcal{H}_f(a, b; 0, 0) < \mathcal{H}_f(a, b; 1, 0) < \mathcal{H}_f\left(a, b; 1, \frac{1}{2}\right) \\ < \mathcal{H}_f(a, b; 1, 1) < \mathcal{H}_f(a, b; 1, 2).$$

From it we can obtain the following inequalities respectively, that are

$$(4.2) \quad \sqrt{ab} < L(a, b) < \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 < E(a, b) < \frac{a + b}{2};$$

$$(4.3) \quad \sqrt{ab} < \frac{a + b}{2} < \left(\frac{a + b}{\sqrt{a} + \sqrt{b}}\right)^2 < Z(a, b) < \frac{a^2 + b^2}{a + b};$$

$$(4.4) \quad \sqrt{ab} < E(a, b) < \left[\frac{E(a, b)}{E(\sqrt{a}, \sqrt{b})}\right]^2 < Y(a, b) < \frac{E(a^2, b^2)}{E(a, b)}.$$

Notice $\frac{E(a^2, b^2)}{E(a, b)} = Z(a, b)$, then (4.4) can be written into that

$$(4.5) \quad \sqrt{ab} < E(a, b) < Z^2(\sqrt{a}, \sqrt{b}) < E \exp\left(1 - \frac{G^2}{L^2}\right) < Z(a, b).$$



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The inequality (4.2) was proved by [13], which shows that can be inserted L , $\frac{A+G}{2}$ and E between G and A , so we call (4.2) the G - A inequality chain. (4.3) and (4.4) are the same in form completely, so we call (4.1) the G - A inequality chain for homogeneous functions.

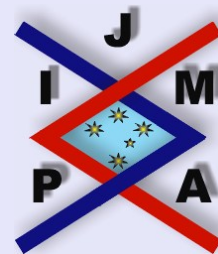
Remark 3. That $\frac{E(a^2, b^2)}{E(a, b)} = Z(a, b)$ is a new identical equation for mean. In fact,

$$\begin{aligned} E(a, b)Z(a, b) &= e^{-1} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} b^{\frac{b}{b+a}} a^{\frac{a}{b+a}} \\ &= e^{-1} b^{\frac{b}{b+a} + \frac{b}{b-a}} a^{\frac{a}{b+a} - \frac{a}{b-a}} \\ &= e^{-1} b^{\frac{2b^2}{b^2-a^2}} a^{\frac{-2a^2}{b^2-a^2}} \\ &= e^{-1} \left(\frac{(b^2)^{b^2}}{(a^2)^{a^2}} \right)^{\frac{1}{b^2-a^2}} = E(a^2, b^2). \end{aligned}$$

It shows that $Z(a, b)$ is not only one “geometric mean”, but also one ratio of one exponential mean to another. Thus inequalities involving $Z(a, b)$ may be translated into inequalities involving exponential mean.

Example 4.2 (An estimation for upper bound of Stolarsky mean). From 2) of Corollary 2.2, we can prove expediently an estimation for the upper bound of the Stolarsky mean presented by [12]:

$$S_p(a, b) < p^{\frac{1}{1-p}}(a + b) \text{ with } p > 2, \text{ where } S_p(a, b) = \left(\frac{b^p - a^p}{p(b - a)} \right)^{\frac{1}{p-1}}.$$



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In fact, from 2) of Corollary 2.2, when $p, q \in (-\infty, 0) \cup (0, +\infty)$, $\mathcal{H}_D(p, q)$ is strictly decreasing in both p and q , so when $p > 2$, we have $\mathcal{H}_D(a, b; 1, p) < \mathcal{H}_D(a, b; 1, 2)$.

Notice

$$(4.6) \quad \mathcal{H}_D(a, b; p, 1) = \left(\frac{a^p - b^p}{a - b} \right)^{\frac{1}{p-1}} = p^{\frac{1}{p-1}} S_p(a, b) \quad (p > 0),$$

thus when $p > 2$, we obtain $p^{\frac{1}{p-1}} S_p(a, b) < 2^{\frac{1}{2-1}} S_2(a, b) = a + b$, i.e. $S_p(a, b) < p^{\frac{1}{1-p}}(a + b)$.

Example 4.3 (Reversed inequalities and estimations for exponential mean).

By 1) of Corollary 2.2, $\mathcal{H}_L(p, q)$ is strictly increasing in both p and q , so when $p_1 \in (0, 1)$, $p_2 \in (1, +\infty)$, we have

$$\mathcal{H}_L(a, b; p_1, 1) < \mathcal{H}_L(a, b; 1, 1) < \mathcal{H}_L(a, b; p_2, 1),$$

i.e.

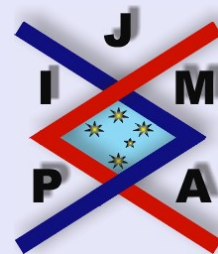
$$(4.7) \quad S_{p_1}(a, b) < E(a, b) < S_{p_2}(a, b).$$

On the other hand, By 2) of Corollary 2.2, when $p, q \in (-\infty, 0) \cup (0, +\infty)$, $\mathcal{H}_D(p, q)$ is strictly monotone decreasing in both p and q . So when $p_1 \in (0, 1)$, $p_2 \in (1, +\infty)$, we have

$$(4.8) \quad \mathcal{H}_D(a, b; p_1, 1) > \mathcal{H}_D(a, b; 1, 1) > \mathcal{H}_D(a, b; p_2, 1).$$

From (4.6), (4.8) can be written into

$$p_2^{\frac{1}{p_2-1}} S_{p_2}(a, b) < eE(a, b) < p_1^{\frac{1}{p_1-1}} S_{p_1}(a, b)$$



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or

$$(4.9) \quad \frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a, b) < E(a, b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a, b).$$

Combining (4.7) with (4.9), we have

$$(4.10) \quad S_{p_1}(a, b) < E(a, b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a, b), \text{ where } p_1 \in (0, 1),$$

$$(4.11) \quad \frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a, b) < E(a, b) < S_{p_2}(a, b), \text{ where } p_2 \in (1, +\infty).$$

In particular, when $p_1 = \frac{1}{2}, p_2 = 2$, by (4.10), (4.11), we get

$$(4.12) \quad \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 < E(a, b) < \frac{4}{e} \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2,$$

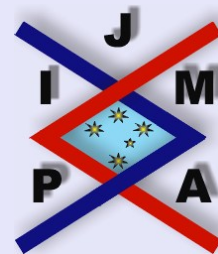
$$(4.13) \quad \frac{2}{e} \left(\frac{a+b}{2} \right) < E(a, b) < \frac{a+b}{2}.$$

The inequalities (4.12) and (4.13) may be denoted simply by

$$(4.14) \quad \frac{A+G}{2} < E < \frac{4}{e} \frac{A+G}{2},$$

$$(4.15) \quad \frac{2}{e} A < E < A.$$

The inequalities (4.14) and (4.15) make certain a bound of error that exponential mean E are estimated by A or $\frac{A+G}{2}$.



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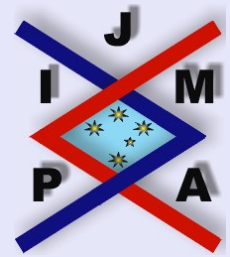
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