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ON L'HOSPITAL-TYPE RULES FOR MONOTONICITY

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ABSTRACT. Elsewhere we developed rules for the monotonicity pattern of the ratio $r := f/g$ of two differentiable functions on an interval (a, b) based on the monotonicity pattern of the ratio $\rho := f'/g'$ of the derivatives. Those rules are applicable even more broadly than l'Hospital's rules for limits, since in general we do not require that both f and g, or either of them, tend to 0 or ∞ at an endpoint or any other point of (a, b). Here new insight into the nature of the rules for monotonicity is provided by a key lemma, which implies that, if ρ is monotonic, then $\tilde{\rho} := r' \cdot g^2/|g'|$ is so; hence, r' changes sign at most once. Based on the key lemma, a number of new rules are given. One of them is as follows: Suppose that $f(a+) = g(a+) = 0$; suppose also that $\rho \nearrow\searrow$ on (a, b) – that is, for some $c \in (a, b)$, $\rho \nearrow$ (ρ is increasing) on (a, c) and $\rho \searrow$ on (c, b). Then $r \nearrow$ or $\nearrow\searrow$ on (a, b). Various applications and illustrations are given.

Key words and phrases: L'Hospital-type rules, Monotonicity, Borwein-Borwein-Rooin ratio, Becker-Stark inequalities, Anderson-Vamanamurthy-Vuorinen inequalities, log-concavity, Maclaurin series, Hyperbolic geometry, Right-angled triangles.

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1. INTRODUCTION

Let $-\infty \le a < b \le \infty$. Let f and g be differentiable functions defined on the interval (a, b), and let

$$r := \frac{f}{g}.$$

It is assumed throughout (unless specified otherwise) that g and g' do not take on the zero value and do not change their respective signs on (a, b). In [16], general "rules" for monotonicity patterns, resembling the usual l'Hospital rules for limits, were given. In particular, according to [16, Proposition 1.9], one has the dependence of the monotonicity pattern of r (on (a, b)) on

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that of

$$\rho := \frac{f'}{g'}$$

(and also on the sign of gg') as given by Table 1.1. The vertical double line in the table separates the conditions (on the left) from the corresponding conclusions (on the right).

ρ	gg'	r
$\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$
$\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\searrow$	< 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$

Table 1.1: Basic general rules for monotonicity.

Here, for instance, $r \searrow\nearrow$ means that there is some $c \in (a, b)$ such that $r \searrow$ (that is, r is decreasing) on (a, c) and $r \nearrow$ on (c, b). Now suppose that one also knows whether $r \nearrow$ or $r \searrow$ in a right neighborhood of a and in a left neighborhood of b; then Table 1.1 uniquely determines the monotonicity pattern of r.

Clearly, the stated l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are so, and even beyond that, because these monotonicity rules do not require that both f and g (or either of them) tend to 0 or ∞ at any point.

The proof of these rules is very easy if one additionally assumes that the derivatives f' and g' are continuous and r' has only finitely many roots in (a, b) (which will be the case if, for instance, r is not a constant while f and g are real-analytic functions on $[a, b]$). Such an easy proof [21, Section 1] is based on the identity

$$g^2\, r' = (\rho - r)\, g\, g', \tag{1.1}$$

which is easy to check. A proof without using the additional conditions (that the derivatives f' and g' are continuous and r' has only finitely many roots) was given in [16].

Based on Table 1.1, one can generally infer the monotonicity pattern of r given that of ρ, however complicated the latter is. In particular, one has the rules given by Table 1.2.

ρ	gg'	r
$\nearrow\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$
$\searrow\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\nearrow\searrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\searrow\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$

Table 1.2: Derived general rules for monotonicity.

Each monotonicity pattern of r in Tables 1.1 and 1.2 does actually occur; see Remark 5.12 for details.

In the special case when both f and g vanish at an endpoint of the interval (a, b), l'Hospital-type rules for monotonicity and their applications can be found, in different forms and with different proofs, in [11, 12, 13, 10, 2, 3, 1, 4, 5, 15, 16, 17, 18].

The *special-case* rule can be stated as follows: Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$; suppose also that ρ is increasing or decreasing on the entire interval (a, b); then, respectively, r is increasing or decreasing on (a, b). When the condition $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$ does hold, the special-case rule may be more convenient, because then one does not have to investigate the monotonicity pattern of ratio r near the endpoints of the interval (a, b).

A unified treatment of the monotonicity rules, applicable whether or not f and g vanish at an endpoint of (a, b), can be found in [16].

L'Hospital's rule for limits when the denominator tends to ∞ does not have a "special-case" analogue for monotonicity; see e.g. [21, Section 1] for details.

In view of what has been said here, it should not be surprising that a very wide variety of applications of these l'Hospital-type rules for monotonicity patterns were given: in areas of analytic inequalities [5, 15, 16, 19], approximation theory [17], differential geometry [10, 11, 12, 21], information theory [15, 16], (quasi)conformal mappings [1, 2, 3, 4], statistics and probability [13, 16, 17, 18], etc.

Clearly, the stated rules for monotonicity could be helpful when f' or g' can be expressed simpler than f or g, respectively. Such functions f and g are essentially the same as the functions that could be taken to play the role of u in the integration-by-parts formula $\int u\,dv = uv - \int v\,du$; this class of functions includes polynomial, logarithmic, inverse trigonometric and inverse hyperbolic functions, and as well as non-elementary "anti-derivative" functions of the form $x \mapsto c + \int_a^x h(u)\,du$ or $x \mapsto c + \int_x^b h(u)\,du$.

"Discrete" analogues, for f and g defined on $\mathbb{Z}$, of the l'Hospital-type rules for monotonicity are available as well [20].

Let us conclude this Introduction by a brief description of the contents of the paper.

Section 2 contains what is referred to in this paper as the key lemma (Lemma 2.1). This lemma provides new insight into the nature of the l'Hospital-type rules for monotonicity, as well as a basis for further developments. The key lemma states that the monotonicity pattern of function $\tilde{\rho} := r' \cdot g^2/|g'|$ is the same as that of ρ if $gg' > 0$, and opposite to the pattern of ρ if $gg' < 0$. Clearly, from this lemma, such rules as the ones given by Table 1.1 are easily deduced, since $\operatorname{sign}(r') = \operatorname{sign}\tilde{\rho}$. We present two proofs of the key lemma: one proof is short and self-contained, even if somewhat cryptic; the other proof is longer but apparently more intuitive.

In Section 3, certain shortcuts are given for the monotonicity rules based on the key lemma. As stated above, Table 1.1 uniquely determines the monotonicity pattern ($\nearrow$ or $\searrow$) of r on (a, b) provided that one knows (i) the monotonicity pattern of ρ on (a, b), (ii) the sign of gg' on (a, b), and also (iii) whether $r \nearrow$ or $r \searrow$ in a right neighborhood of a and in a left neighborhood of b. In Section 3, it is noted (Corollary 3.2) that, instead of these assumptions (i)–(iii), it suffices to know simply the signs of the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ in order to determine uniquely the monotonicity pattern of r on (a, b) – provided that ρ is monotonic on (a, b). However, if the sign of gg' on (a, b) is taken into account as well as whether ρ is increasing or decreasing on (a, b), then (Corollary 3.3) one needs to determine the sign of only one of the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$.

In Section 4, the stated special-case rule for monotonicity (with f and g both vanishing at an endpoint of the interval (a, b)) is extended (Propositions 4.3 and 4.4) to include the cases when ρ is not monotonic on (a, b) but rather has one of the patterns $\nearrow\searrow$ or $\searrow\nearrow$. Moreover, it can be allowed that both f and g vanish at an interior point, rather than at an endpoint, of the interval (Proposition 4.5). These developments are based on the key lemma, as well.

In Section 5, a general discussion concerning the interplay between the functions r, ρ, and $\tilde{\rho}$ is presented as viewed from different angles.

Finally, in Section 6, a number of applications and illustrations of the rules for monotonicity are given.

2. KEY LEMMA

Lemma 2.1 (Key lemma)**.** *The monotonicity pattern ($\nearrow$ or $\searrow$) of the function*

$$\tilde{\rho} := g^2 \frac{r'}{|g'|} \tag{2.1}$$

on (a,b) is determined by the monotonicity pattern of ρ and the sign of gg', according to Table 2.1.

ρ	gg'	$\tilde{\rho}$
$\nearrow$	> 0	$\nearrow$
$\searrow$	> 0	$\searrow$
$\nearrow$	< 0	$\searrow$
$\searrow$	< 0	$\nearrow$

Table 2.1: The monotonicity pattern of $\tilde{\rho}$ is the same as that of ρ if $gg' > 0$, and opposite to the pattern of ρ if $gg' < 0$.

Proof of Lemma 2.1. Let us verify the first line of Table 2.1. So, it is assumed that $\rho \nearrow$ and $gg' > 0$. This verification follows very closely the lines of the proof of [16, Proposition 1.2].

Fix any x and y such that

$$a < x < y < b$$

and consider the function h defined by the formula

$$h(u) := h_y(u) := f'(y)\, g(u) - g'(y)\, f(u).$$

For all $u \in (a, y)$, one has

$$h'(u) = f'(y)\, g'(u) - g'(y)\, f'(u) = g'(y)\, g'(u)\, (\rho(y) - \rho(u)) > 0,$$

because g' is nonzero and does not change sign on (a,b) and $\rho \nearrow$ on (a,b). Hence, $h \nearrow$ on (a,y); moreover, being continuous, h is increasing on $(a,y]$.

Next, one has a key identity

$$(\tilde{\rho}(y) - \tilde{\rho}(x))\, |g'(y)| = \big(h(y) - h(x)\big) + \big(\rho(y) - \rho(x)\big)\, g(x)\, g'(y);$$

here it is taken into account that g' is nonzero and does not change sign on (a,b), so that $|g'(y)|/|g'(x)| = g'(y)/g'(x)$. The first summand, $h(y) - h(x)$, on the right-hand side of this identity is positive — because $h \nearrow$ on $(a,y]$; the second summand, $[\rho(y) - \rho(x)]\; g(x)\, g'(y)$ is also positive — because $\rho \nearrow$ on (a,b) while $gg' > 0$ on (a,b) and g' does not change sign on (a,b). Thus, $\tilde{\rho}(y) > \tilde{\rho}(x)$.

This verifies the first line of Table 2.1. Its second line can be deduced from the first one by the "vertical reflection"; that is, by replacing f by $-f$ (and hence r by $-r$, while keeping g the same). The third line can be deduced from the second one by the "horizontal reflection"; that is, by "changing the variable" from x to $-x$. Finally, the fourth line can be deduced from the third one by the "vertical reflection". □

While the above proof is short and self-contained, it may seem somewhat cryptic. Let us give another version of the proof, which is longer but perhaps more illuminating (especially its Step 1). The latter proof makes use of the following technical lemma.

Lemma 2.2. *Let h be any real function h on (a, b) such that for all $x \in (a, b)$*

$$h(x) \geq h(x-) \quad \textit{and} \quad (D_+h)(x) \geq 0, \tag{2.2}$$

$$\textit{where} \quad (D_+h)(x) := \liminf_{\Delta x \downarrow 0} \frac{\Delta h}{\Delta x} \tag{2.3}$$

is the lower right Dini derivative (possibly infinite) of the function h at point x, and

$$\Delta h := (\Delta h)(x; \Delta x) := h(x + \Delta x) - h(x).$$

Then h is nondecreasing on (a, b).

Proof. This statement is essentially well known, at least when the function h is continuous; cf., e.g., [22, Example 11.3 (IV)]. The following proof is provided for the readers' convenience. For any $x \in (a, b)$ and any $\varepsilon > 0$, consider the set

$$E := E_{x,\varepsilon} := \{y \in [x, b) \colon h(u) \geq h(x) - \varepsilon \cdot (u - x) \ \forall u \in [x, y)\}.$$

Then $E \neq \emptyset$, since $x \in E$. Therefore, there exists $c := c_{x,\varepsilon} := \sup E$, and $c \in [x, b] \subseteq [x, \infty]$. It suffices to show that $c = b$ for every $\varepsilon > 0$; indeed, then one will have $h(u) \geq h(x) - \varepsilon \cdot (u - x)$ for all $u \in [x, b)$ and all $\varepsilon > 0$, whence $h(u) \geq h(x)$ for all $x \in (a, b)$ and $u \in [x, b)$.

To obtain a contradiction, assume that $c \neq b$ for some $\varepsilon > 0$. Then it is easy to see that $c \in E$, and so, $h(u) \geq h(x) - \varepsilon \cdot (u - x)$ for all $u \in [x, c)$ and hence for $u = c$ (since $h(c) \geq h(c-)$). Thus, $h(c) \geq h(x) - \varepsilon \cdot (c - x)$. On the other hand, the condition $c \neq b$ implies that $(D_+h)(c) \geq 0$, and so, there exists some $d \in (c, b)$ such that $h(u) \geq h(c) - \varepsilon \cdot (u - c)$ for all $u \in [c, d)$. It follows that $h(u) \geq h(x) - \varepsilon \cdot (u - x)$ for all $u \in [c, d)$ and hence for all $u \in [x, d)$. That is, $d \in E$ while $d > c$, which contradicts the condition $c = \sup E$. ☐

The other proof of Lemma 2.1. Again, it suffices to verify the first line of Table 2.1, so that it is assumed that $\rho \nearrow$ and $gg' > 0$ on (a, b). Note first that

$$\tilde{\rho} = (\rho\, g - f)\ \mathrm{sign}(g'). \tag{2.4}$$

Recall that $\mathrm{sign}(g')$ is constant on (a, b). The proof will be done in two steps.
Step 1: Here the first line of Table 2.1 will be verified under the additional condition that ρ is differentiable on (a, b). Then (2.4) implies

$$\tilde{\rho}' = \rho' \cdot g \cdot \mathrm{sign}(g'), \quad \text{whence} \tag{2.5}$$

$$\mathrm{sign}(\tilde{\rho}') = \mathrm{sign}(\rho'). \tag{2.6}$$

Since $\rho \nearrow$, one has $\rho' \geq 0$ and hence, by (2.6), $\tilde{\rho}' \geq 0$, so that $\tilde{\rho}$ is nondecreasing (on (a, b)). To obtain a contradiction, suppose now that the condition $\tilde{\rho} \nearrow$ fails (that is, $\tilde{\rho}$ is not *strictly* increasing on (a, b)). Then $\tilde{\rho}$ must be constant and hence $\tilde{\rho}' = 0$ on some non-empty interval $(c, d) \subset (a, b)$. It follows by (2.6) that $\rho' = 0$ on (c, d), which contradicts the condition $\rho \nearrow$.
Step 2: Here the first line of Table 2.1 will be verified without the additional condition. In view of (2.4), one has the obvious identity

$$\Delta\tilde{\rho} = \big((\Delta\rho) \cdot (g + \Delta g) + \rho \cdot \Delta g - \Delta f\big) \cdot \mathrm{sign}(g'). \tag{2.7}$$

Dividing both sides of this identity by Δx and letting $\Delta x \downarrow 0$, one has (cf. (2.5))

$$D_+\tilde{\rho} = (D_+\rho) \cdot g \cdot \mathrm{sign}(g') \geq 0,$$

because (i) the function g is differentiable and hence continuous; (ii) $gg' > 0$; (iii) $\rho\, g' = f'$; and (iv) $\rho \nearrow$ and hence $D_+\rho \geq 0$. It also follows from (2.7) that for all $x \in (a, b)$

$$\begin{aligned}\tilde{\rho}(x-) - \tilde{\rho}(x) &= \lim_{\Delta x \uparrow 0} \Delta\tilde{\rho}(x; \Delta x) \\ &= \lim_{\Delta x \uparrow 0} \Delta\rho(x; \Delta x) \cdot g(x) \cdot \operatorname{sign}(g'(x)) \leq 0,\end{aligned}$$

since $\rho \nearrow$ and $gg' > 0$. Hence, $\tilde{\rho}(x) \geq \tilde{\rho}(x-)$ for all $x \in (a, b)$. Thus, by Lemma 2.2, $\tilde{\rho}$ is nondecreasing on (a, b).

Therefore, if the condition $\tilde{\rho} \nearrow$ fails, then $\tilde{\rho}$ is constant on some non-empty interval $(c, d) \subset (a, b)$. It follows by (2.4) that $\rho\, g - f = K$ on (c, d) for some constant K, whence $\rho = (f + K)/g$ is differentiable on (c, d). Thus, according to Step 1, $\tilde{\rho} \nearrow$ on (c, d), which is a contradiction. □

3. REFINED GENERAL RULES FOR MONOTONICITY

As before, the term "general rules for monotonicity" refers to the rules valid without the special condition that both f and g vanish at an endpoint of the interval (a, b).

From the key lemma (Lemma 2.1), the general l'Hospital-type rules for monotonicity given by Table 1.1 easily follow.

Corollary 3.1. *The rules given by Table 1.1 are true.*

Proof. Indeed, consider the first line of Table 1.1. Thus, it is assumed that $\rho \nearrow$ and $gg' > 0$ on (a, b). Then, by the first line of Table 2.1, $\tilde{\rho} \nearrow$ on (a, b). Therefore, $\tilde{\rho}(x)$ may change sign only from $-$ to $+$ as x increases from a to b. In view of (2.1), the same holds with r' instead of $\tilde{\rho}$. More formally, there exists some $c \in [a, b]$ such that $r' < 0$ on (a, c) and $r' > 0$ on (c, b). Thus, either $r \nearrow$ on (a, b) (when $c = a$) or $r \searrow$ on (a, b) (when $c = b$) or $r \searrow\!\nearrow$ on (a, b) (when $c \in (a, b)$). This verifies the first line of Table 1.1. The other three lines of Table 1.1 can be verified similarly; alternatively, they can be deduced from the first line (cf. the end of the first proof of Lemma 2.1). □

As was stated in the Introduction, if one also knows whether $r \nearrow$ or $r \searrow$ in a right neighborhood of a and in a left neighborhood of b, then Table 1.1 uniquely determines the monotonicity pattern of r. Sometimes it is very easy to determine the monotonicity patterns of r near an endpoint, a or b. For example, if $r(b-) = \infty$, then it follows immediately that $r \nearrow$ in a left neighborhood of b (given the knowledge that $r \nearrow$ or $\searrow$ or $\searrow\!\nearrow$ or $\nearrow\!\searrow$ on (a, b)). Or, if it is known that $r(a+) = 0$ while $r > 0$ on (a, b), then it follows immediately that $r \nearrow$ in a right neighborhood of a.

However, in some other cases it may be not so easy to determine the monotonicity patterns of r near a or b, especially when the functions f and g depend on a number of parameters. In such situations, any additional shortcuts may prove useful. With this in mind, let us present the following corollaries to the key lemma.

Corollary 3.2. *If $\rho \nearrow$ or $\searrow$ on (a, b), then the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ always exist in $[-\infty, \infty]$, and $\tilde{\rho}(a+) \neq \tilde{\rho}(b-)$. At that, the rules given by Table 3.1 are true.*

Corollary 3.3. *The rules given by Table 3.2 are true.*

The message conveyed by Corollary 3.2 is the following. If $\rho \nearrow$ or $\searrow$ on (a, b), then the monotonicity patterns of r near the endpoints a and b (and hence on the entire interval (a, b)) are completely determined by the signs of the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$. (In particular, at that the sign of gg' is no longer relevant. Note also that the four cases in Table 3.1 concerning the signs of $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ are exhaustive. Moreover, the four cases are pairwise mutually exclusive — because $\tilde{\rho}(a+) \neq \tilde{\rho}(b-)$ and hence $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ cannot be simultaneously zero.)

$\tilde{\rho}(a+)$	$\tilde{\rho}(b-)$	r
≥ 0	≥ 0	$\nearrow$
> 0	< 0	$\nearrow\searrow$
< 0	> 0	$\searrow\nearrow$
≤ 0	≤ 0	$\searrow$

Table 3.1: If $\rho \nearrow$ or $\searrow$, then the signs of $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ determine the pattern of r on (a, b).

ρ	gg'	$\tilde{\rho}(a+)$	$\tilde{\rho}(b-)$	r'	r
$\nearrow$	> 0	≥ 0		> 0	$\nearrow$
$\nearrow$	> 0		≤ 0	< 0	$\searrow$
$\searrow$	> 0		≥ 0	> 0	$\nearrow$
$\searrow$	> 0	≤ 0		< 0	$\searrow$
$\searrow$	< 0	≥ 0		> 0	$\nearrow$
$\searrow$	< 0		≤ 0	< 0	$\searrow$
$\nearrow$	< 0		≥ 0	> 0	$\nearrow$
$\nearrow$	< 0	≤ 0		< 0	$\searrow$

Table 3.2: The content of the blank cells is not needed, and easy to restore.

On the other hand, by Corollary 3.3, if the sign of gg' *is* taken into account, then — in 8 of the $2^4 = 16$ possible cases concerning the signs of $D_+\rho$, gg', $\tilde{\rho}(a+)$, and $\tilde{\rho}(b-)$ — one needs to determine only one of the two signs, $\operatorname{sign} \tilde{\rho}(a+)$ or $\operatorname{sign} \tilde{\rho}(b-)$, depending on the case.

Note that lines 1, 4, 6, and 7 of Table 3.2 correspond to parts (1), (2), (3), and (4) of [16, Corollary 1.3], where limits superior or inferior to $\tilde{\rho}(x)$ as $x \downarrow a$ or $x \uparrow b$ are used in place of the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ (which latter we now know always exist, by Corollary 3.2, provided that $\rho \nearrow$ or $\searrow$ on (a, b)).

Proof of Corollary 3.2. If $\rho \nearrow$ or $\searrow$ then, by Table 2.1, $\tilde{\rho}$ is (strictly) monotonic (on (a, b)). Hence, the limits $\tilde{\rho}(a+)$ and $\tilde{\rho}(b-)$ exist and differ from each other. Now the rules of Table 3.1 immediately follow by Lemma 2.1 (cf. the proof of Corollary 3.1). □

Proof of Corollary 3.3. It suffices to consider only the first line of Table 3.2, so that it is assumed that $\rho \nearrow$, $gg' > 0$, and $\tilde{\rho}(a+) \geq 0$. By the first line of Table 2.1, $\tilde{\rho} \nearrow$. Hence, $\tilde{\rho}(b-) > \tilde{\rho}(a+) \geq 0$. It remains to refer to the first line of Table 3.1. □

4. DERIVED SPECIAL-CASE RULES FOR MONOTONICITY

A slightly stronger version of the basic special-case rule for monotonicity mentioned in Section 1 is

Proposition 4.1 ([15, Proposition 1.1], [16, Proposition 1.1])**.** *Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$.*

(1) *If $\rho \nearrow$ on (a, b), then $r' > 0$ and hence $r \nearrow$ on (a, b).*
(2) *If $\rho \searrow$ on (a, b), then $r' < 0$ and hence $r \searrow$ on (a, b).*

Developments presented in Section 2 provide further insight into this special-case rule as well. Indeed, in view of (2.1), Proposition 4.1 can be restated as follows.

Proposition 4.2. *Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$.*

(1) *If $\rho \nearrow$ on (a, b), then $\tilde{\rho} > 0$ on (a, b).*
(2) *If $\rho \searrow$ on (a, b), then $\tilde{\rho} < 0$ on (a, b).*

To prove Proposition 4.2, one may observe that for all $y \in (a, b)$

$$\tilde{\rho}(y) = h_y(y)/|g'(y)|,$$

where $h_y(u) = f'(y)\, g(u) - g'(y)\, f(u)$, as defined in the first proof of Lemma 2.1. In that proof, it was shown that the function h_y is increasing on $(a, y]$.

On the other hand, the condition $f(a+) = g(a+) = 0$ implies that $h_y(a+) = 0$. It follows that $h_y(y) > h_y(a+) = 0$. Hence, $\tilde{\rho}(y) > 0$ for all $y \in (a, b)$. Now (2.1) shows that indeed $r' > 0$ and hence $r \nearrow$ on (a, b). The case $f(b-) = g(b-) = 0$ is similar. The above reasoning is very close to the lines of the proof of [15, Proposition 1.1].

Whenever it is indeed the case that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$, the special-case rules are more convenient, because then one need not further investigate the behavior of ratio r near the endpoints, a and b.

The main question in this section is the following: under the same special condition — $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$, can the derived general rules given by Table 1.2 be similarly simplified?

Proposition 4.3 below shows that the answer to this question is yes. Moreover, we shall also consider the case when f and g both vanish at an interior point of the interval, rather than at one of its endpoints. To obtain these "derived" special-case rules, we shall again rely mainly on the key lemma, Lemma 2.1. We shall also rely here on the "basic" special-case rules given by Proposition 4.1 or, rather, on their re-formulation given by Proposition 4.2.

Proposition 4.3. *The special-case rules given by Table 4.1 are true.*

endpoint condition	ρ	r
$f(a+) = g(a+) = 0$	$\nearrow\searrow$	$\nearrow$ *or* $\nearrow\searrow$
$f(a+) = g(a+) = 0$	$\searrow\nearrow$	$\searrow$ *or* $\searrow\nearrow$
$f(b-) = g(b-) = 0$	$\nearrow\searrow$	$\searrow$ *or* $\nearrow\searrow$
$f(b-) = g(b-) = 0$	$\searrow\nearrow$	$\nearrow$ *or* $\searrow\nearrow$

Table 4.1: Derived special rules for monotonicity, when f and g both vanish at an endpoint.

Proof of Proposition 4.3. It suffices to consider the first line of Table 4.1, so that it is assumed that $f(a+) = g(a+) = 0$ and $\rho \nearrow\searrow$ on (a, b); that is, there exists some $c \in (a, b)$ such that $\rho \nearrow$ on (a, c) and $\rho \searrow$ on (c, b). The condition $g(a+) = 0$ implies that $gg' > 0$ on (a, b). Then, by the second line of Table 2.1, $\tilde{\rho} \searrow$ on (c, b). Also, by part (1) of Proposition 4.2, $\tilde{\rho} > 0$ on (a, c). Hence, there exists some $d \in [c, b]$ such that $\tilde{\rho} > 0$ on $(a, c) \cup (c, d)$ and $\tilde{\rho} < 0$ on (d, b). (At that, $d = b$ if $\tilde{\rho}(b-) \geq 0$ (and hence $\tilde{\rho}(c+) > 0$), and $d \in [c, b)$ if $\tilde{\rho}(b-) < 0$.) Therefore and in view of (2.1), $r' > 0$ on $(a, c) \cup (c, d)$ and $r' < 0$ on (d, b). Since r is differentiable and hence continuous on (a, b), it follows that $r \nearrow$ on (a, d) and $r \searrow$ on (d, b). Thus, if $d = b$ then $r \nearrow$ on (a, b); and if $d \in [c, b)$ then $r \nearrow\searrow$ on (a, b). $\square$

In the course of the proof of Proposition 4.3, a little more was established than stated in Proposition 4.3. Namely, based on the sign of $\tilde{\rho}(b-)$, one can discriminate between the two alternative monotonicity patterns of r given in the first line of Table 4.1; similarly, for the other three lines of Table 4.1. Thus, one has the following.

Proposition 4.4. *The special-case rules given by Table 4.2 are true.*

endpoint condition	ρ	$\tilde{\rho}(a+)$	$\tilde{\rho}(b-)$	r
$f(a+) = g(a+) = 0$	$\nearrow\searrow$		≥ 0	$\nearrow$
$f(a+) = g(a+) = 0$	$\nearrow\searrow$		< 0	$\nearrow\searrow$
$f(a+) = g(a+) = 0$	$\searrow\nearrow$		≤ 0	$\searrow$
$f(a+) = g(a+) = 0$	$\searrow\nearrow$		> 0	$\searrow\nearrow$
$f(b-) = g(b-) = 0$	$\nearrow\searrow$	≤ 0		$\searrow$
$f(b-) = g(b-) = 0$	$\nearrow\searrow$	> 0		$\nearrow\searrow$
$f(b-) = g(b-) = 0$	$\searrow\nearrow$	≥ 0		$\nearrow$
$f(b-) = g(b-) = 0$	$\searrow\nearrow$	< 0		$\searrow\nearrow$

Table 4.2: Specific derived special-case rules for monotonicity, when f and g both vanish at an endpoint.

Let us also consider the case when both f and g vanish at an interior point of the interval.

Proposition 4.5. *Suppose that the following conditions hold:*

- $-\infty \leq a < b < c \leq \infty$;
- f *and* g *are differentiable functions defined on the set* $(a, c) \setminus \{b\}$;
- *on each of the intervals* (a, b) *and* (b, c), *the functions* g *and* g' *do not take on the zero value and do not change their respective signs;*
- $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$;
- *there exists a finite limit* $\rho(b) := \lim_{x\to b} \rho(x)$ *and hence, by l'Hospital's rule, the limit* $r(b) := \lim_{x\to b} r(x) = \rho(b)$, *where* $r(x) := f(x)/g(x)$ *and* $\rho(x) := f'(x)/g'(x)$ *for* $x \in (a, c) \setminus \{b\}$, *so that the functions* r *and* ρ *are extended from* $(a, c) \setminus \{b\}$ *to* (a, c).

Then the special-case rules given by Table 4.3 concerning the monotonicity patterns of ρ *and* r *on* (a, c) *are true.*

ρ	r
$\nearrow$	$\nearrow$
$\searrow$	$\searrow$
$\searrow\nearrow$	$\nearrow$ *or* $\searrow$ *or* $\searrow\nearrow$
$\nearrow\searrow$	$\nearrow$ *or* $\searrow$ *or* $\nearrow\searrow$

Table 4.3: Derived special-case rules for monotonicity, when f and g both vanish at an interior point.

Proof of Proposition 4.5. Lines 1 and 2 of Table 4.3 follow immediately from Proposition 4.1. Line 4 can be deduced from line 3 by the "vertical reflection", that is, by replacing f by $-f$. It

remains to consider line 3. Thus, it is assumed that there exists some $\xi \in (a, c)$ such that $\rho \searrow$ on (a, ξ) and $\rho \nearrow$ on (ξ, c). One of the following three cases must occur.
Case 1: $\xi = b$. Then, by Proposition 4.1, $r \searrow$ on (a, b) and $r \nearrow$ on (b, c), so that $r \searrow\nearrow$ on (a, c).
Case 2: $\xi \in (b, c)$. Then $\rho \searrow$ on (a, b) (since $\rho \searrow$ on (a, ξ)). Hence, by Proposition 4.1, one has $r \searrow$ on (a, b). On the other hand, $\rho \searrow$ on (b, ξ) and $\rho \nearrow$ on (ξ, c). Hence, by Proposition 4.3 (line 2 of Table 4.1), $r \searrow$ or $\searrow\nearrow$ on (b, c). It follows that $r \searrow$ or $\searrow\nearrow$ on (a, c).
Case 3: $\xi \in (a, b)$. This case is similar to Case 2, but here one will conclude that $r \nearrow$ or $\searrow\nearrow$ on (a, c).

This verifies line 3 of Table 4.3. □

5. DISCUSSION

Remark 5.1. It is easy to see from the proofs of the key lemma and the rules based on it that, instead of the requirement for f and g to be differentiable on (a, b) it would be enough to assume, for instance, only that f and g are continuous and both have finite right derivatives f'_+ and g'_+ (or finite left derivatives f'_- and g'_-) on (a, b), and then use these one-side derivatives in place of f' and g'. (Cf. [15, Remark 1.2].)

One corollary of Remark 5.1 is as follows.

Corollary 5.2. *Take any $c \in (a, b)$, and let f be any convex real function on (a, b). Then the ratio $f(x)/(x-c)$ switches at most once from decreasing to increasing when x increases from c to b. Similarly, this ratio switches at most once from increasing to decreasing when x increases from a to c.*

Remark 5.3. Here Corollary 5.2 appears as a particular application of Corollary 3.1 (enhanced in accordance with Remark 5.1). However, one could, vice versa, deduce Corollary 3.1 from Corollary 5.2 by "changing the variable" from x to $X := g(x)$, so that $f(x) = F(X) := f(g^{-1}(X))$, $g(x) = X$, $r(x) = F(X)/X$, and $\rho(x) = F'(X)$.

An obvious special case of Corollary 5.2 is:

Corollary 5.4. *Take any $c \in (a, b)$, and let f be any convex real function on (a, b). Let $r_c(x) := (f(x) - f(c))/(x - c)$ for $x \in (a, b) \setminus \{c\}$, and $r_c(c) := k$, where k is an arbitrary point in the interval $[f'_-(c), f'_+(c)]$. Then the ratio $r_c(x)$ increases when x increases from a to b.*

Corollary 5.4 is immediate from Proposition 4.5 enhanced in accordance with Remark 5.1.

Remark 5.5. This remark complements Remark 5.1, which allowed using one-side derivatives of f and g in place of f' and g'. However, if g *is* differentiable on (a, b), then the phrase "and do not change their respective signs" in the assumption "g and g' do not take on the zero value and do not change their respective signs on (a, b)" stated in the beginning of Section 1 is superfluous. Indeed, if g is differentiable, then it is continuous and therefore does not change sign, since it does not take on the zero value. As for the implication

$$g' \text{ does not change sign provided that } g' \text{ does not take on the zero value},$$

it follows by the intermediate value theorem for the derivative (see e.g. [6, Theorem 5.16]), as was pointed out in [5].

Remark 5.6. Moreover, if f and g are differentiable on (a, b) and ρ is monotonic on (a, b), then ρ and $\tilde{\rho}$ are continuous on (a, b). Indeed, take any $c \in (a, b)$. Since ρ is monotonic, there exist limits $\rho(c-)$ and $\rho(c+)$. On the other hand, the ratio

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{(f(x)-f(c))/(x-c)}{(g(x)-g(c))/(x-c)}$$

tends to $\rho(c)$ as $x \to c$. Next, by the Cáuchy mean value theorem, this ratio tends to $\rho(c-)$ as $x \uparrow c$ and to $\rho(c+)$ as $x \downarrow c$. Thus, $\rho(c-) = \rho(c) = \rho(c+)$, for each $c \in (a,b)$, so that ρ is continuous on (a,b). Now it is seen that $\tilde{\rho}$ is continuous as well, since $\tilde{\rho} = (\rho g - f)\,\operatorname{sign}(g')$.

Remark 5.7. All the stated rules for monotonicity have natural "non-strict" analogues, with strict inequalities and terms "increasing" and "decreasing" replaced by the corresponding non-strict inequalities and terms "non-decreasing" and "non-increasing".

Remark 5.8. Lemma 2.1 shows that (given the sign of gg') the monotonicity pattern of $\tilde{\rho}$ is completely determined by the monotonicity pattern of ρ. It is readily seen — especially from the second proof of Lemma 2.1 — that the relation between the patterns of ρ and $\tilde{\rho}$ is reversible, so that, given the monotonicity pattern of $\tilde{\rho}$ and the sign of gg', the monotonicity pattern of ρ can be completely restored. That is, each line of Table 2.1 can be read right-to-left. For instance, if $\tilde{\rho} \nearrow$ and $gg' > 0$, then $\rho \nearrow$. Thus, given the sign of gg', the monotonicity pattern of $\tilde{\rho}$ carries the same amount of information as the monotonicity pattern of ρ.

In contrast, it should now be clear that the relation between the monotonicity patterns of r and ρ is not reversible in any reasonable sense. The pattern of ρ can be anything even if the pattern of r and the sign of gg' are given. For instance, if $\tilde{\rho}$ is positive on (a,b) then, by (2.1), $r \nearrow$ on (a,b); at that, $\tilde{\rho}$ and hence ρ can be made as "wavy" as desired. To be even more specific, let $(a,b) := (0,\infty)$ or $(-\infty,0)$, $g(x) := 1/x$, and $\tilde{\rho}(x) := 2+\sin x$, so that $\tilde{\rho} > 0$ everywhere. Next, in accordance with (2.1), let

$$\begin{aligned} r(x) &:= \int_0^x \frac{|g'(u)|}{g(u)^2}\,\tilde{\rho}(u)\,\mathrm{d}u \\ &= 1+2x-\cos x, \quad \text{whence} \\ f(x) &= g(x)\,r(x) = (1+2x-\cos x)/x \quad \text{and} \\ \rho(x) &= 1-\cos x - x\sin x, \end{aligned} \tag{5.1}$$

$x \in (-\infty,0)\cup(0,\infty)$, so that r, ρ, and $\tilde{\rho}$ can be extended to $\mathbb{R}$, by continuity. Then $r' > 0$ and hence $r \nearrow$ on $\mathbb{R}$, while ρ is "infinitely wavy" on $\mathbb{R}$, just as $\tilde{\rho}$ is; see Figures 5.1 and 5.2.

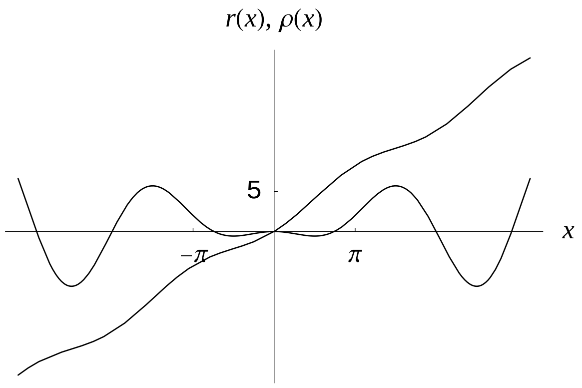

Figure 5.1: Graphs of r and ρ: r, increasing; ρ, non-monotonic, "infinitely wavy".

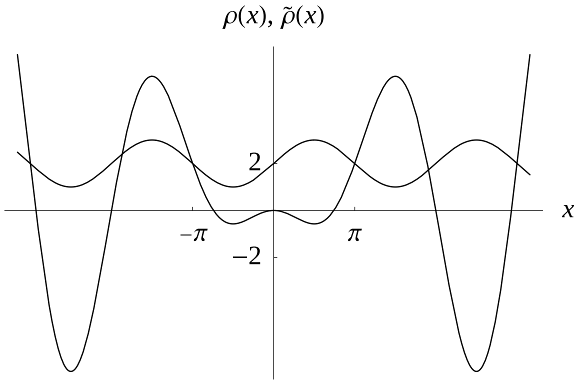

Figure 5.2: The monotonicity pattern of $\tilde\rho$ exactly follows that of ρ, and vice versa, in accordance with Table 2.1. Recall that here $\tilde\rho(x) = 2 + \sin x > 0$ for all $x \in \mathbb{R}$.

Remark 5.9. As was pointed out in [16] (see Remark 1.21 and Examples 1.2 and 1.3 therein), "the waves of r may be thought of as obtained from the waves of ρ by a certain kind of delaying and smoothing down procedure." Here, at least the "smoothing down" part is explicit in view of (5.1), since the "waves" of $\tilde\rho$ are in perfect unison with those of ρ, and hence vice versa. In this connection, one can also consider the representation

$$r(x) = \frac{r(c)g(c) + \int_c^x \rho(u)g'(u)\,du}{g(c) + \int_c^x g'(u)\,du} \quad \text{for } x \in [c,d] \subset (a,b)$$

of r on $[c,d]$, which is (in the case when $gg' > 0$) a weighted-average of the "initial" value $r(c)$ and the values of ρ on $[c,d]$.

As for the waves of r being "delayed" relative to the waves of ρ, it should be assumed that two particles are moving, one along the graph of r and the other one along the graph of ρ, left-to-right if $gg' > 0$ and right-to-left if $gg' < 0$; at that, the abscissas of the two particles are always staying equal to each other.

Remark 5.10. One can see that, under certain general conditions, ρ *must* be non-monotonic on an interval while r is monotonic on it. Indeed, suppose that $gg' > 0$ on (a,b) and r forms an increasing "half-wave" on an interval $[c,d] \subset (a,b)$; that is, $r' > 0$ on (c,d) and $r'(c) = r'(d) = 0$. Assume also that f and g are twice differentiable on (a,b), $r''(c) \neq 0$, and $r''(d) \neq 0$. It follows that $r''(c) > 0$ and $r''(d) < 0$. It is easy to check that

$$\rho = r + r'\,v, \quad \text{where} \quad v := g/g';$$

cf. [16, (1.8), (1.7)]. Then one can see that the conditions $r'(c) = r'(d) = 0$ imply $\rho(c) = r(c)$ and $\rho(d) = r(d)$. Moreover, $\rho'(c) = r''(c)\,v(c) > 0$ and $\rho'(d) = r''(d)\,v(d) < 0$, so that ρ is necessarily non-monotonic on (c,d).

See Figure 5.3, where $[c,d] := [-\pi/2, \pi/2]$, $f(x) := e^x \sin x$, and $g(x) := e^x$, so that $r(x) = \sin x$ and $\rho(x) = \sqrt{2}\,\sin(x + \pi/4)$, for all $x \in \mathbb{R}$; cf. [16, Example 1.2].

Remark 5.11. The latter example also illustrates a general situation. Indeed, without loss of generality, $g > 0$. "Changing the variable" x to $X := \ln g(x)$, one has $g(x) = e^X$, so that one may assume that $g(x) = e^x$ and hence $v(x) = 1$ for all x. Next, if r is smooth enough on a finite interval $[c,d]$ then, for any $T > d - c$, one can extend r from the interval $[c,d]$ to a smooth

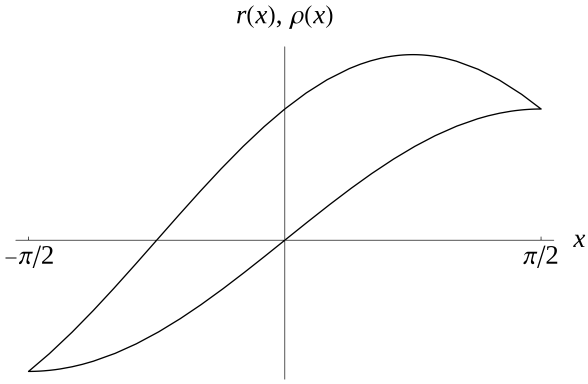

Figure 5.3: r, increasing; ρ, non-monotonic.

periodic function of period T on $\mathbb{R}$, so that one has the Fourier series representations

$$r(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nkx + B_n \sin nkx) \quad \text{and hence}$$

$$\rho(x) = A_0 + \sum_{n=1}^{\infty} \sqrt{1+n^2k^2} \left(A_n \cos(nk(x+\psi_n)) + B_n \sin(nk(x+\psi_n))\right)$$

for some real sequences (A_n) and (B_n) and all $x \in \mathbb{R}$, where $k := \frac{2\pi}{T}$ and $\psi_n := \frac{\arctan(nk)}{nk}$. Thus, with the variable x transformed into $X = \ln g(x)$, the nth harmonic component $A_n \cos nkx + B_n \sin nkx$ of r has a $\sqrt{1+n^2k^2}$ times smaller amplitude and a phase delayed by ψ_n, as compared with the amplitude and phase of the nth harmonic component of ρ, for every natural n. It also follows that ρ conveys a more powerful signal than r does, in the sense that

$$\int_c^d \rho(x)^2 \, |d \ln |g(x)|| \geq \int_c^d r(x)^2 \, |d \ln |g(x)||.$$

Remark 5.12. Note that each monotonicity pattern of r in Tables 1.1 and 1.2 does actually occur, for each set of conditions on ρ and gg'. Here let us provide a rather general description of how this can happen, suggested by the weighted-average representation of r given in Remark 5.9. For instance, consider the first line of Table 1.1, where it is assumed that $\rho \nearrow$ and $gg' > 0$ on (a, b). Suppose here also that $g > 0$, $f = f_0 + C$ for some constant C, $f_0(a+) \in \mathbb{R}$, $g(a+) \in (0, \infty)$, $\rho(a+) \in \mathbb{R}$, and $\rho(b-) = \infty$ (for example, one can take $a = 0$, $b = \infty$, $g(x) = 1 + x$, and $f_0(x) = e^x$ for all $x > 0$). Let $C_0 := \rho(a+)\, g(a+) - f_0(a+)$. If $C > C_0$, then $\rho(a+) < r(a+)$, so that, in view of identity (1.1), $r' < 0$ and hence $r \searrow$ in a right neighborhood of a. Now the first line of Table 1.1 implies that $r \searrow$ or $\searrow\nearrow$ on (a, b). Moreover, since $\rho \nearrow$ and $\rho(b-) = \infty$, the pattern $r \searrow$ on (a, b) would imply that in a left neighborhood of b one has $\rho > r$ and hence, by (1.1), $r \nearrow$, which is a contradiction. This leaves the pattern $r \searrow\nearrow$ on (a, b) as the only possibility; that is, $r \searrow$ on (a, c) and $r \nearrow$ on (c, b), for some $c \in (a, b)$, so that each of the patterns $\searrow\nearrow$, $\searrow$, and $\nearrow$ does occur for r.

6. APPLICATIONS AND ILLUSTRATIONS

6.1. **Monotonicity properties of a ratio considered by Borwein, Borwein and Rooin.** Borwein *et al.* [9] showed that the ratio

$$\frac{a^x - b^x}{c^x - d^x}, \tag{6.1}$$

$x \neq 0$ (extended to $x = 0$ by continuity), is convex in $x \in \mathbb{R}$ provided that

$$a > b \geq c > d > 0. \tag{6.2}$$

They also determined the values of a, b, c, and d for which ratio (6.1) is log-convex.

Moreover, it was shown in [9] that ratio (6.1) is increasing in $x \in \mathbb{R}$ under condition (6.2). Here the monotonicity pattern of ratio (6.1) will be determined for any positive values of a, b, c, and d, whether condition (6.2) holds or not. Dividing both the numerator and denominator of ratio (6.1) by d^x, one may assume without loss of generality that $d = 1$. Denoting then c^x by y, one rewrites ratio (6.1) as

$$r(y) := \frac{y^\beta - y^\alpha}{y - 1} \tag{6.3}$$

for $y \in (0,1) \cup (1,\infty)$ and $r(1) := \lim_{y\to 1} r(y) = \beta - \alpha$, where $\alpha := \frac{\ln b}{\ln c}$ and $\beta := \frac{\ln a}{\ln c}$. Without loss of generality, it will be assumed that

$$\beta > \alpha.$$

Proposition 6.1. *The monotonicity pattern of ratio r in* (6.3) *is given by Table 6.1, where the trivial case with $\alpha = 0$ and $\beta = 1$ must be excluded.*

Case	r
I. $\alpha \leq 0$, $\beta \leq 1$	$\searrow$
II. $\alpha < 0$, $\beta > 1$	$\searrow\nearrow$
III. $\alpha > 0$, $\beta < 1$	$\nearrow\searrow$
IV. $\alpha \geq 0$, $\beta \geq 1$	$\nearrow$

Table 6.1: The monotonicity pattern of ratio r in (6.3).

Note that condition (6.2) corresponds to the case when $\beta > \alpha \geq 1$, which is a subcase of Case IV of Table 6.1.

Proof of Proposition 6.1. Let $f(y) := y^\beta - y^\alpha$ and $g(y) := y - 1$, so that f/g equals the ratio r in (6.3). Then

$$\rho(y) = f'(y)/g'(y) = \beta y^{\beta-1} - \alpha y^{\alpha-1} \qquad \text{and}$$
$$\rho'(y) = \big(\beta(\beta-1)y^{\beta-\alpha} - \alpha(\alpha-1)\big)\, y^{\alpha-2}.$$

Hence,

$$y_* := \left(\frac{\alpha(\alpha-1)}{\beta(\beta-1)}\right)^{\frac{1}{\beta-\alpha}}$$

is the only root of ρ' in $(0,\infty)$ provided that $\alpha(\alpha-1)\beta(\beta-1) > 0$; otherwise, ρ' has no root in $(0,\infty)$.

For each of the Cases I and IV in Table 6.1, two subcases will be considered. At that, remember the assumption $\beta > \alpha$.

Subcase I.1: $\alpha \le 0$ and $\beta \le 0$, so that $\alpha < \beta \le 0$. Here $\alpha(\alpha-1) > 0$ and $\beta(\beta-1) \ge 0$. Hence, for all $y > 0$, one has $\rho'(y) < 0$ iff $y < y_*$ (letting $y_* := \infty$ if $\beta = 0$). Therefore, $\rho \searrow\nearrow$ on $(0,\infty)$ ($\rho \searrow$ on $(0,\infty)$ if $\beta = 0$). It follows by Proposition 4.5 that $r \nearrow$ or $\searrow$ or $\searrow\nearrow$ on $(0,\infty)$. Also, $r(\infty-) = 0$ while $r > 0$ on $(1,\infty)$, so that $r \searrow$ in a left neighborhood of ∞. Thus, $r \searrow$ on $(0,\infty)$ in Subcase I.1.
Subcase I.2: $\alpha \le 0$ and $0 < \beta \le 1$, so that $\alpha \le 0 < \beta \le 1$ (but $(\alpha,\beta) \ne (0,1)$). Here $\rho' < 0$ and hence $\rho \searrow$ on $(0,\infty)$. Thus, by Proposition 4.5, $r \searrow$ on $(0,\infty)$ in Subcase I.2 as well.
Case II. $\alpha < 0$ and $\beta > 1$. Here, for all $y > 0$, one has $\rho'(y) < 0$ iff $y < y_*$. Therefore, $\rho \searrow\nearrow$ on $(0,\infty)$. It follows by Proposition 4.5 that $r \nearrow$ or $\searrow$ or $\searrow\nearrow$ on $(0,\infty)$. Also, here $r(0+) = r(\infty-) = \infty$. Thus, $r \searrow\nearrow$ on $(0,\infty)$ in Case II.
Case III. $\alpha > 0$ and $\beta < 1$, so that $0 < \alpha < \beta < 1$. Here, for all $y > 0$, one has $\rho'(y) > 0$ iff $y < y_*$. Therefore, $\rho \nearrow\searrow$ on $(0,\infty)$. It follows by Proposition 4.5 that $r \nearrow$ or $\searrow$ or $\nearrow\searrow$ on $(0,\infty)$. Also, here $r(0+) = r(\infty-) = 0$ and $r > 0$ on $(0,\infty)$. Thus, $r \nearrow\searrow$ on $(0,\infty)$ in Case III.
Subcase IV.1: $0 \le \alpha < 1$ and $\beta \ge 1$, so that $0 \le \alpha < 1 \le \beta$ (but $(\alpha,\beta) \ne (0,1)$). Here $\rho' > 0$ and hence $\rho \nearrow$ on $(0,\infty)$. Thus, by Proposition 4.5, $r \nearrow$ on $(0,\infty)$ in Subcase IV.1.
Subcase IV.2: $\alpha \ge 1$ and $\beta \ge 1$, so that $1 \le \alpha < \beta$. Here, for all $y > 0$, one has $\rho'(y) < 0$ iff $y < y_*$. Therefore, $\rho \searrow\nearrow$ on $(0,\infty)$ ($\rho \nearrow$ on $(0,\infty)$ if $\alpha = 1$). It follows by Proposition 4.5 that $r \nearrow$ or $\searrow$ or $\searrow\nearrow$ on $(0,\infty)$. Also, here $r(0+) = 0$ and $r > 0$ on $(0,\infty)$. Thus, $r \nearrow$ on $(0,\infty)$ in Subcase IV.2 as well. □

The matter of the convexity of ratio (6.1) without condition (6.2) is more complicated and will not be pursued here.

6.2. Monotonicity and log-concavity properties of the partial sum of the Maclaurin series for e^x.

For $x \in \mathbb{R}$ and $k \in \{0,1,\dots\}$, consider

$$S_k(x) := \sum_{j=0}^{k-1} \frac{x^j}{j!},$$

the kth partial sum for the Maclaurin series for e^x, where $0^0 := 1$ and $S_0 := 0$. For all $k \in \{1,2,\dots\}$, one has $S_k' = S_{k-1}$ and $S_k(x) > 0$ if $x \ge 0$.

Consider the ratio

$$s_k := \frac{S_{k+1}}{S_k}$$

on $(0,\infty)$. Applying Proposition 4.1 to this ratio k times and observing that $s_1(x) = 1 + x$ is increasing in x, one obtains

Proposition 6.2. *For each $k \in \{1,2,\dots\}$, one has $s_k' > 0$ and hence $s_k \nearrow$ on $(0,\infty)$.*

Since $s_k' = 1 - S_{k+1}S_{k-1}/S_k^2$, one obtains

Corollary 6.3. *For each $x > 0$, the partial sum $S_k(x)$ is strictly log-concave in $k \in \{1,2,\dots\}$.*

Corollary 6.3 also follows from results of [20].

6.3. Monotonicity and log-concavity properties of the remainder in the Maclaurin series for e^x.

For $x \in \mathbb{R}$ and $k \in \{0,1,\dots\}$, consider

$$R_k(x) := e^x - \sum_{j=0}^{k-1} \frac{x^j}{j!},$$

the kth remainder for the Maclaurin series for e^x. For all $k \in \{1, 2, \dots\}$, one has $R_k' = R_{k-1}$ and $R_k(0) = 0$; also, $R_0(x) = e^x > 0$, so that $\operatorname{sign} R_k(x) = 1$ if $x > 0$ and $\operatorname{sign} R_k(x) = (-1)^k$ if $x < 0$.

Consider the ratio

$$r_k := \frac{R_{k+1}}{R_k},$$

extended from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ by continuity. Applying Proposition 4.5 to this ratio k times and observing that $r_0(x) = 1 - e^{-x}$ is increasing in $x \in \mathbb{R}$, one obtains

Proposition 6.4. *For each $k \in \{0, 1, \dots\}$, the ratio r_k is increasing on $\mathbb{R}$.*

Since $r_k' = 1 - R_{k+1}R_{k-1}/R_k^2$, one has

Corollary 6.5. *For each $x \neq 0$, the remainder $|R_k(x)|$ is log-concave in $k \in \{0, 1, \dots\}$.*

Following along the lines of the proof of Proposition 4.5, one can show that $|R_k(x)|$ is actually strictly log-concave in $k \in \{0, 1, \dots\}$ for each real $x \neq 0$. Corollary 6.5 also follows from results of [14, 20].

6.4. **Becker-Stark and Anderson-Vamanamurthy-Vuorinen inequalities and related monotonicity properties.** Using series expansions based on complex analysis, Becker and Stark [8] obtained the inequalities

$$\frac{4}{\pi}\,\frac{x}{1-x^2} < \tan\left(\frac{\pi x}{2}\right) < \frac{\pi}{2}\,\frac{x}{1-x^2} \quad \text{for} \quad x \in (0,1) \tag{6.4}$$

as a two-sided rational approximation to the tangent function. This approximation is rather tight, since the ratio of the upper and lower bounds in (6.4) is $\frac{\pi}{2}/\frac{4}{\pi} = 1.233\dots$. Moreover, as noted in [8], the constant factors $\frac{4}{\pi}$ and $\frac{\pi}{2}$ in (6.4) are the best possible ones.

Anderson, Vamanamurthy and Vuorinen [5] obtained another nice inequality:

$$\left(\frac{\sin x}{x}\right)^3 > \cos x \quad \text{for} \quad x \in (0, \pi/2), \tag{6.5}$$

whose hyperbolic counterpart,

$$\left(\frac{\sinh x}{x}\right)^3 > \cosh x \quad \text{for} \quad x > 0, \tag{6.6}$$

was implicit in [5].

Here we provide monotonicity properties for appropriate ratios, which imply inequalities (6.4), (6.5), and (6.6) in a quite elementary way. As will be seen from our proof, inequalities (6.4) turn out to be indirectly related with (6.5) and (6.6).

Let us begin with the monotonicity properties pertaining to inequalities (6.5) and (6.6).

Proposition 6.6. *The ratio*

$$\frac{\left(\frac{\sin x}{x}\right)^3}{\cos x}$$

increases from 1 to ∞ as x increases from 0 to $\pi/2$.

Proof. The cubic root of this ratio is the ratio $r(x) := \frac{\sin x\,\cos^{-1/3} x}{x}$, whose derivative ratio $\rho(x) = \frac{2}{3}\cos^{2/3} x + \frac{1}{3}\cos^{-4/3} x$ is increasing in $x \in (0, \pi/2)$. It remains to refer to the special-case rule for monotonicity (Proposition 4.1). □

Quite similarly one can prove

Proposition 6.7. *The ratio*

$$\frac{\left(\frac{\sinh x}{x}\right)^3}{\cosh x}$$

increases from 1 *to* ∞ *as* x *increases from* 0 *to* ∞.

Clearly, inequalities (6.5) and (6.6) immediately follow from Propositions 6.6 and 6.7, respectively.

Now one is prepared to consider the monotonicity property pertaining to inequalities (6.4).

Proposition 6.8. *The ratio*

$$r(x) := \frac{\frac{x}{1-x^2}}{\tan(\pi x/2)}$$

increases from $2/\pi$ *to* $\pi/4$ *as* x *increases from* 0 *to* $\pi/2$. *Hence, one has inequalities* (6.4) *and also the mentioned fact that the constant factors* $\frac{4}{\pi}$ *and* $\frac{\pi}{2}$ *in* (6.4) *are the best possible ones.*

Proof. Let $f(x) := \cot(\pi x/2)$ and $g(x) := (1-x^2)/x$ for $x \in (0,1)$, so that $f/g = r$. Let

$$r_1(x) := \rho(x) = \frac{f'(x)}{g'(x)} = \frac{f_1(x)}{g_1(x)},$$

where $f_1(x) := \pi \sin^{-2}(\pi x/2)$ and $g_1(x) := 2 + 2x^{-2}$, $x \in (0,1)$. Consider also

$$\tilde{\rho} = g^2 \frac{r'}{|g'|}, \quad \tilde{\rho}_1 := g_1^2 \frac{r_1'}{|g_1'|}, \quad \text{and} \quad \rho_1(x) := \frac{f_1'(x)}{g_1'(x)} = \frac{2}{\pi} \frac{\cos t}{\left(\frac{\sin t}{t}\right)^3},$$

where $x \in (0,1)$ and $t := \pi x/2$, so that $\rho_1 \searrow$ on $(0,1)$, by Proposition 6.6. Also, $\tilde{\rho}_1(0+) = \frac{\pi}{3} - \frac{4}{\pi} < 0$ and $\tilde{\rho}_1(1-) = \pi > 0$. Hence, by Corollary 3.2 (Table 3.1, line 3), $r_1 \searrow\nearrow$ on $(0,1)$; that is, $\rho \searrow\nearrow$ on $(0,1)$. Next, $\tilde{\rho}(0+) = 0$. Therefore, by Proposition 4.4 (Table 4.2, line 7), $r \nearrow$ on $(0,1)$. □

This proof of Proposition 6.8 provides a good illustration of the monotonicity rules developed in Sections 3 and 4.

6.5. **A monotonicity property of right-angled triangles in hyperbolic geometry.** The Pythagoras theorem for the Poincaré model of hyperbolic geometry (see e.g. [7, Theorem 7.11.1]) states that for any right-angled (geodesic) triangle with a hypotenuse (of geodesic length) c and catheti a and b one has

$$\cosh c = \cosh a \, \cosh b.$$

Proposition 6.9. *For the isosceles (with* $a = b$*) right-angled hyperbolic triangle, the ratio* c/a *increases from* $\sqrt{2}$ *to* 2 *as* a *increases from* 0 *to* ∞.

Proof. For $a > 0$, let $f(a) := \operatorname{arccosh}(\cosh^2 a)$ and $g(a) := a$, so that

$$\frac{c}{a} = \frac{f(a)}{g(a)} = r(a) \quad \text{and hence} \quad \rho(a) = \frac{f'(a)}{g'(a)} = \frac{2\cosh a}{\sqrt{1+\cosh^2 a}}.$$

Therefore, $\rho(a)$ increases from $\sqrt{2}$ to 2 as a increases from 0 to ∞. The same holds for $r(a)$, by the special-case rule for monotonicity (Proposition 4.1) and l'Hospital's rules for limits. □

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