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SHARP CONSTANTS FOR SOME INEQUALITIES CONNECTED TO THE P-LAPLACE OPERATOR

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Abstract

In this paper we investigate a set of structure conditions used in the existence theory of differential equations. More specific, we find best constants for the corresponding inequalities in the special case when the differential operator is the p-Laplace operator.

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Contents

1. Introduction

When dealing with certain nonlinear boundary value problems of the kind

$$\begin{cases} -\operatorname{div}\left(A\left(x,\nabla u\right)\right)=f \text{ on } \Omega\subset\mathbb{R}^n, \\ u\in H_0^{1,p}\left(\Omega\right),\ 1<p<\infty, \end{cases}$$

it is common to assume that the function $A:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies suitable continuity and monotonicity conditions in order to prove existence and uniqueness of solutions, see e.g. the books [6], [9], [11] and [12]. For C_1^* and C_2^* finite and positive constants, a popular set of such structure conditions are the following:

$$\left|A\left(x,\xi_1\right)-A\left(x,\xi_2\right)\right|\leq C_1^*\left(\left|\xi_1\right|+\left|\xi_2\right|\right)^{p-1-\alpha}\left|\xi_1-\xi_2\right|^{\alpha},$$
$$\left\langle A\left(x,\xi_1\right)-A\left(x,\xi_2\right),\xi_1-\xi_2\right\rangle\geq C_2^*\left(\left|\xi_1\right|+\left|\xi_2\right|\right)^{p-\beta}\left|\xi_1-\xi_2\right|^{\beta},$$

where $0\leq\alpha\leq\min\left(1,p-1\right)$ and $\max\left(p,2\right)\leq\beta<\infty$. See for instance the articles [1], [2], [3], [4], [7], [8] and [10], where these conditions (or related variants) are used in the theory of homogenization. It is well known that the corresponding function

$$A\left(x,\nabla u\right)=\left|\nabla u\right|^{p-2}\nabla u$$

for the p-Poisson equation satisfies these conditions, see e.g. [12], but the best possible constants C_1^* and C_2^* are in general not known. In this article we prove that the best constants C_1 and C_2 for the inequalities

$$\left|\left|\xi_1\right|^{p-2}\xi_1-\left|\xi_2\right|^{p-2}\xi_2\right|\leq C_1\left(\left|\xi_1\right|+\left|\xi_2\right|\right)^{p-1-\alpha}\left|\xi_1-\xi_2\right|^{\alpha},$$
$$\left\langle\left|\xi_1\right|^{p-2}\xi_1-\left|\xi_2\right|^{p-2}\xi_2,\xi_1-\xi_2\right\rangle\geq C_2\left(\left|\xi_1\right|+\left|\xi_2\right|\right)^{p-\beta}\left|\xi_1-\xi_2\right|^{\beta},$$

are

$$C_1 = \max\left(1, 2^{2-p}, (p-1)\,2^{2-p}\right),$$
$$C_2 = \min\left(2^{2-p}, (p-1)\,2^{2-p}\right),$$

see Figure 1.

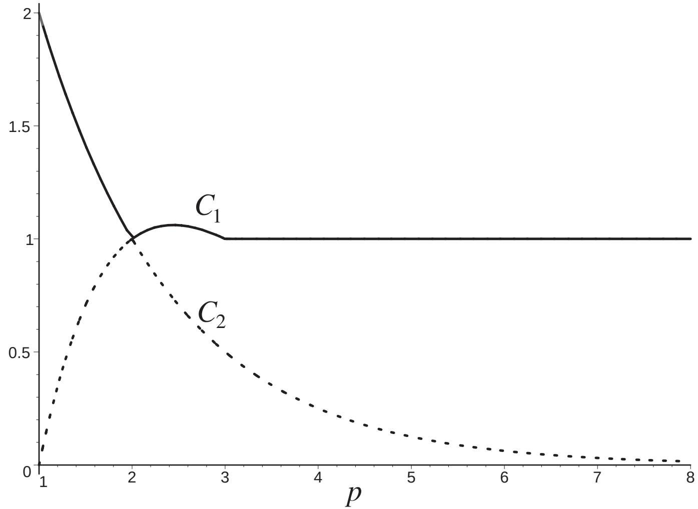

Figure 1: The constants C_1 and C_2 plotted for different values of p.

2. Main Results

Let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product on $\mathbb{R}^n$ and let p be a real constant, $1 < p < \infty$. Moreover, we will assume that $|\xi_1| \geq |\xi_2| > 0$, which poses no restriction due to symmetry reasons. The main results of this paper are collected in the following two theorems:

Theorem 2.1. *Let $\xi_1, \xi_2 \in \mathbb{R}^n$ and assume that the constant α satisfies*

$$0 \leq \alpha \leq \min(1, p-1).$$

Then it holds that

$$\left| |\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2 \right| \leq C_1 \left(|\xi_1| + |\xi_2| \right)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha},$$

with equality if and only if

$$\begin{cases} \xi_1 = -\xi_2, & \text{for } 1 < p < 2, \\ \forall \xi_1, \xi_2 \in \mathbb{R}^n, & \text{for } p = 2, \\ \xi_1 = \xi_2, & \text{for } 2 < p < 3 \text{ and } \alpha = 1, \\ \xi_1 = k\xi_2,\ 1 \leq k < \infty, & \text{for } p = 3, \\ \xi_1 = k\xi_2 \text{ when } k \to \infty, & \text{for } 3 < p < \infty. \end{cases}$$

The constant C_1 is sharp and given by

$$C_1 = \max\left(2^{2-p}, (p-1)\, 2^{2-p}, 1\right).$$

Theorem 2.2. *Let $\xi_1, \xi_2 \in \mathbb{R}^n$ and assume that the constant β satisfies*

$$\max(p, 2) \leq \beta < \infty.$$

Then it holds that

$$\left\langle |\xi_1|^{p-2}\,\xi_1 - |\xi_2|^{p-2}\,\xi_2, \xi_1 - \xi_2 \right\rangle \geq C_2\left(|\xi_1| + |\xi_2|\right)^{p-\beta}|\xi_1 - \xi_2|^{\beta},$$

with equality if and only if

$$\begin{cases} \xi_1 = \xi_2, & \text{for } 1 < p < 2 \text{ and } \beta = 2, \\ \forall \xi_1, \xi_2 \in \mathbb{R}^n, & \text{for } p = 2, \\ \xi_1 = -\xi_2, & \text{for } 2 < p < \infty. \end{cases}$$

The constant C_2 is sharp and given by

$$C_2 = \min\left(2^{2-p}, (p-1)\,2^{2-p}\right).$$

3. Some Auxiliary Lemmas

In this section we will prove the four inequalities

$$\left| |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \right| \le c_1 |\xi_1 - \xi_2|^{p-1}, \quad 1 < p \le 2,$$
$$\left\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2 \right\rangle \ge c_2 (|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|^2, \quad 1 < p \le 2,$$
$$\left| |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \right| \le c_1 (|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2|, \quad 2 \le p < \infty,$$
$$\left\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2 \right\rangle \ge c_2 |\xi_1 - \xi_2|^p, \quad 2 \le p < \infty.$$

Note that, by symmetry, we can assume that $|\xi_1| \ge |\xi_2| > 0$. By putting

$$\eta_1 = \tfrac{\xi_1}{|\xi_1|}, \ |\eta_1| = 1, \qquad \eta_2 = \tfrac{\xi_2}{|\xi_2|}, \ |\eta_2| = 1,$$
$$\gamma = \langle \eta_1, \eta_2 \rangle, \ -1 \le \gamma \le 1, \quad k = \tfrac{|\xi_1|}{|\xi_2|} \ge 1,$$

we see that the four inequalities above are in turn equivalent with

$$\left| k^{p-1}\eta_1 - \eta_2 \right| \le c_1 |k\eta_1 - \eta_2|^{p-1}, \quad 1 < p \le 2, \tag{3.1}$$
$$\left\langle k^{p-1}\eta_1 - \eta_2, k\eta_1 - \eta_2 \right\rangle \ge c_2 (k+1)^{p-2} |k\eta_1 - \eta_2|^2, \quad 1 < p \le 2, \tag{3.2}$$
$$\left| k^{p-1}\eta_1 - \eta_2 \right| \le c_1 (k+1)^{p-2} |k\eta_1 - \eta_2|, \quad 2 \le p < \infty, \tag{3.3}$$
$$\left\langle k^{p-1}\eta_1 - \eta_2, k\eta_1 - \eta_2 \right\rangle \ge c_2 |k\eta_1 - \eta_2|^p, \quad 2 \le p < \infty. \tag{3.4}$$

Before proving these inequalities, we need one lemma.

Lemma 3.1. *Let $k \ge 1$ and $p > 1$. Then the function*

$$h(k) = (3-p)\left(1 - k^{p-1}\right) + (p-1)\left(k - k^{p-2}\right)$$

satisfies $h(1) = 0$. *When* $k > 1$, $h(k)$ *is positive and strictly increasing for* $p \in (1,2) \cup (3,\infty)$, *and negative and strictly decreasing for* $p \in (2,3)$. *Moreover,* $h(k) \equiv 0$ *for* $p = 2$ *or* $p = 3$.

Proof. We easily see that $h(1) = 0$. Two differentiations yield

$$h'(k) = (p-1)\left((p-3)k^{p-2} + 1 - (p-2)k^{p-3}\right),$$
$$h''(k) = (p-1)(p-2)(p-3)k^{p-3}\left(1 - \frac{1}{k}\right),$$

with $h'(1) = 0$ and $h''(1) = 0$. When $p \in (1,2) \cup (3,\infty)$, we have that $h''(k) > 0$ for $k > 1$ which implies that $h'(k) > 0$ for $k > 1$, which in turn implies that $h(k) > 0$ for $k > 1$. When $p \in (2,3)$, a similar reasoning gives that $h'(k) < 0$ and $h(k) < 0$ for $k > 1$. Finally, the lemma is proved by observing that $h(k) \equiv 0$ for $p = 2$ or $p = 3$. □

Remark 3.2. *The special case* $p = 2$ *is trivial, with equality (*$c_1 = c_2 \equiv 1$*) for all* $\xi_i \in \mathbb{R}^n$ *in all four inequalities (3.1) – (3.4). Hence this case will be omitted in all the proofs below.*

Lemma 3.3. *Let* $1 < p < 2$ *and* $\xi_1, \xi_2 \in \mathbb{R}^n$. *Then*

$$\left||\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2\right| \leq c_1 |\xi_1 - \xi_2|^{p-1},$$

with equality if and only if $\xi_1 = -\xi_2$. *The constant* $c_1 = 2^{2-p}$ *is sharp.*

Proof. We want to prove (3.1) for $k \geq 1$. By squaring and putting $\gamma = \langle \eta_1, \eta_2 \rangle$, we see that this is equivalent with proving

$$k^{2(p-1)} + 1 - 2k^{p-1}\gamma \leq c_1^2\left(k^2 + 1 - 2k\gamma\right)^{p-1},$$

where $-1 \leq \gamma \leq 1$. Now construct

$$f_1(k,\gamma) = \frac{k^{2(p-1)} + 1 - 2k^{p-1}\gamma}{(k^2+1-2k\gamma)^{p-1}} = \frac{(k^{p-1}-1)^2 + 2k^{p-1}(1-\gamma)}{\left((k-1)^2 + 2k(1-\gamma)\right)^{p-1}}.$$

Then

$$f_1(k,\gamma) < \infty.$$

Moreover,

$$\frac{\partial f_1}{\partial \gamma} = -\frac{2k\left((1-k^{p-2})(k^p-1) + (2-p)\left(2k^{p-1}(1-\gamma) + (k^{p-1}-1)^2\right)\right)}{(k^2+1-2k\gamma)^p} < 0.$$

Hence we attain the maximum for $f_1(k,\gamma)$ (and thus also for $\sqrt{f_1(k,\gamma)}$) on the border $\gamma = -1$. We therefore examine

$$g_1(k) = \sqrt{f_1(k,-1)} = \frac{k^{p-1}+1}{(k+1)^{p-1}}.$$

We have

$$g_1'(k) = -\frac{p-1}{(k+1)^p}\left(1-k^{p-2}\right) \leq 0,$$

with equality if and only if $k=1$. The smallest possible constant c_1 for which inequality (3.1) will always hold is the maximum value of $g_1(k)$, which is thus attained for $k=1$. Hence

$$c_1 = g_1(1) = 2^{2-p}.$$

This constant is attained for $k=1$ and $\gamma=-1$, that is, when $\xi_1 = -\xi_2$. $\square$

Lemma 3.4. *Let $1 < p < 2$ and $\xi_1, \xi_2 \in \mathbb{R}^n$. Then*

$$\left\langle |\xi_1|^{p-2}\,\xi_1 - |\xi_2|^{p-2}\,\xi_2, \xi_1 - \xi_2 \right\rangle \geq c_2\left(|\xi_1| + |\xi_2|\right)^{p-2}|\xi_1 - \xi_2|^2,$$

with equality if and only if $\xi_1 = \xi_2$. The constant $c_2 = (p-1)\,2^{2-p}$ is sharp.

Proof. We want to prove (3.2) for $k \geq 1$. By putting $\gamma = \langle \eta_1, \eta_2 \rangle$, we see that this is equivalent with proving

$$k^p + 1 - \left(k^{p-2} + 1\right)k\gamma \geq c_2\,(k+1)^{p-2}\left(k^2 + 1 - 2k\gamma\right),$$

where $-1 \leq \gamma \leq 1$. Now construct

$$\begin{aligned} f_2(k,\gamma) &= \frac{k^p + 1 - \left(k^{p-2}+1\right)k\gamma}{(k+1)^{p-2}\left(k^2+1-2k\gamma\right)} \\ &= \frac{\left(k^{p-1}-1\right)(k-1) + \left(k^{p-1}+k\right)(1-\gamma)}{(k+1)^{p-2}\left((k-1)^2 + 2k(1-\gamma)\right)}. \end{aligned}$$

Then

$$f_2(k,\gamma) > 0.$$

Moreover,

$$\frac{\partial f_2}{\partial \gamma} = -\frac{k\left(1-k^{p-2}\right)\left(k^2-1\right)}{(k+1)^{p-2}\left(k^2+1-2k\gamma\right)^2} \leq 0,$$

with equality for $k = 1$. Hence we attain the minimum for $f_2(k,\gamma)$ on the border $\gamma = 1$. We therefore examine

$$g_2(k) = f_2(k,1) = \frac{k^{p-1}-1}{(k-1)(k+1)^{p-2}}.$$

By Lemma 3.1 we have that

$$g_2'(k) = \frac{(3-p)\left(1-k^{p-1}\right)+(p-1)\left(k-k^{p-2}\right)}{(k-1)^2(k+1)^{p-1}} \geq 0,$$

with equality if and only if $k=1$. The largest possible constant c_2 for which inequality (3.2) always will hold is the minimum value of $g_2(k)$, which thus is attained for $k=1$. Hence

$$c_2 = \lim_{k\to 1} g_2(k) = \lim_{k\to 1} \frac{k^{p-1}-1}{(k-1)(k+1)^{p-2}} = (p-1)\,2^{2-p}.$$

This constant is attained for $k=1$ and $\gamma=1$, that is, when $\xi_1=\xi_2$. □

Lemma 3.5. *Let $2<p<\infty$ and $\xi_1,\xi_2\in\mathbb{R}^n$. Then*

$$\left||\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2\right| \leq c_1\left(|\xi_1|+|\xi_2|\right)^{p-2}|\xi_1-\xi_2|,$$

with equality if and only if

$$\begin{cases} \xi_1=\xi_2, & \text{for } 2<p<3,\\ \xi_1=k\xi_2 \text{ when } 1\leq k<\infty, & \text{for } p=3,\\ \xi_1=k\xi_2 \text{ when } k\to\infty, & \text{for } 3<p<\infty. \end{cases}$$

The constant c_1 is sharp, where $c_1=(p-1)\,2^{2-p}$ for $2<p<3$ and $c_1=1$ for $3\leq p<\infty$.

Proof. We want to prove (3.3) for $k \geq 1$. By squaring and putting $\gamma = \langle \eta_1, \eta_2 \rangle$, we see that this is equivalent with proving

$$k^{2(p-1)} + 1 - 2k^{p-1}\gamma \leq c_1^2 (k+1)^{2(p-2)} \left(k^2 + 1 - 2k\gamma\right),$$

where $-1 \leq \gamma \leq 1$. Now construct

$$\begin{aligned} f_3(k,\gamma) &= \frac{k^{2(p-1)} + 1 - 2k^{p-1}\gamma}{(k+1)^{2(p-2)} \left(k^2 + 1 - 2k\gamma\right)} \\ &= \frac{\left(k^{p-1} - 1\right)^2 + 2k^{p-1}(1-\gamma)}{(k+1)^{2(p-2)} \left((k-1)^2 + 2k(1-\gamma)\right)}. \end{aligned}$$

Then

$$f_3(k,\gamma) < \infty.$$

Moreover,

$$\frac{\partial f_3}{\partial \gamma} = \frac{2k\left(k^{p-2} - 1\right)\left(k^p - 1\right)}{(k+1)^{2(p-2)} \left(k^2 + 1 - 2k\gamma\right)} \geq 0,$$

with equality for $k = 1$. Hence we attain the maximum for $f_3(k,\gamma)$ (and thus also for $\sqrt{f_3(k,\gamma)}$) on the border $\gamma = 1$. We therefore examine

$$g_3(k) = \sqrt{f_3(k,1)} = \frac{k^{p-1} - 1}{(k-1)(k+1)^{p-2}}.$$

First we note that

$$g_3(k) \equiv 1$$

when $p = 3$, implying that $c_1 = 1$ with equality for all $\xi_1 = k\xi_2$, $1 \le k < \infty$. Moreover, we have that

$$g_3'(k) = \frac{(3-p)\left(1-k^{p-1}\right) + (p-1)\left(k-k^{p-2}\right)}{(k-1)^2 (k+1)^{p-1}}.$$

By Lemma 3.1 it follows that $g_3(k) \le 0$ for $2 < p < 3$ with equality if and only if $k = 1$. The smallest possible constant c_1 for which inequality (3.3) will always hold is the maximum value of $g_3(k)$, which thus is attained for $k = 1$. Hence

$$c_1 = \lim_{k \to 1} g_3(k) = \lim_{k \to 1} \frac{k^{p-1} - 1}{(k-1)(k+1)^{p-2}} = (p-1)\,2^{2-p}, \text{ for } 2 < p < 3.$$

This constant is attained for $k = 1$ and $\gamma = 1$, that is, when $\xi_1 = \xi_2$.

Again using Lemma 3.1, we see that $g_3(k) \ge 0$ for $3 < p < \infty$, with equality if and only if $k = 1$. The smallest possible constant c_1 for which inequality (3.3) will always hold is the maximum value of $g_3(k)$, which thus is attained when $k \to \infty$. Hence

$$c_1 = \lim_{k \to \infty} g_3(k) = \lim_{k \to \infty} \frac{k^{p-1} - 1}{(k-1)(k+1)^{p-2}} = 1, \text{ for } 3 < p < \infty.$$

This constant is attained when $k \to \infty$ and $\gamma = 1$, that is, when $\xi_1 = k\xi_2$, $k \to \infty$. $\square$

Lemma 3.6. *Let $2 < p < \infty$ and $\xi_1, \xi_2 \in \mathbb{R}^n$. Then*

$$\left\langle |\xi_1|^{p-2}\,\xi_1 - |\xi_2|^{p-2}\,\xi_2, \xi_1 - \xi_2 \right\rangle \ge c_2\,|\xi_1 - \xi_2|^p,$$

with equality if and only if $\xi_1 = -\xi_2$. The constant $c_2 = 2^{2-p}$ is sharp.

Proof. We want to prove (3.4) for $k \geq 1$. By squaring and putting $\gamma = \langle \eta_1, \eta_2 \rangle$, we see that this is equivalent to proving

$$\left(k^p + 1 - \left(k^{p-2} + 1\right) k\gamma\right)^2 \geq c_2^2 \left(k^2 + 1 - 2k\gamma\right)^p,$$

where $-1 \leq \gamma \leq 1$. Now construct

$$\begin{aligned} f_4(k,\gamma) &= \frac{\left(k^p + 1 - \left(k^{p-2}+1\right)k\gamma\right)^2}{\left(k^2+1-2k\gamma\right)^p} \\ &= \frac{\left(\left(k^{p-1}-1\right)(k-1) + \left(k^{p-1}+k\right)(1-\gamma)\right)^2}{\left((k-1)^2 + 2k(1-\gamma)\right)^p}. \end{aligned}$$

Then

$$f_4(k,\gamma) > 0.$$

Moreover,

$$\frac{\partial f_4}{\partial \gamma} = \frac{2k\left((p-2)A(k) + B(k)\right)A(k)}{\left(k^2+1-2k\gamma\right)^{p+1}} > 0,$$

where

$$\begin{aligned} A(k) &= \left(k^{p-1}-1\right)(k-1) + \left(k^{p-1}+k\right)(1-\gamma), \\ B(k) &= \left(k^{p-2}-1\right)\left(k^2-1\right). \end{aligned}$$

Hence we attain the minimum for $f_4(k,\gamma)$ (and thus also for $\sqrt{f_4(k,\gamma)}$) on the border $\gamma = -1$. We therefore examine

$$g_4(k) = \sqrt{f_4(k,-1)} = \frac{k^{p-1}+1}{(k+1)^{p-1}}.$$

We have

$$g_4'(k) = \frac{(p-1)\left(k^{p-2}-1\right)}{(k+1)^p} \geq 0,$$

with equality if and only if $k = 1$. The largest possible constant c_2 for which inequality (3.4) will always hold is the minimum value of $g_4(k)$, which thus is attained for $k = 1$. Hence

$$c_2 = g_4(1) = 2^{2-p}.$$

This constant is attained for $k = 1$ and $\gamma = -1$, that is, when $\xi_1 = -\xi_2$. $\square$

4. Proof of the Main Theorems

Proof of Theorem 2.1. Let $1 < p < 2$. Then the condition $0 \leq \alpha \leq \min(1, p-1) = p-1$ implies that $p - 1 - \alpha \geq 0$. From Lemma 3.3 it follows that

$$\begin{aligned}\left| |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \right| &\leq c_1 |\xi_1 - \xi_2|^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha} \\ &\leq c_1 (|\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha},\end{aligned}$$

with $c_1 = 2^{2-p}$, and we have equality when $\xi_1 = -\xi_2$. Now let $2 < p < \infty$. Then $0 \leq \alpha \leq \min(1, p-1) = 1$ implies that $1 - \alpha \geq 0$. From Lemma 3.5 it follows that

$$\begin{aligned}\left| |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \right| &\leq c_1 \frac{(|\xi_1| + |\xi_2|)^{p-1-\alpha}}{(|\xi_1| + |\xi_2|)^{1-\alpha}} |\xi_1 - \xi_2| \\ &\leq c_1 (|\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^{\alpha},\end{aligned}$$

with

a) $c_1 = (p-1)\,2^{2-p}$, equality for $\xi_1 = \xi_2$ when $\alpha = 1$ for $2 < p < 3$,

b) $c_1 = 1$, equality for $\xi_1 = k\xi_2$ when $k \to \infty$ for $3 < p < \infty$.

The case $p = 2$ is trivial and the case $p = 3$ has equality for $\xi_1 = k\xi_2$, $1 \leq k < \infty$, both cases with constant $c_1 = 1$. The theorem follows by taking these two inequalities together. □

Proof of Theorem 2.2. Let $1 < p < 2$. Then the condition $2 = \max(p, 2) \leq \beta < \infty$ implies that $\beta - 2 \geq 0$. From Lemma 3.4 it follows that

$$\begin{aligned}\left\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2 \right\rangle &\geq c_2 (|\xi_1| + |\xi_2|)^{p-\beta} (|\xi_1| + |\xi_2|)^{\beta-2} |\xi_1 - \xi_2|^2 \\ &\geq c_2 (|\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^{\beta},\end{aligned}$$

with $c_2 = (p-1)\,2^{2-p}$ and equality for $\xi_1 = \xi_2$ when $\beta = 2$. Now let $2 < p < \infty$. Then $p = \max(p,2) \leq \beta < \infty$ implies that $\beta - p \geq 0$. From Lemma 3.6 it follows that

$$\begin{aligned}\left\langle |\xi_1|^{p-2}\,\xi_1 - |\xi_2|^{p-2}\,\xi_2, \xi_1 - \xi_2 \right\rangle &\geq c_2\,|\xi_1 - \xi_2|^{p-\beta}\,|\xi_1 - \xi_2|^{\beta} \\ &\geq c_2\,(|\xi_1| + |\xi_2|)^{p-\beta}\,|\xi_1 - \xi_2|^{\beta},\end{aligned}$$

with $c_2 = 2^{2-p}$ and equality for $\xi_1 = -\xi_2$. The case $p = 2$ is trivial, with constant $c_2 = 1$. The theorem is proven by taking these two inequalities together. □

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