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# AN INEQUALITY BETWEEN COMPOSITIONS OF WEIGHTED ARITHMETIC AND GEOMETRIC MEANS

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ABSTRACT. Let $\mathbb{P}$ denote the collection of positive sequences defined on $\mathbb{N}$. Fix $w \in \mathbb{P}$. Let $s,\ t$, respectively, be the sequences of partial sums of the infinite series $\sum w_k$ and $\sum s_k$, respectively. Given $x \in \mathbb{P}$, define the sequences $A(x)$ and $G(x)$ of weighted arithmetic and geometric means of $x$ by

$$A_n(x) = \sum_{k=1}^{n} \frac{w_k}{s_n} x_k,\ G_n(x) = \prod_{k=1}^{n} x_k^{w_k/s_n},\ n = 1, 2, \dots$$

Under the assumption that $\log t$ is concave, it is proved that $A(G(x)) \leq G(A(x))$ for all $x \in \mathbb{P}$, with equality if and only if $x$ is a constant sequence.

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## 1. INTRODUCTION

In [13], Kedlaya proved the following theorem.

**Theorem 1.1.** *Let $x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_n$ be positive real numbers, and define $s_i = w_1 + w_2 + \cdots + w_i$, $i = 1, 2, \dots, n$. Assume that*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots \geq \frac{w_n}{s_n}. \tag{1.1}$$

*Then*

$$\prod_{i=1}^{n} \left( \sum_{j=1}^{i} \frac{w_j}{s_i} x_j \right)^{w_i/s_n} \geq \sum_{j=1}^{n} \frac{w_j}{s_n} \prod_{i=1}^{j} x_i^{w_i/s_j}, \tag{1.2}$$

*with equality if and only if $x_1 = x_2 = \cdots = x_n$.*

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Choosing $w$ to be a constant sequence, we recover the inequality

$$\sqrt[n]{\prod_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}x_j\right)} \geq \frac{1}{n}\sum_{j=1}^{n}\sqrt[j]{\prod_{i=1}^{j}x_i}, \tag{1.3}$$

which Kedlaya [12] had previously established, thereby confirming a conjecture of the author [9]. The strict inequality prevails in (1.3) unless $x_1 = x_2 = \cdots = x_n$. Evidently, inequality (1.3) is a sharp refinement of Carleman's well-known one [4, 7]. (Indeed, as a tribute to Carleman, the author was led to formulate (1.3) in an attempt to design a suitable problem for the IMO when it was held in Sweden in 1991. However, unbeknownst to him at the time, two stronger versions of it had already been stated, without proof, by Nanjundiah [17].)

In passing, we note that (1.3) is also a simple consequence of more general results found by Bennett [2, 3], and Mond and Pečarić [16].

Also in [13], Kedlaya deduced a weighted version of Carleman's inequality from Theorem 1.1, viz.,

**Theorem 1.2.** *Let $w_1, w_2, \ldots$ be a sequence of positive real numbers, and define $s_i = w_1 + w_2 + \cdots + w_i$, for $i = 1, 2, \ldots$. Assume that*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots . \tag{1.4}$$

*Then, for any sequence $a_1, a_2, \ldots$ of positive real numbers with $\sum_k w_k a_k < \infty$,*

$$\sum_{k=1}^{\infty} w_k\, a_1^{w_1/s_k} \cdots a_k^{w_k/s_k} < e\sum_{k=1}^{\infty} w_k a_k.$$

Carleman's classical inequality is obtained from this by setting $w_i = 1$, $i = 1, 2, \ldots$. This beautiful result has attracted the attention of many authors, and has been proved in a variety of ways. It has also been extended in different directions by a host of people. Anyone interested in knowing the history of Carleman's inequality, and such matters, is urged to consult [11], which has an extensive bibliography. In addition, the fascinating monograph by Bennett [1] contains some very interesting developments of it, and mentions, *inter alia*, the significant extensions of it made by Cochran and Lee [5], Heinig [8] and Love [14, 15]. Readers interested in its continuous analogues should also read [18].

Kedlaya expressed a doubt that the monotonicity condition (1.4) was needed in Theorem 1.2. His suspicions were well-founded, for, already in 1925, Hardy [6, 7], following a suggestion made to him by Pólya, proved this statement without any extra hypothesis on the weights. In fact, in the presence of condition (1.4), a much stronger conclusion can be drawn, as the author has recently discovered [10]. This begs the question: does Theorem 1.1 also hold under less stringent conditions on the weights than (1.1)? It is trivially true when $n = 1$, and a convexity argument shows it also holds without any restriction on the weights when $n = 2$. However, as Kedlaya himself pointed out, the result is false in general. As he mentions, a necessary condition for the truth of Theorem 1.1 is that

$$\left(\frac{w_n}{s_n}\right)^{s_{n-1}} \leq \left(\frac{w_1}{s_1}\right)^{w_1}\left(\frac{w_2}{s_2}\right)^{w_2}\cdots\left(\frac{w_{n-1}}{s_{n-1}}\right)^{w_{n-1}}.$$

On the other hand, examples show that the sufficient assumption (1.1) is not necessary. For instance, with $n = 3, w_1 = 2, w_2 = 1, w_3 = 3$, then $w_2/s_2 < w_3/s_3$, so that condition (1.1) fails, yet

$$\frac{2a + \sqrt[3]{a^2 b} + 3\sqrt[6]{a^2 bc^3}}{6} \leq \sqrt[6]{a^2\left(\frac{2a+b}{3}\right)\left(\frac{2a+b+3c}{6}\right)^3},$$

for all $a, b, c > 0$, with equality if and only if $a = b = c$. (This is a simple consequence of the fact that, if

$$F(x, y) = \frac{(2 + x + 3\sqrt{x}y)^6}{(2 + x^3)(2 + x^3 + 3y^2)^3},$$

then

$$\begin{aligned}\max_{x\geq 0}\max_{y\geq 0} F(x, y) &= \max_{x\geq 0}\left[\frac{1}{2 + x^3}\left(\max_{y\geq 0}\frac{(2 + x + 3\sqrt{x}y)^2}{2 + x^3 + 3y^2}\right)^3\right]\\ &= \max_{x\geq 0}\frac{(4 + 10x + x^2 + 3x^4)^3}{(2 + x^3)^4}\\ &= 72,\end{aligned}$$

which can be verified in a routine manner, even by non-calculus arguments. Alternatively, it can be inferred as a special case of Theorem 2.1 which follows. Moreover, there is equality if and only if $x = y = 1$.)

As an examination of his proof of Theorem 1.1 reveals, Kedlaya actually proved something stronger than (1.2) under the hypothesis (1.1), namely, denoting by $L_n$, $R_n$ the left-hand and right-hand sides of (1.2), then

$$\left(\frac{L_1}{R_1}\right)^{s_1} \leq \left(\frac{L_2}{R_2}\right)^{s_2} \leq \cdots \leq \left(\frac{L_n}{R_n}\right)^{s_n}. \tag{1.5}$$

However, this statement is false in general, and, in particular, is not implied by (1.2). To see this, note that, with $n = 3$, and the same choice of weights $w_1 = 2, w_2 = 1, w_3 = 3$ as before, so that (1.2) holds, the claim that $(L_3/R_3)^{s_3} \geq (L_2/R_2)^{s_2}$ is equivalent to the statement that

$$2(2a + b + 3c)(2a + \sqrt[3]{a^2b}) \geq (2a + \sqrt[3]{a^2b} + 3\sqrt[6]{a^2bc^3})^2, \quad \forall a, b, c > 0.$$

However, this is not true generally, as may be seen by taking $a = 1, b = 64, c = 121$. So, Kedlaya proved a stronger statement with the hypothesis that the sequence $s_i/w_i$ is increasing. By adopting a different proof-strategy, we show here that (1.2) holds under a weaker hypothesis than this.

## 2. THE MAIN RESULT

The purpose of this note is to present the following result which strengthens Theorem 1.1.

**Theorem 2.1.** *Let $x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n$ be positive real numbers. Define $s_i = w_1 + w_2 + \cdots + w_i$, $i = 1, 2, \ldots, n$. Assume that*

$$\frac{s_k^2}{w_{k+1}} \geq \sum_{j=1}^{k-1} s_j, \quad k = 2, 3, \ldots, n-1. \tag{2.1}$$

*Then*

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{w_j}{s_i}x_j\right)^{w_i/s_n} \geq \sum_{j=1}^{n}\frac{w_j}{s_n}\prod_{i=1}^{j}x_i^{w_i/s_j}.$$

*Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.*

**Remark 2.2.** In terms of the sequence $t_i = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \ldots, n$, it is not difficult to see that (2.1) is equivalent to the statement

$$t_i^2 \geq t_{i-1}t_{i+1}, \quad i = 2, 3, \ldots, n-1,$$

i.e., that $\log t_i$ is concave, whereas (1.1) is equivalent to the assertion that $\log s_i$ is concave. But we make no use of this alternative description of (2.1).

Before turning to the proof of Theorem 2.1 we show that (2.1) is implied by (1.1).

**Lemma 2.3.** *Let* $w_1, w_2, \dots$ *be a sequence of positive numbers, and define the sequence* $s_1, s_2, \dots$ *by*

$$s_i = w_1 + w_2 + \cdots + w_i, \quad i = 1, 2, \dots .$$

*Suppose*

$$\frac{w_1}{s_1} \geq \frac{w_2}{s_2} \geq \cdots \geq \frac{w_n}{s_n} \geq \cdots .$$

*Then*

$$s_k^2 - w_{k+1} \sum_{j=1}^{k-1} s_j > 0, \quad k = 2, 3, \dots .$$

*Proof.* The proof is by induction. To begin with, since $w_2 s_2 - w_3 s_1 = w_2 s_3 - w_s s_2 \geq 0$, we have that

$$s_2^2 - w_3 s_1 = w_1 s_2 + w_2 s_2 - w_3 s_1 \geq w_1 s_2 > 0.$$

So, suppose the claimed result holds for some $m \geq 2$. Then, noting that, for $i \geq 2$, $w_i s_i - w_{i+1} s_{i-1} = w_i s_{i+1} - w_{i+1} s_i \geq 0$, we see that

$$\begin{aligned}
s_{m+1}^2 - w_{m+2} \sum_{j=1}^{m} s_j &\geq \frac{w_{m+2}}{w_{m+1}} s_{m+1} s_m - w_{m+2} \sum_{j=1}^{m} s_j \\
&= \frac{w_{m+2}}{w_{m+1}} \left( s_{m+1} s_m - w_{m+1} \sum_{j=1}^{m} s_j \right) \\
&= \frac{w_{m+2}}{w_{m+1}} \left( s_m^2 + w_{m+1} s_m - w_{m+1} \sum_{j=1}^{m} s_j \right) \\
&= \frac{w_{m+2}}{w_{m+1}} \left( s_m^2 - w_{m+1} \sum_{j=1}^{m-1} s_j \right) \\
&> 0,
\end{aligned}$$

by the induction assumption. The result follows. □

We prove Theorem 2.1 by induction, and, to make productive use of the induction hypothesis, we need the following elementary result.

**Lemma 2.4.** *Let* $A, B > 0$. *Let* $p > 1, q = p/(p-1)$. *Then, for all* $s \geq 0$,

$$(A + Bs)^p \leq (A^q + B^q)^{p-1}(1 + s^p),$$

*with equality if and only if* $s = (B/A)^{q-1}$.

*Proof.* The inequality is trivial if $s = 0$. Suppose $s > 0$. Exploiting the strict convexity of $t \to t^q$, it is clear that

$$\begin{aligned}
\left( \frac{A + Bs}{1 + s^p} \right)^q &= \left( \frac{A + (Bs^{1-p}) s^p}{1 + s^p} \right)^q \\
&\leq \frac{A^q + (Bs^{1-p})^q s^p}{1 + s^p} \\
&= \frac{A^q + B^q}{1 + s^p},
\end{aligned}$$

with equality if and only if $A = Bs^{1-p}$. The stated result follows quickly from this. □

**Corollary 2.5.** *Let $p > 1$, $q = p/(p-1)$. Let $A, B, C, D > 0$. Then, for all $t \geq 0$,*

$$\frac{(A+Bt)^p}{C+Dt^p} \leq \frac{1}{CD}(A^q D^{q-1} + B^q C^{q-1})^{p-1},$$

*with equality if and only if $t = (BC/AD)^{q-1}$.*

We are now ready to deal with the proof of Theorem 2.1.

For convenience, define the sequences of weighted averages $A_k$, $G_k$ of $x_1, x_2, \ldots, x_n$ by

$$A_k = \sum_{i=1}^{k} \frac{w_i}{s_k} x_i, \qquad G_k = \prod_{i=1}^{k} x_i^{w_i/s_k}, \quad k = 1, 2, \ldots, n.$$

We are required to prove that

$$\sum_{i=1}^{n} \frac{w_i}{s_n} G_i \leq \prod_{i=1}^{n} A_i^{w_i/s_n},$$

holds under condition (2.1), with equality if and only if

$$x_1 = x_2 = \cdots = x_n.$$

*Proof.* We prove this by induction. The result clearly holds for $n = 1$. Moreover, as we mentioned in the introduction, a simple convexity argument establishes that it also holds when $n = 2$. We continue, therefore, with the assumption that $n \geq 3$. Suppose the result holds for some positive integer $m$, with $1 \leq m \leq n-1$, so that, with

$$X = \prod_{i=1}^{m} A_i^{w_i/s_m},$$

then

$$\begin{aligned}\sum_{i=1}^{m+1} \frac{w_i}{s_{m+1}} G_i &= \frac{s_m \sum_{i=1}^{m} \frac{w_i}{s_m} G_i + w_{m+1} G_{m+1}}{s_{m+1}} \\ &\leq \frac{s_m X + w_{m+1} G_{m+1}}{s_{m+1}} \\ &= (1-\alpha)X + \alpha Y\, x_{m+1}^{\alpha},\end{aligned}$$

where $\alpha = w_{m+1}/s_{m+1}$ and

$$Y = \prod_{i=1}^{m} x_i^{w_i/s_{m+1}} = G_m^{s_m/s_{m+1}} = G_m^{1-\alpha}.$$

In addition,

$$A_{m+1} = \frac{s_m A_m + w_{m+1} x_{m+1}}{s_{m+1}} = (1-\alpha)A_m + \alpha x_{m+1}.$$

We claim now that

$$\begin{aligned}(1-\alpha)X + \alpha Y\, x_{m+1}^{\alpha} &\leq X^{s_m/s_{m+1}} {A_{m+1}}^{w_{m+1}/s_{m+1}} \\ &= X^{1-\alpha}((1-\alpha)A_m + \alpha x_{m+1})^{\alpha},\end{aligned}$$

i.e.,

$$\frac{((1-\alpha)X + \alpha Y x_{m+1}^{\alpha})^{1/\alpha}}{(1-\alpha)A_m + \alpha x_{m+1}} \leq X^{(1-\alpha)/\alpha}.$$

By Corollary 2.5, with $p = 1/\alpha$, $A = (1-\alpha)X$, $B = \alpha Y$, $C = (1-\alpha)A_m$, $D = \alpha, q = 1/(1-\alpha)$, the left-hand side does not exceed

$$\frac{\left((1-\alpha)X^{1/(1-\alpha)} + \alpha Y^{1/(1-\alpha)}\, A_m^{\alpha/(1-\alpha)}\right)^{(1-\alpha)/\alpha}}{A_m},$$

with equality if and only if

$$x_{m+1} = \left(\frac{YA_m}{X}\right)^{1/(1-\alpha)}.$$

Thus, to finish the proof, we must establish that

$$(1-\alpha)X^{1/(1-\alpha)} + \alpha Y^{1/(1-\alpha)} A_m^{\alpha/(1-\alpha)} \le X A_m^{\alpha/(1-\alpha)},$$

i.e., that

$$s_m \left(\frac{X}{A_m}\right)^{\alpha/(1-\alpha)} + w_{m+1}\frac{Y^{1/(1-\alpha)}}{X} \le s_{m+1}.$$

In other words,

$$s_m \left(\frac{\prod_{i=1}^m A_i^{w_i/s_m}}{A_m}\right)^{w_{m+1}/s_m} + w_{m+1}\prod_{i=1}^m \left(\frac{x_i}{A_i}\right)^{w_i/s_m} \le s_{m+1}, \tag{2.2}$$

with the additional assertion that there is equality if and only if $x_1 = x_2 = \cdots = x_m$. This inequality is of independent interest, and can be considered for its own sake. To prove it, consider the second term on the left-hand side of (2.2). This is equal to

$$\frac{w_{m+1}G_m}{X} = w_{m+1}\sqrt[s_m]{\prod_{i=1}^m \left(\frac{x_i}{A_i}\right)^{w_i}},$$

whence, by the convexity of the exponential function, bearing in mind that $s_m = \sum_{i=1}^m w_i$, we see that this does not exceed

$$\frac{w_{m+1}}{s_m}\sum_{i=1}^m \frac{w_i x_i}{A_i}.$$

Moreover, there is equality if and only if

$$1 = \frac{x_1}{A_1} = \frac{x_i}{A_i}, \qquad i = 1, 2, \ldots, m,$$

i.e., $x_1 = x_2 = \cdots = x_m$.

Now we focus on the first term. To begin with, observe that

$$\begin{aligned}\frac{X}{A_m} &= \sqrt[s_m]{\frac{\prod_{i=1}^m A_i^{w_i}}{A_m^{s_m}}}\\ &= \sqrt[s_m]{\frac{\prod_{i=1}^{m-1} A_i^{w_i}}{A_m^{s_{m-1}}}}\\ &= \sqrt[s_m]{\prod_{i=1}^{m-1}\left(\frac{A_i}{A_{i+1}}\right)^{s_i}}.\end{aligned}$$

Hence, once more by the convexity of the exponential function,

$$\begin{aligned}
s_m \left(\frac{X}{A_m}\right)^{\alpha/(1-\alpha)} &= s_m \left(1^{c_m} \prod_{i=1}^{m-1} \left(\frac{A_i}{A_{i+1}}\right)^{s_i}\right)^{w_{m+1}/s_m^2} \\
&\le \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=1}^{m-1} \frac{s_i A_i}{A_{i+1}}\right) \\
&= \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=2}^{m} \frac{s_{i-1} A_{i-1}}{A_i}\right),
\end{aligned}$$

where

$$c_m = \frac{s_m^2}{w_{m+1}} - \sum_{i=1}^{m-1} s_i \ge 0,$$

by hypothesis. Equality holds here if and only if

$$1 = \frac{A_i}{A_{i+1}}, \qquad i = 1, 2, \dots, m-1,$$

i.e.,

$$s_i \sum_{j=1}^{i+1} w_j x_j = s_{i+1} \sum_{j=1}^{i} w_j x_j, \qquad i = 1, 2, \dots, m-1,$$

equivalently, if and only if $x_m = \cdots = x_2 = x_1$.

Combining our estimates we see that

$$\begin{aligned}
s_m \left(\frac{X}{A_m}\right)^{\alpha/(1-\alpha)} + w_{m+1} \frac{Y^{1/(1-\alpha)}}{X} &\le \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=2}^{m} \frac{s_{i-1} A_{i-1}}{A_i} + \sum_{i=1}^{m} \frac{w_i x_i}{A_i}\right) \\
&= \frac{w_{m+1}}{s_m} \left(c_m + w_1 + \sum_{i=2}^{m} \frac{s_{i-1} A_{i-1} + w_i x_i}{A_i}\right) \\
&= \frac{w_{m+1}}{s_m} \left(c_m + w_1 + \sum_{i=2}^{m} \frac{s_i A_i}{A_i}\right) \\
&= \frac{w_{m+1}}{s_m} \left(c_m + \sum_{i=1}^{m-1} s_i + s_m\right) \\
&= \frac{w_{m+1}}{s_m} \left(\frac{s_m^2}{w_{m+1}} + s_m\right) \\
&= s_{m+1}.
\end{aligned}$$

Thus (2.2) holds. Moreover, equality holds in (2.2) if and only if $x_1 = x_2 = \cdots = x_m$. Of course, (2.2) implies the inequality in Theorem 2.1, by induction. It therefore only remains to discuss the case of equality in this. But, if $x_1 = x_2 = \cdots = x_m$, then $A_m = X = x_1$, and $Y = x_1^{s_m/s_{m+1}}$, whence equality holds throughout only if, in addition, $x_{m+1} = Y^{1/(1-\alpha} = x_1$ also. But, clearly, the equality holds if all the $x$'s are equal. This finishes the proof. □

## REFERENCES

[1] G. BENNETT, Factorizing the classical inequalities, *Mem. Amer. Math. Soc.*, **120** (1996), *no. 576*, viii+130 pp.

[2] G. BENNETT, An inequality for Hausdorff means, *Houston J. Math.*, **25**(4) (1996), 709–744.

[3] G. BENNETT, Summability matrices and random walk, *Houston J. Math.*, **28**(4) (2002), 865–898.

[4] T. CARLEMAN, Sur les functions quasi-analytiques, *Fifth Scand. Math. Congress* (1923), 181–196.

[5] J.A. COCHRAN AND C.S. LEE, Inequalities related to Hardy's and Heinig's, *Math. Proc. Camb. Phil. Soc.*, **96**(1) (1984), 1–7.

[6] G.H. HARDY, Notes on some points of the integral calculus (LX), *Messenger of Math.*, **54** (1925), 150–156.

[7] G.H. HARDY, J.E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.

[8] H. HEINIG, Some extensions of Hardy's inequality, *SIAM J. Math. Anal.*, **6** (1975), 698–713.

[9] F. HOLLAND, On a mixed arithmetic-mean, geometric-mean inequality, *Math. Competitions*, **5** (1992), 60–64.

[10] F. HOLLAND, A strengthening of the Carleman-Hardy-Pólya inequality, *Proc. Amer. Math. Soc.*, To appear.

[11] M. JOHANSSON, L.E. PERSSON AND A. WEDESTIG, Carleman's inequality—history, proofs and some new generalizations, *J. Inequal. Pure & Appl. Math.*, **4**(3) (2003), Art. 53. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=291`].

[12] K. KEDLAYA, Proof of a mixed arithmetic-mean, geometric-mean inequality, *Amer. Math. Monthly*, **101** (1994), 355–357.

[13] K. KEDLAYA, A Weighted Mixed-Mean Inequality, *Amer. Math. Monthly*, **106** (1999), 355–358.

[14] E.R. LOVE, Inequalities related to those of Hardy and of Cochran and Lee, *Math. Proc. Camb. Phil. Soc.*, **99**(1) (1986), 395–408.

[15] E.R. LOVE, Inequalities related to Carleman's inequality, *Inequalities*, (Birmingham, 1987), 135–141, *Lecture Notes in Pure and Appl. Math*. **129**, Dekker, New York, 1991.

[16] B. MOND AND J.E. PEČARIĆ, A mixed means inequality, *Austral.Math. Soc. Gaz.*, **23**(2) (1996) 67–70.

[17] T.S. NANJUNDIAH, Sharpening of some classical inequalities, *Math Student*, **20** (1952), 24–25.

[18] B. OPIC AND P. GURKA, Weighted inequalities for geometric means, *Proc. Amer. Math. Soc.*, **120** (1994), 771–779.