

# Journal of Inequalities in Pure and Applied Mathematics

## ON INVERSES OF TRIANGULAR MATRICES WITH MONOTONE ENTRIES

KENNETH S. BERENHAUT AND PRESTON T. FLETCHER

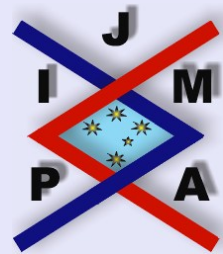
Department of Mathematics  
Wake Forest University  
Winston-Salem, NC 27106, USA.

*EMail:* [berenhks@wfu.edu](mailto:berenhks@wfu.edu)

*URL:* <http://www.math.wfu.edu/Faculty/berenhaut.html>

*EMail:* [fletpt1@wfu.edu](mailto:fletpt1@wfu.edu)

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Abstract

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## Abstract

This note employs recurrence techniques to obtain entry-wise optimal inequalities for inverses of triangular matrices whose entries satisfy some monotonicity constraints. The derived bounds are easily computable.

*2000 Mathematics Subject Classification:* 15A09, 39A10, 26A48.

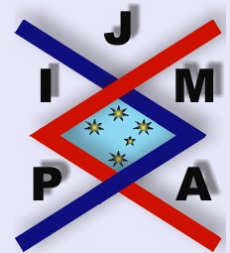
*Key words:* Explicit bounds, Triangular matrix, Matrix inverse, Monotone entries, Off-diagonal decay, Recurrence relations.

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# 1. Introduction

Much work has been done in the recent past to understand off-diagonal decay properties of structured matrices and their inverses (cf. Benzi and Golub [1], Demko, Moss and Smith [4], Eijkhout and Polman [5], Jaffard [6], Nabben [7] and [8], Peluso and Politi [9], Robinson and Wathen [10], Strohmer [11], Vecchio [12] and the references therein).

This paper studies nonnegative triangular matrices with off-diagonal decay. In particular, let

$$\mathbf{L}_n = \begin{bmatrix} l_{1,1} & & & & \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & l_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & l_{n,n} \end{bmatrix}$$

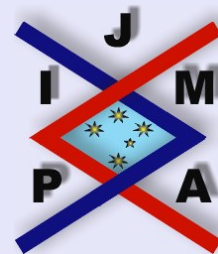
be an invertible lower triangular matrix, and

$$\mathbf{X}_n = \mathbf{L}_n^{-1} = \begin{bmatrix} x_{1,1} & & & & \\ x_{2,1} & x_{2,2} & & & \\ x_{3,1} & x_{3,2} & x_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{bmatrix},$$

be its inverse.

We are interested in obtaining bounds on the entries in  $\mathbf{X}_n$  under the row-wise monotonicity assumption

$$(1.1) \quad 0 \leq l_{i,1} \leq l_{i,2} \leq \cdots \leq l_{i,i-1} \leq l_{i,i}$$



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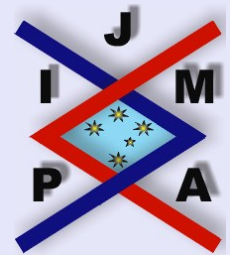
for  $2 \leq i \leq n$ .

As an added generalization, we will consider  $[l_{i,j}]$  satisfying

$$(1.2) \quad 0 \leq \frac{l_{i,1}}{l_{i,i}} \leq \frac{l_{i,2}}{l_{i,i}} \leq \dots \leq \frac{l_{i,i-1}}{l_{i,i}} \leq \kappa_{i-1},$$

for some nondecreasing sequence  $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots)$ .

The paper proceeds as follows. Section 2 contains some recurrence-type lemmas, while the main result, Theorem 3.1, and its proof are contained in Section 3. The paper closes with some illustrative examples.



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## 2. Preliminary Lemmas

In establishing our main results, we will employ recurrence techniques. In particular, suppose  $\{b_i\}$  and  $\{\alpha_{i,j}\}$  satisfy the linear recurrence

$$(2.1) \quad b_i = \sum_{k=0}^{i-1} (-\alpha_{i,k}) b_k, \quad (1 \leq i \leq n),$$

with  $b_0 = 1$  and

$$(2.2) \quad 0 \leq \alpha_{i,0} \leq \alpha_{i,1} \leq \alpha_{i,2} \leq \cdots \leq \alpha_{i,i-1} \leq A_i,$$

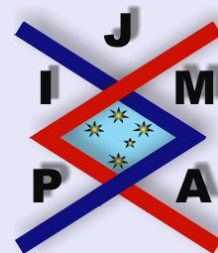
for  $i \geq 1$ .

We will employ the following lemma, which reduces the scope of consideration in bounding solutions to (2.1).

**Lemma 2.1.** *Suppose that  $\{b_i\}$  and  $\{\alpha_{i,j}\}$  satisfy (2.1) and (2.2). Then, there exists a sequence  $a_1, a_2, \dots, a_n$ , with  $0 \leq a_i \leq i$  for  $1 \leq i \leq n$ , such that  $|b_n| \leq |d_n|$ , where  $\{d_i\}$  satisfies  $d_0 = 1$ , and for  $1 \leq i \leq n$ ,*

$$(2.3) \quad d_i = \begin{cases} \sum_{j=a_i}^{i-1} (-A_j) d_j, & \text{if } a_i < i \\ 0, & \text{otherwise} \end{cases}.$$

In proving Lemma 2.1, we will refer to the following result on inner products.



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**Lemma 2.2.** Suppose that  $\mathbf{p} = (p_1, \dots, p_n)'$  and  $\mathbf{q} = (q_1, \dots, q_n)'$  are  $n$ -vectors with

$$(2.4) \quad 0 \geq p_1 \geq p_2 \geq \dots \geq p_n \geq -A.$$

Define

$$(2.5) \quad \mathbf{p}_n^*(\nu, A) = (\overbrace{0, 0, \dots, 0}^{\nu}, \overbrace{-A, \dots, -A, -A}^{n-\nu})$$

for  $0 \leq \nu \leq n$ . Then,

$$(2.6) \quad \min_{0 \leq \nu \leq n} \{\mathbf{p}_n^*(\nu, A) \cdot \mathbf{q}\} \leq \mathbf{p} \cdot \mathbf{q} \leq \max_{0 \leq \nu \leq n} \{\mathbf{p}_n^*(\nu, A) \cdot \mathbf{q}\},$$

where  $\mathbf{p} \cdot \mathbf{q}$  denotes the standard dot product  $\sum_{i=1}^n p_i q_i$ .

*Proof.* Suppose  $\mathbf{p}$  is of the form

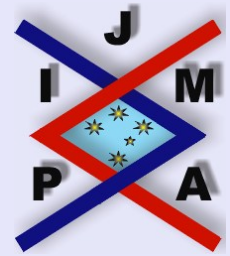
$$(2.7) \quad (p_1, \dots, p_j, \overbrace{-k, \dots, -k}^{e_1}, \overbrace{-A, \dots, -A}^{e_2}),$$

with  $0 \geq p_1 \geq p_2 \geq \dots \geq p_j > -k > -A$ ,  $e_1 \geq 1$  and  $e_2 \geq 0$ . First, assume that  $\mathbf{p} \cdot \mathbf{q} > 0$ , and consider  $S = \sum_{i=j+1}^{e_1+j} q_i$ . If  $S < 0$  then, since  $k < A$ ,

$$(2.8) \quad (p_1, p_2, \dots, p_{j-1}, p_j, \overbrace{-A, \dots, -A}^{e_1} \overbrace{-A, \dots, -A}^{e_2}) \cdot \mathbf{q} \geq \mathbf{p} \cdot \mathbf{q}.$$

Otherwise, since  $-k < p_j$ ,

$$(2.9) \quad (p_1, p_2, \dots, p_{j-1}, p_j, \overbrace{p_j, \dots, p_j}^{e_1}, \overbrace{-A, \dots, -A}^{e_2}) \cdot \mathbf{q} \geq \mathbf{p} \cdot \mathbf{q}.$$



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In either case, there is a vector of the form in (2.7) with strictly less distinct values, whose inner product with  $\mathbf{q}$  is at least as large as  $\mathbf{p} \cdot \mathbf{q}$ . Inductively, there exists a vector of the form in (2.7) with  $e_2 + e_1 = n$ , with as large, or larger, inner product. Hence, we have reduced to the case where  $\mathbf{p} =$

$\left( \overbrace{-k, \dots, -k}^{e_1}, \overbrace{-A, \dots, -A}^{e_2} \right)$ , where  $e_1 = 0$  and  $e_2 = 0$  are permissible. If  $k = 0$  or  $e_1 = 0$ , then  $\mathbf{p} = \mathbf{p}_n^*(e_1, A)$ . Otherwise, consider  $S = \sum_{i=1}^{e_1} q_i$ . If  $S < 0$ , then

$$(2.10) \quad \mathbf{p}_n^*(0, A) \cdot \mathbf{q} \geq \mathbf{p} \cdot \mathbf{q}.$$

If  $S \geq 0$ ,

$$(2.11) \quad \mathbf{p}_n^*(e_1, A) \cdot \mathbf{q} \geq \mathbf{p} \cdot \mathbf{q}.$$

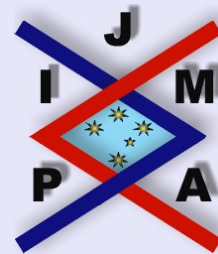
The result for the case  $\mathbf{p} \cdot \mathbf{q} > 0$  now follows from (2.10) and (2.11).

The case when  $\mathbf{p} \cdot \mathbf{q} \leq 0$  is handled similarly, and the lemma follows.  $\square$

We now turn to a proof of Lemma 2.1.

*Proof of Lemma 2.1.* The proof, here, involves applying Lemma 2.2 to successively “scale” the rows of the coefficient matrix

$$\begin{bmatrix} -\alpha_{1,0} & 0 & \dots & 0 \\ -\alpha_{2,0} & -\alpha_{2,1} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n,0} & -\alpha_{n,1} & \dots & -\alpha_{n,n-1} \end{bmatrix},$$



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while not decreasing the value of  $|b_n|$  at any step.

First, define the sequences

$$\bar{\alpha}_i = (-\alpha_{i,0}, \dots, -\alpha_{i,i-1}) \quad \text{and}$$

$$\mathbf{b}^{k,j} = (b_k, \dots, b_j),$$

for  $0 \leq k \leq j \leq n-1$  and  $1 \leq i \leq n$ .

Now, note that applying Lemma 2.2 to the vectors  $\mathbf{p} = \bar{\alpha}_n$  and  $\mathbf{q} = \mathbf{b}^{0,n-1}$  yields a vector  $\mathbf{p}^*(\nu_n, A_n)$  (as in (2.5)) such that either

$$(2.12) \quad \mathbf{p}^*(\nu_n, A_n) \cdot \mathbf{b}^{0,n-1} \geq \bar{\alpha}_n \cdot \mathbf{b}^{0,n-1} = b_n > 0$$

or

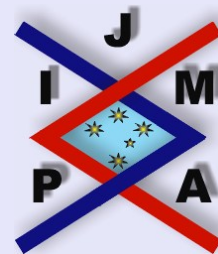
$$(2.13) \quad \mathbf{p}^*(\nu_n, A_n) \cdot \mathbf{b}^{0,n-1} \leq \bar{\alpha}_n \cdot \mathbf{b}^{0,n-1} = b_n \leq 0$$

Hence, suppose that the entries of the  $k^{\text{th}}$  through  $n^{\text{th}}$  rows of the coefficient matrix are of the form in (2.5), and express  $b_n$  as a linear combination of  $b_1, b_2, \dots, b_k$  i.e.

$$(2.14) \quad \begin{aligned} b_n &= \sum_{i=1}^k C_i^k b_i \\ &= C_k^k b_k + \sum_{i=1}^{k-1} C_i^k b_i. \end{aligned}$$

Now, suppose  $C_k^k > 0$ . As before, applying Lemma 2.2 to the vectors  $\mathbf{p} = \bar{\alpha}_k$  and  $\mathbf{q} = \mathbf{b}^{0,k-1}$  yields a vector  $\mathbf{p}_k^*(\nu_k, A_k)$ , such that

$$(2.15) \quad \mathbf{p}_k^*(\nu_k, A_k) \cdot \mathbf{b}^{0,k-1} \geq \bar{\alpha}_k \cdot \mathbf{b}^{0,k-1} = b_k.$$



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Similarly, if  $C_k^k \leq 0$ , we obtain a vector  $\mathbf{p}_k^*(\nu_k, A_k)$ , such that

$$(2.16) \quad \mathbf{p}_k^*(\nu_k, A_k) \cdot \mathbf{b}^{0,k-1} \leq \bar{\alpha}_k \cdot \mathbf{b}^{0,k-1} = b_k.$$

Using the respective entries in  $\mathbf{p}_k^*(\nu_k, A_k)$  in place of those in  $\bar{\alpha}_k$  in (2.1) will not decrease the value of  $b_n$ . This completes the induction for the case  $b_n > 0$ ; the case  $b_n \leq 0$  is similar, and the lemma follows.  $\square$

**Remark 1.** A version of Lemma 2.3 for  $A_i \equiv 1$  was recently applied in proving that all symmetric Toeplitz matrices generated by monotone convex sequences have off-diagonal decay preserved through triangular decompositions (see [2]).

Now, For  $\mathbf{a} = (A_1, A_2, A_3, \dots)$ , with

$$(2.17) \quad 0 \leq A_1 \leq A_2 \leq A_3 \leq \dots$$

define

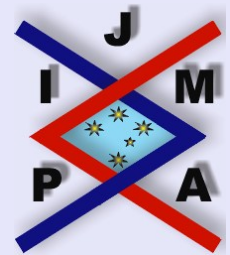
$$(2.18) \quad Z_i(\mathbf{a}) \stackrel{\text{def}}{=} \max \left\{ \prod_{v=j}^i A_v : 1 \leq j \leq i \right\},$$

for  $i \geq 1$ .

We have the following result on bounds for linear recurrences.

**Lemma 2.3.** Suppose that  $\mathbf{a} = (A_j)$  satisfies the monotonicity constraint in (2.17). Then, for  $i \geq 1$ ,

$$(2.19) \quad \sup\{|b_i| : \{b_j\} \text{ and } \{\alpha_{i,j}\} \text{ satisfy (2.1) and (2.2)}\} = Z_i(\mathbf{a}).$$




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*Proof.* Suppose that  $\{b_i\}$  satisfies (2.1) and (2.2), and set  $\zeta_i = Z_i(\mathbf{a})$  and  $M_i = \max\{1, \zeta_i\}$ , for  $i \geq 1$ . From (2.18), we have

$$(2.20) \quad A_{i+1}M_i = \zeta_{i+1},$$

for  $i \geq 1$ . By Lemma 2.1, we may find sequences  $\{d_i\}$  and  $\{a_i\}$  satisfying (2.3) such that

$$(2.21) \quad |d_n| \geq |b_n|.$$

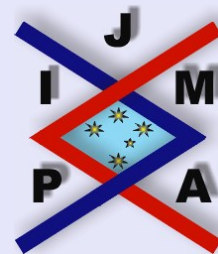
We will show that  $\{d_i\}$  satisfies the inequality

$$(2.22) \quad |d_l + d_{l+1} + \cdots + d_i| \leq M_i,$$

for  $0 \leq l \leq i$ .

Note that (2.22) (for  $i = n - 1$ ) and (2.3) imply that  $d_n = 0$  or  $a_n \leq n - 1$  and

$$(2.23) \quad \begin{aligned} |d_n| &= \left| \sum_{j=a_n}^{n-1} (-A_n)d_j \right| \\ &= A_n \left| \sum_{j=a_n}^{n-1} d_j \right| \\ &\leq A_n M_{n-1} \\ &= \zeta_n. \end{aligned}$$




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Since  $d_0 = 1$ ,  $d_1 \in \{0, -A_1\}$  and

$$\begin{aligned}
 \max\{|d_1|, |d_0 + d_1|\} &= \max\{1, A_1, |1 - A_1|\} \\
 &= \max\{1, A_1\} \\
 (2.24) \qquad \qquad \qquad &= M_1,
 \end{aligned}$$

i.e. the inequality in (2.22) holds for  $i = 1$ . Hence, suppose that (2.22) holds for  $i < N$ . Rewriting  $d_N$ , with  $v = a_N$ , we have for  $0 \leq x \leq N - 1$ ,

$$\begin{aligned}
 &d_x + d_{x+1} + \cdots + d_N \\
 &= (d_x + d_{x+1} + \cdots + d_{N-1}) - A_N(d_v + \cdots + d_{N-1}) \\
 (2.25) \quad &= \begin{cases} (1 - A_N)(d_v + \cdots + d_{N-1}) + (d_x + \cdots + d_{v-1}), & \text{if } v > x \\ (1 - A_N)(d_x + \cdots + d_{N-1}) \\ \qquad \qquad \qquad - A_N(d_v + \cdots + d_{x-1}), & \text{if } v \leq x \end{cases} .
 \end{aligned}$$

Let

$$S_1 = \begin{cases} d_v + \cdots + d_{N-1}, & \text{if } v > x \\ d_x + \cdots + d_{N-1}, & \text{if } v \leq x \end{cases} ,$$

and

$$S_2 = \begin{cases} d_x + \cdots + d_{v-1}, & \text{if } v > x \\ d_v + \cdots + d_{x-1}, & \text{if } v \leq x \end{cases} .$$

In showing that  $|d_x + d_{x+1} + \cdots + d_N| \leq M_N$ , we will consider several cases depending on whether  $A_N > 1$  or  $A_N \leq 1$ , and the signs of  $S_1$  and  $S_2$ .

**Case 1** ( $A_N > 1$  and  $S_1 S_2 > 0$ )



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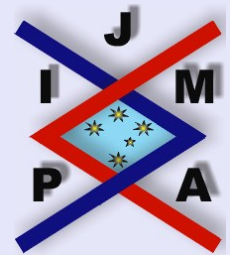
1.  $v > x$ .

$$\begin{aligned}
 |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\
 &\leq \max\{A_N|S_1|, A_N|S_2|\} \\
 &\leq A_N \max\{M_{N-1}, M_{v-1}\} \\
 &\leq A_N M_{N-1} \\
 &= \zeta_N \\
 (2.26) \qquad &= M_N,
 \end{aligned}$$

where the first inequality follows since  $(1 - A_N)S_1$  and  $S_2$  are of opposite signs and  $A_n > 1$ . The second inequality follows from induction. The last equalities are direct consequences of the definition of  $M_N$  and the fact that  $A_N > 1$ . The monotonicity of  $\{M_i\}$  is employed in obtaining the third inequality.

2.  $v \leq x$ .

$$\begin{aligned}
 |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\
 &\leq |A_N S_1 + A_N S_2| \\
 &= A_N |S_1 + S_2| \\
 &= A_N |d_v + d_{v+1} + \cdots + d_{N-1}| \\
 &\leq A_N M_{N-1} \\
 &= \zeta_N \\
 (2.27) \qquad &= M_N.
 \end{aligned}$$




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In (2.27), the first inequality follows since  $(1 - A_N)S_1$  and  $-A_N S_2$  are of the same sign.

**Case 2** ( $A_N > 1$  and  $S_1 S_2 \leq 0$ )

1.  $v > x$ .

$$(2.28) \quad \begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\ &= |-A_N S_1 + (S_1 + S_2)|. \end{aligned}$$

If  $S_1$  and  $S_1 + S_2$  are of the same sign, then

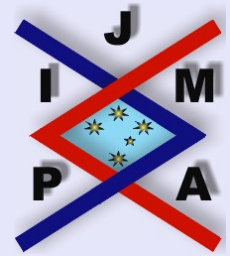
$$(2.29) \quad \begin{aligned} |-A_N S_1 + (S_1 + S_2)| &\leq \max\{A_N |S_1|, |S_1 + S_2|\} \\ &\leq A_N M_{N-1} \\ &= M_N. \end{aligned}$$

Otherwise,

$$(2.30) \quad \begin{aligned} |-A_N S_1 + (S_1 + S_2)| &\leq |-A_N S_1 + A_N(S_1 + S_2)| \\ &= A_N |S_2| \\ &\leq A_N M_{N-1} \\ &= M_N. \end{aligned}$$

2.  $v \leq x$ .

$$(2.31) \quad \begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\ &\leq \max\{A_N |S_1|, A_N |S_2|\} \\ &\leq A_N M_{N-1} \\ &= M_N \end{aligned}$$



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**Case 3** ( $A_N \leq 1$  and  $S_1 S_2 > 0$ )

Note that for  $A_N \leq 1$ ,  $M_i = 1$  for all  $i$ .

1.  $v > x$ .

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\ &\leq |S_1 + S_2| \\ &\leq M_{N-1} \\ (2.32) \qquad \qquad \qquad &= M_N. \end{aligned}$$

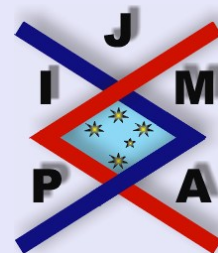
2.  $v \leq x$ .

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\ &\leq \max\{|S_1|, |S_2|\} \\ &\leq M_{N-1} \\ (2.33) \qquad \qquad \qquad &= M_N. \end{aligned}$$

**Case 4** ( $A_N \leq 1$  and  $S_1 S_2 \leq 0$ )

1.  $v > x$ .

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\ &\leq \max\{|S_1|, |S_2|\} \\ &\leq \max\{M_{N-1}, M_{v-1}\} \\ (2.34) \qquad \qquad \qquad &= M_N. \end{aligned}$$



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2.  $v \leq x$ .

$$\begin{aligned}
 |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\
 &\leq |S_1 + S_2| \\
 &\leq M_{N-1} \\
 (2.35) \qquad \qquad \qquad &= M_N.
 \end{aligned}$$

Thus, in all cases  $|d_x + d_{x+1} + \cdots + d_N| \leq M_N$  and hence by (2.23),  $|d_N| \leq \zeta_N$ . Equation (2.19) now follows since, for  $1 \leq h \leq n$ ,  $|b_n| = A_h A_{h+1} \cdots A_n$  is attained for  $[\alpha_{i,j}]$  defined by

$$(2.36) \qquad \alpha_{i,j} = \begin{cases} -A_h, & \text{if } i = h \\ -A_i, & \text{if } i > h, j = i \\ 0, & \text{otherwise} \end{cases} .$$

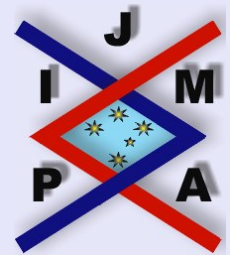
□

We close this section with an elementary result (without proof) which will serve to connect entries in  $L_n^{-1}$  with solutions to (2.1).

**Lemma 2.4.** *Suppose  $M = [m_{i,j}]_{n \times n}$  and  $\mathbf{y} = [y_i]_{n \times 1}$ , satisfy  $M\mathbf{y} = (1, 0, \dots, 0)'$ , with  $M$  an invertible lower triangular matrix. Then,  $y_1 = 1/m_{1,1}$ , and*

$$(2.37) \qquad y_i = \sum_{j=1}^{i-1} \left( -\frac{m_{i,j}}{m_{i,i}} \right) y_j,$$

for  $2 \leq i \leq n$ .



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### 3. The Main Result

We are now in a position to prove our main result.

**Theorem 3.1.** *Suppose  $\kappa = (\kappa_i)$  satisfies*

$$(3.1) \quad 0 \leq \kappa_1 \leq \kappa_2 \leq \kappa_3 \leq \dots,$$

and set

$$(3.2) \quad S \stackrel{\text{def}}{=} \{i : \kappa_i > 1\}.$$

As well, define  $\{W_{i,j}\}$  by

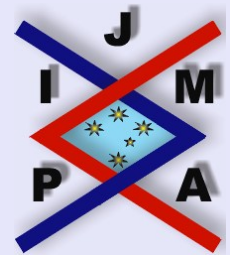
$$(3.3) \quad W_{i,j} \stackrel{\text{def}}{=} \prod_{v \in (S \cap \{j, j+1, \dots, i-2\}) \cup \{i-1\}} \kappa_v.$$

Then, for  $1 \leq i \leq n$ ,  $|x_{i,i}| \leq 1/l_{i,i}$  and for  $1 \leq j < i \leq n$ ,

$$(3.4) \quad |x_{i,j}| \leq \frac{W_{i,j}}{l_{j,j}}.$$

*Proof.* Suppose that  $n \geq 1$  and  $\mathbf{X}_n = \mathbf{L}_n^{-1}$ . Solving for the sub-diagonal entries in the  $p^{\text{th}}$  column of  $\mathbf{X}_n$  leads to the matrix equation

$$\begin{pmatrix} l_{p,p} & & & & \\ l_{p+1,p} & l_{p+1,p+1} & & & \\ \vdots & \vdots & \ddots & & \\ l_{n,p} & l_{n,p+1} & \cdots & l_{n,n} & \end{pmatrix} \begin{pmatrix} x_{p,p} \\ x_{p+1,p} \\ \vdots \\ x_{n,p} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$



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Applying Lemma 2.4 gives  $x_{p,p} = 1/l_{p,p}$ , and

$$(3.5) \quad x_{p+i,p} = \sum_{j=0}^{i-1} \left( -\frac{l_{p+i,p+j}}{l_{p+i,p+i}} \right) x_{p+j,p},$$

for  $1 \leq i \leq n - p$ .

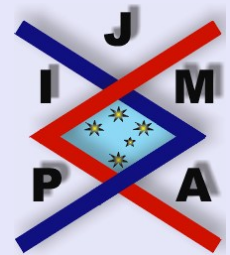
Now, note that (1.2) gives

$$(3.6) \quad 0 \leq \frac{l_{p+i,p}}{l_{p+i,p+i}} \leq \frac{l_{p+i,p+1}}{l_{p+i,p+i}} \leq \dots \leq \frac{l_{p+i,p+i-1}}{l_{p+i,p+i}} \leq \kappa_{p+i-1}.$$

Hence by Lemma 2.3,

$$(3.7) \quad \begin{aligned} |x_{p+i,p}| &\leq |x_{p,p}| Z_i((\kappa_p, \kappa_{p+1}, \dots, \kappa_{p+i-1})) \\ &= \frac{1}{l_{p,p}} W_{p+i,p}, \end{aligned}$$

for  $1 \leq i \leq n - p$ , and the theorem follows. □




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## 4. Examples

In this section, we provide examples to illustrate some of the structural information contained in Theorem 3.1.

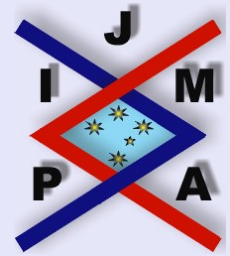
**Example 4.1 (Equally spaced  $A_i$ ).** Suppose that  $A_i = Ci$  for  $i \geq 1$ , where  $C > 0$ . Then, for  $n \geq 1$ ,

$$Z_n(\mathbf{a}) = \begin{cases} nC, & C \in (0, \frac{1}{n-1}]; \\ (n)_k C^k, & C \in (\frac{1}{n-k+1}, \frac{1}{n-k}], (2 \leq k \leq n-1); \\ n!C^n, & C \in (1, \infty), \end{cases}$$

where  $(n)_k = n(n-1) \cdots (n-k+1)$ .

Consider the matrix

$$L_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0.75 & 0.75 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1.25 & 1.25 & 1.25 & 1.25 & 1 & 0 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1 \end{pmatrix},$$



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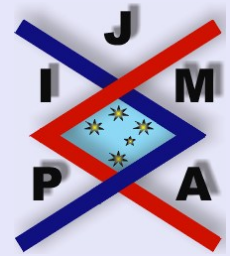
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with (rounded to three decimal places)

$$(4.1) \quad X_7 = L_7^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.375 & -0.5 & 1 & 0 & 0 & 0 & 0 \\ -0.281 & -0.375 & -0.75 & 1 & 0 & 0 & 0 \\ -0.094 & -0.125 & -0.25 & -1 & 1 & 0 & 0 \\ 1.25 & 0 & 0 & 0 & -1.25 & 1 & 0 \\ -1.875 & 0 & 0 & 0 & 0.375 & -1.5 & 1 \end{pmatrix}.$$

Applying Theorem 3.1, with  $\kappa = (.25, .50, .75, 1.00, 1.25, 1.50, \dots)$  gives the entry-wise bounds

$$(4.2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0.75 & 0.75 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1 & 0 \\ 1.875 & 1.875 & 1.875 & 1.875 & 1.875 & 1.5 & 1 \end{pmatrix}.$$




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Comparing (4.1) and (4.2), the absolute values of entry-wise ratios are

$$(4.3) \quad \begin{pmatrix} 1 & & & & & & & & \\ 1 & 1 & & & & & & & \\ 0.75 & 1 & 1 & & & & & & \\ 0.375 & 0.5 & 1 & 1 & & & & & \\ 0.094 & 0.125 & 0.25 & 1 & 1 & & & & \\ 1 & 0 & 0 & 0 & 1 & 1 & & & \\ 1 & 0 & 0 & 0 & 0.2 & 1 & 1 & & \end{pmatrix}.$$

Note that here  $\mathbf{L}_7$  was constructed so that  $|x_{7,1}| = W_{7,1}$ . In fact, as suggested by (2.19), for each 4-tuple  $(\kappa, I, J, n)$  with  $1 \leq J \leq I \leq n$ , there exists a pair  $(\mathbf{L}_n, \mathbf{X}_n)$  satisfying (1.2) with  $\mathbf{X}_n = (x_{i,j}) = \mathbf{L}_n^{-1}$ , such that  $|x_{I,J}| = W_{I,J}$ .

**Example 4.2 (Constant  $A_i$ ).** Suppose that  $A_i = C$  for  $i \geq 1$ , where  $C > 0$ . Then, for  $n \geq 1$ ,

$$Z_n(\mathbf{a}) = \begin{cases} C, & \text{if } C \leq 1 \\ C^n, & \text{if } C > 1 \end{cases}.$$

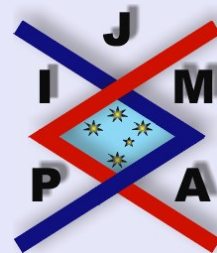
In [3], the following theorem was obtained when (2.2) is replaced with

$$(4.4) \quad 0 \leq \alpha_{i,j} \leq A,$$

for  $0 \leq j \leq i - 1$  and  $i \geq 1$ .

**Theorem 4.1.** Suppose that  $A > 0$  and  $m = [1/A]$ , where square brackets indicate the greatest integer function. If  $\{\Lambda_j\}_{j=1}^\infty$  is defined by

$$(4.5) \quad \Lambda_n = \max\{|b_i| : \{b_i\} \text{ and } [\alpha_{i,j}] \text{ satisfy (2.1) and (4.4)}\},$$



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for  $n \geq 1$ , then

$$(4.6) \quad \Lambda_n = \begin{cases} A, & \text{if } n = 1 \\ \max(A, A^2), & \text{if } n = 2 \\ \left[ \frac{n-2}{2} \right] \left[ \frac{n-1}{2} \right] A^3 + A, & \text{if } 3 \leq n \leq 2m + 1 \\ (n-2)A^2, & \text{if } n = 2m + 2 \\ A\Lambda_{n-1} + \Lambda_{n-2}, & \text{if } n \geq 2m + 3 \end{cases} .$$

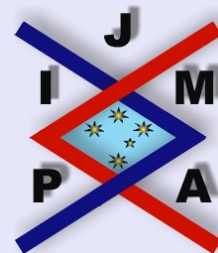
*Proof.* See [3]. □

Thus, if the monotonicity assumption in (2.2) is dropped the scenario is much different. In fact, in (4.6),  $\{\Lambda_n\}$  increases at an exponential rate for all  $A > 0$ . This leads to the following question.

**Open Question.** Set

$$(4.7) \quad \Lambda_n^* = \max\{|b_n| : \{b_i\} \text{ and } [\alpha_{i,j}]\} \\ \text{satisfy (2.1) and } \alpha_{i,j} \leq A_i \text{ for } 0 \leq j \leq i-1\}.$$

What is the value of  $\Lambda_n^*$  in terms of the sequence  $\{A_i\}$  and its assorted properties (eg. monotonicity, convexity etc.)?



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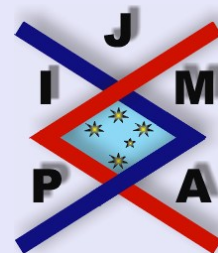
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## References

- [1] M. BENZI, AND G. GOLUB, Bounds for the entries of matrix functions with applications to preconditioning, *BIT*, **39**(3) (1999), 417–438.
- [2] K.S. BERENHAUT AND D. BANDYOPADHYAY, Monotone convex sequences and Cholesky decomposition of symmetric Toeplitz matrices, *Linear Algebra and Its Applications*, **403** (2005), 75–85.
- [3] K.S. BERENHAUT AND D.C. MORTON, Second order bounds for linear recurrences with negative coefficients, in press, *J. of Comput. and App. Math.*, (2005).
- [4] S. DEMKO, W. MOSS, AND P. SMITH, Decay rates for inverses of band matrices, *Math. Comp.*, **43** (1984), 491–499.
- [5] V. EIJKHOUT AND B. POLMAN, Decay rates of inverses of banded  $m$ -matrices that are near to Toeplitz matrices, *Linear Algebra Appl.*, **109** (1988), 247–277.
- [6] S. JAFFARD, Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **7**(5) (1990), 461–476.
- [7] R. NABBEN, Decay rates of the inverse of nonsymmetric tridiagonal and band matrices, *SIAM J. Matrix Anal. Appl.*, **20**(3) (1999), 820–837.
- [8] R. NABBEN, Two-sided bounds on the inverses of diagonally dominant tridiagonal matrices, Special issue celebrating the 60th birthday of Ludwig Elsner, *Linear Algebra Appl.*, **287**(1-3) (1999), 289–305.



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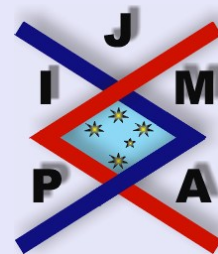
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- [9] R. PELUSO, AND T. POLITI, Some improvements for two-sided bounds on the inverse of diagonally dominant tridiagonal matrices, *Linear Algebra Appl.*, **330**(1-3) (2001), 1–14.
- [10] P.D. ROBINSON AND A.J. WATHEN, Variational bounds on the entries of the inverse of a matrix, *IMA J. Numer. Anal.*, **12**(4) (1992), 463–486.
- [11] T. STROHMER, Four short stories about Toeplitz matrix calculations, *Linear Algebra Appl.*, **343/344** (2002), 321–344.
- [12] A. VECCHIO, A bound for the inverse of a lower triangular Toeplitz matrix, *SIAM J. Matrix Anal. Appl.*, **24**(4) (2003), 1167–1174.




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