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AN EXTENSION OF THE HERMITE-HADAMARD INEQUALITY AND AN APPLICATION FOR GINI AND STOLARSKY MEANS

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ABSTRACT. In this paper we extend the Hermite-Hadamard inequality

$$f\left(\frac{p+q}{2}\right) \le \frac{1}{q-p}\int_p^q f(x)dx \le \frac{f(p)+f(q)}{2}$$

for convex-concave symmetric functions. As consequences some new inequalities for Gini and Stolarsky means are also derived.

Key words and phrases: Hadamard's Inequality, Gini means, Stolarsky means.

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1. INTRODUCTION

The so-called Hermite-Hadamard inequality [7] is one of the most investigated classical inequalities concerning convex functions. It reads as follows:

Theorem 1.1. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $f : \mathcal{I} \to \mathbb{R}$ be a convex function. Then, for all subintervals $[p, q] \subset \mathcal{I}$,*

$$f\left(\frac{p+q}{2}\right) \le \frac{1}{q-p}\int_p^q f(x)dx \le \frac{f(p)+f(q)}{2}, \tag{1.1}$$

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while in the case when f is concave all the inequalities are reversed, i.e.,

$$f\left(\frac{p+q}{2}\right) \geq \frac{1}{q-p}\int_p^q f(x)dx \geq \frac{f(p)+f(q)}{2}. \tag{1.2}$$

holds.

An account on the history of this inequality can be found in [9]. Surveys on various generalizations and developments can be found in [10] and [4]. The description of best possible inequalities of Hadamard-Hermite type are due to Fink [5]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to a two dimensional linear space of continuous functions.

In this form (1.1) and (1.2) are valid only for functions that are purely convex or concave on their whole domain. In Section 2 we will see that under appropriate conditions the same inequalities can be stated for a much larger family of functions. It will turn out that the results, obtained for this situation, can be applied for the Gini and Stolarsky means. In this way, we will obtain new inequalities for these classes of two variable homogeneous means.

2. THE EXTENSION OF THE HERMITE-HADAMARD INEQUALITY

Let $\mathcal{I} \subset \mathbb{R}$ be an arbitrary real interval and $m \in \overline{\mathcal{I}}$. A function $f : \mathcal{I} \to \mathbb{R}$ is called *symmetric with respect to point* m if the equation

$$f(m-t) + f(m+t) = 2f(m) \tag{2.1}$$

holds for all $t \in (\mathcal{I} - m) \cap (m - \mathcal{I})$. Observe that when m is one of the endpoints of the interval $\mathcal{I}$, then $(\mathcal{I} - m) \cap (m - \mathcal{I})$ is either empty or the singleton $\{m\}$, therefore (2.1) does not mean any restriction on f.

Concerning symmetric functions, we have the following obvious statement.

Lemma 2.1. *Let $f : \mathcal{I} \to \mathbb{R}$ be symmetric with respect to an element $m \in \overline{\mathcal{I}}$. Then*

$$\int_{m-\alpha}^{m+\alpha} f(x)dx = 2\alpha f(m)$$

for any positive α in $(\mathcal{I} - m) \cap (m - \mathcal{I})$.

Proof. By splitting the integral at the point m and applying substitutions $x = m - t$ and $x = m + t$, respectively, we get that

$$\begin{aligned}\int_{m-\alpha}^{m+\alpha} f(x)dx &= \int_{\alpha}^{0} -f(m-t)dt + \int_0^{\alpha} f(m+t)dt \\ &= \int_0^{\alpha} \big(f(m-t) + f(m+t)\big)dt.\end{aligned}$$

Due to the symmetry of f, the integrand equals $2f(m)$, which completes the proof. □

Theorem 2.2. *Let $f : \mathcal{I} \to \mathbb{R}$ be symmetric with respect to an element $m \in \overline{\mathcal{I}}$, furthermore, suppose that f is convex over the interval $\mathcal{I} \cap (-\infty, m]$ and concave over $\mathcal{I} \cap [m, \infty)$. Then, for any interval $[p, q] \subset \mathcal{I}$*

$$f\left(\frac{p+q}{2}\right) \underset{(\leq)}{\geq} \frac{1}{q-p}\int_p^q f(x)dx \underset{(\leq)}{\geq} \frac{f(p)+f(q)}{2} \tag{2.2}$$

holds if $\frac{p+q}{2} \underset{(\leq)}{\geq} m$.

(In (2.2) the reversed inequalities are valid if f is concave over the interval $\mathcal{I} \cap (-\infty, m]$ and convex over $\mathcal{I} \cap [m, \infty)$).

Proof. Suppose first that $\frac{p+q}{2} \geq m$. The case $p, q \geq m$ has no interest, since then Theorem 1.1 could be applied. Therefore, we may assume that $p < m < q$.

First we show the left hand side inequality in (2.2)

$$f\left(\frac{p+q}{2}\right) \geq \frac{1}{q-p}\int_p^q f(x)dx.$$

For this purpose, we split the integral into two parts:

$$\int_p^q f(x)dx = \int_p^{2m-p} f(x)dx + \int_{2m-p}^q f(x)dx.$$

Applying Lemma 2.1 with $\alpha = m - p$, the first integral is equal to $2(m-p)f(m)$. Moreover, due to the assumptions $p < m < q$ and $\frac{p+q}{2} \geq m$, we have that $m < 2m - p \leq q$. Therefore the function f is concave over the interval $[2m-p, q]$, thus, by Theorem 1.1, the value of the second integral is less than or equal to $(q-2m+p)f(\frac{q+2m-p}{2})$. That is, we have shown that

$$\int_p^q f(x)dx \leq 2(m-p)f(m) + (q-2m+p)f\left(\frac{q+2m-p}{2}\right).$$

Using the concavity of f over the interval $[m, \frac{q+2m-p}{2}]$, we obtain

$$\begin{aligned}2\frac{m-p}{q-p}f(m) &+ \frac{q-2m+p}{q-p}f\left(\frac{q+2m-p}{2}\right) \\ &\leq f\left(2\frac{m-p}{q-p}\cdot m + \frac{q-2m+p}{q-p}\cdot\frac{q+2m-p}{2}\right) = f\left(\frac{p+q}{2}\right).\end{aligned}$$

This inequality combined with previous one, immediately yields (2.2) and thus proof of the first part is complete.

Now we prove the right hand side inequality in (2.2). Using the symmetry of f and the concavity over the interval $[2m-p, q]$, Lemma 2.1 and Theorem 1.1 yield

$$\begin{aligned}\int_p^q f(x)dx &= \int_p^{2m-p} f(x)dx + \int_{2m-p}^q f(x)dx \\ &\geq 2(m-p)f(m) + (q-2m+p)\frac{f(2m-p)+f(q)}{2}.\end{aligned}$$

To complete the proof of (2.2), it is enough to show that

$$(2m-2p)f(m) + (q-2m+p)\frac{f(2m-p)+f(q)}{2} \geq (q-p)\frac{f(p)+f(q)}{2}. \tag{2.3}$$

For, we use, again, the concavity of f over the interval $[m, q]$. Thus,

$$\begin{aligned}f(2m-p) &= f\left(\frac{q-2m+p}{q-m}\cdot m + \frac{m-p}{q-m}\cdot q\right) \\ &\geq \frac{q-2m+p}{q-m}f(m) + \frac{m-p}{q-m}f(q).\end{aligned}\tag{2.4}$$

Substituting $f(p)$ by $2f(m) - f(2m-p)$ in (2.3), one can easily check that (2.4) and (2.3) are equivalent inequalities. Consequently, (2.3) follows from (2.4).

An analogous argument leads also to the result in the case $\frac{p+q}{2} \leq m$. Finally, if f is concave over the interval $\mathcal{I} \cap (-\infty, m]$ and convex over $\mathcal{I} \cap [m, \infty)$ then, applying what we have already proven for $-f$, the statement follows. $\square$

Remark 2.3. Theorem 1.1 can be considered as a special case of Theorem 2.2. For, one has to take m to be one of the endpoints of $\mathcal{I}$.

3. AN APPLICATION FOR GINI AND STOLARSKY MEANS

Given two real parameters a, b, if x, y are positive numbers, then their Gini mean $G_{a,b}$ (cf. [6]) is defined by:

$$G_{a,b}(x,y) = \begin{cases} \left(\frac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}} & \text{if } a \neq b, \\ \exp\left(\frac{x^a \log x + y^a \log y}{x^a+y^a}\right) & \text{if } a = b, \end{cases}$$

while their Stolarsky mean $S_{a,b}$ (cf. [14], [15]) is the following:

$$S_{a,b}(x,y) = \begin{cases} \left(\frac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{\frac{1}{a-b}} & \text{if } (a-b)ab \neq 0,\ x \neq y, \\ \exp\left(-\frac{1}{a} + \frac{x^a \log x - y^a \log y}{x^a-y^a}\right) & \text{if } a = b \neq 0,\ x \neq y, \\ \left(\frac{x^a-y^a}{a(\log x - \log y)}\right)^{\frac{1}{a}} & \text{if } a \neq 0,\ b = 0,\ x \neq y, \\ \sqrt{xy} & \text{if } a = b = 0, \\ x, & \text{if } x = y. \end{cases}$$

These definitions create a continuous, moreover, infinitely many times differentiable function

$$(a, b, x, y) \mapsto M_{a,b}(x,y)$$

on the domain $\mathbb{R}^2 \times \mathbb{R}_+^2$, where $M_{a,b}(x,y)$ can stand for either $G_{a,b}(x,y)$ or $S_{a,b}(x,y)$.

Nevertheless the cases in the definitions seem quite different, we will see that they all can be derived from the case of equal parameters, which – in a sense – plays a central role in our treatment. The following lemma is true:

Lemma 3.1. *Let the positive numbers x and y be fixed. Then for any real numbers a, b $(a \neq b)$ the following formula holds:*

$$\log M_{a,b}(x,y) = \frac{1}{a-b}\int_b^a \log M_{t,t}(x,y)dt. \tag{3.1}$$

Proof. For Gini means, we have

$$\begin{aligned} \frac{1}{a-b}\int_b^a \ln G_{t,t}(x,y)dt &= \frac{1}{a-b}\int_b^a \frac{x^t \ln x + y^t \ln y}{x^t+y^t}dt \\ &= \frac{1}{a-b}\Big[\ln\left(x^t+y^t\right)\Big]_b^a \\ &= \frac{1}{a-b}\ln\frac{x^a+y^a}{x^b+y^b} = \ln G_{a,b}(x,y). \end{aligned}$$

In the Stolarsky case we will assume that $x > y$ and $a > b$. If $0 < b < a$ or $b < a < 0$ then

$$\begin{aligned} \frac{1}{a-b}\int_b^a \ln S_{t,t}(x,y)dt &= \frac{1}{a-b}\int_b^a \left(-\frac{1}{t} + \frac{x^t \log x - y^t \log y}{x^t-y^t}\right)dt \\ &= \frac{1}{a-b}\left[\ln\left(\frac{x^t-y^t}{t}\right)\right]_b^a \\ &= \frac{1}{a-b}\ln\frac{\frac{x^a-y^a}{a}}{\frac{x^b-y^b}{b}} = \ln S_{a,b}(x,y). \end{aligned}$$

If $0 = b < a$ or $b < a = 0$ then we can apply the continuity of the integral as the function of its limits. For example,

$$\begin{aligned}\frac{1}{a}\int_0^a \ln S_{t,t}(x,y)dt &= \lim_{b\to 0+}\left(\frac{1}{a-b}\int_b^a\left(-\frac{1}{t}+\frac{x^t\log x - y^t\log y}{x^t-y^t}\right)\right)dt\\ &= \frac{1}{a}\lim_{b\to 0+}\left[\log\left(\frac{x^t-y^t}{t}\right)\right]_b^a\\ &= \frac{1}{a}\left(\log\frac{x^a-y^a}{a} - \lim_{b\to 0+}\log\frac{x^b-y^b}{b}\right)\\ &= \frac{1}{a}\left(\log\frac{x^a-y^a}{a} - \log(\log x - \log y)\right)\\ &= \log S_{a,0}(x,y).\end{aligned}$$

Finally, in the case $b < 0 < a$

$$\begin{aligned}\frac{1}{a-b}\int_b^a \ln S_{t,t}(x,y)dt &= \frac{1}{a-b}\left(\int_b^0 \log S_{t,t}(x,y)dt + \int_0^a \ln S_{t,t}(x,y)dt\right)\\ &= \frac{1}{a-b}\left(a\frac{1}{a}\left(\log\frac{x^a-y^a}{a} - \log(\log x - \log y)\right)\right.\\ &\qquad\left. - b\frac{1}{b}\left(\log\frac{x^b-y^b}{b} - \log(\log x - \log y)\right)\right)\\ &= \log S_{a,b}(x,y).\end{aligned}$$

□

In the sequel, the following results will prove to be useful.

Lemma 3.2. *For any positive* $x \neq 1$,

$$\frac{x(x+1)}{2} < \left(\frac{x-1}{\log x}\right)^3. \tag{3.2}$$

Proof. By Karamata's classical inequality (see [8, p. 272]), we have that

$$\frac{x+x^{1/3}}{1+x^{1/3}} < \frac{x-1}{\log x}. \tag{3.3}$$

Thus, it suffices to show that

$$\frac{x(x+1)}{2} < \left(\frac{x+x^{1/3}}{1+x^{1/3}}\right)^3. \tag{3.4}$$

Dividing both sides by x, then multiplying them by $2(1+x^{1/3})^3$, finally, collecting the terms on the right side, one can easily check that (3.4) becomes

$$0 < (x^{2/3}+x^{1/3}+1)(x^{1/3}-1)^4,$$

which is obviously true for all positive $x \neq 1$. □

The inequality stated in the above lemma can be translated to an inequality concerning the geometric, arithmetic and logarithmic means.

Corollary 3.3. *For all* $x, y > 0$,

$$S_{0,0}^2(x,y)\cdot S_{2,1}(x,y) \le S_{1,0}^3(x,y). \tag{3.5}$$

Proof. If $x = y$, then (3.5) is obvious. If $x \neq 1$ and $y = 1$, then (3.5) is literally the same as (3.2), hence (3.5) holds in this case, too. Now replacing x by x/y in (3.2), and using the homogeneity of the Stolarsky means, we get that (3.5) is valid for all positive $x \neq y$. □

Remark 3.4. Arguing in the same way as in the proof of Corollary 3.4, one can deduce that the inequalities (3.3) and (3.4) are equivalent to

$$S_{0,0}^2(x,y) \cdot G_{\frac{2}{3},\frac{1}{3}}(x,y) \leq S_{1,0}^3(x,y)$$

and

$$S_{2,1}(x,y) = G_{0,1}(x,y) \leq G_{\frac{2}{3},\frac{1}{3}}(x,y)$$

respectively. The latter inequality can also be derived from the comparison theorem of two variable Gini means (cf. [12], [13], [3]).

Our aim is to apply the results in Theorem 2.2 for Gini and Stolarsky means. For this purpose we will show that, for fixed positive x, y, the function

$$\mu_{x,y} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \log M_{t,t}(x,y) \tag{3.6}$$

satisfies the assumptions of Theorem 2.2.

Lemma 3.5. *Let x, y be arbitrary positive numbers. Then the function $\mu_{x,y}$ defined in (3.6) has the following properties:*

(i) $\mu_{x,y}(t) + \mu_{x,y}(-t) = 2\mu_{x,y}(0) \quad (t \in \mathbb{R})$,
(ii) $\mu_{x,y}$ *is convex over* $\mathbb{R}_-$ *and concave over* $\mathbb{R}_+$.

Proof. (i) For Gini means:

$$\begin{aligned}
\mu_{x,y}(t) + \mu_{x,y}(-t) &= \frac{x^t \log x + y^t \log y}{x^t + y^t} + \frac{x^{-t} \log x + y^{-t} \log y}{x^{-t} + y^{-t}} \\
&= \frac{x^t \log x + y^t \log y}{x^t + y^t} + \frac{y^t \log x + x^t \log y}{y^t + x^t} \\
&= \frac{x^t \log(xy) + y^t \log(xy)}{x^t + y^t} \\
&= \log(xy) = 2\mu_{x,y}(0),
\end{aligned}$$

while for Stolarsky means – assuming that $t \neq 0$ –

$$\begin{aligned}
\mu_{x,y}(t) + \mu_{x,y}(-t) &= -\frac{1}{t} + \frac{x^t \log x - y^t \log y}{x^t - y^t} + \frac{1}{t} + \frac{x^{-t} \log x - y^{-t} \log y}{x^{-t} - y^{-t}} \\
&= \frac{x^t \log x - y^t \log y}{x^t - y^t} + \frac{y^t \log x - x^t \log y}{y^t - x^t} \\
&= \frac{x^t \log(xy) - y^t \log(xy)}{x^t - y^t} \\
&= \log(xy) = 2\mu_{x,y}(0).
\end{aligned}$$

(ii) If $x = y$, then $\mu_{x,y}(t) = x$ for all $t \in \mathbb{R}$, hence $\mu_{x,y}$ is convex-concave everywhere. Therefore, we may assume that $x \neq y$.

In the case of Gini means,

$$t^3 \mu''_{x,y}(t) = -\frac{x^t y^t (\log x^t - \log y^t)^3 (x^t - y^t)}{(x^t + y^t)^3}.$$

Since the sign of $x^t - y^t$ is the same as that of $\log x^t - \log y^t$, therefore, $t^3 \mu''_{x,y}(t) \geq 0$ for all $t \in \mathbb{R}$. Thus, $\mu_{x,y}$ is convex over $\mathbb{R}_-$ and concave over $\mathbb{R}_+$.

In the setting of Stolarsky means, we have that

$$\begin{aligned} t^3\mu''_{x,y}(t) &= -2 + \frac{x^t y^t(\log x^t - \log y^t)^3(x^t + y^t)}{(x^t - y^t)^3} \\ &= -2\left(1 - \frac{S^2_{0,0}(x^t, y^t)S_{2,1}(x^t, y^t)}{S^3_{1,0}(x^t, y^t)}\right). \end{aligned}$$

In view of Corollary 3.4, it follows that $t^3\mu''_{x,y}(t) \geq 0$ for all $t \in \mathbb{R}$. Therefore, $\mu_{x,y}$ is convex over $\mathbb{R}_-$ and concave over $\mathbb{R}_+$ in this case, too.

□

As a consequence of Lemma 3.5 and Theorem 2.2, we can provide a lower and an upper estimate for $M_{a,b}$ in terms of the means $M_{\frac{a+b}{2}}$ and $\sqrt{M_{a,a} \cdot M_{b,b}}$.

Theorem 3.6. *Let a, b be real numbers so that $a + b \underset{(\leq)}{\geq} 0$. Then*

$$G_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} G_{a,b}(x,y) \underset{(\leq)}{\geq} \sqrt{G_{a,a}(x,y)G_{b,b}(x,y)}$$

and

$$S_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} S_{a,b}(x,y) \underset{(\leq)}{\geq} \sqrt{S_{a,a}(x,y)S_{b,b}(x,y)}$$

hold for any positive numbers x, y.

Proof. Let x, y be fixed positive numbers. By Lemma 3.5, the function $\mu_{x,y}$ is symmetric with respect to $m = 0$ and is convex (concave) on $\mathbb{R}_-$ (on $\mathbb{R}_+$). Therefore, Theorem 2.2 can be applied to $f := \mu_{x,y}$. Then, by (2.2),

$$\mu_{x,y}\left(\frac{a+b}{2}\right) \underset{(\leq)}{\geq} \frac{1}{a-b}\int_b^a \mu_{x,y}(t)dt \underset{(\leq)}{\geq} \frac{\mu_{x,y}(a) + \mu_{x,y}(b)}{2}$$

if $\frac{a+b}{2} \underset{(\leq)}{\geq} 0$. Thus, by the definition of $\mu_{x,y}$ and in view of Lemma 3.1, the following inequality holds:

$$\log M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} \log M_{a,b}(x,y) \underset{(\leq)}{\geq} \frac{\log M_{a,a}(x,y) + \log M_{b,b}(x,y)}{2}$$

if $a + b \underset{(\leq)}{\geq} 0$. Applying the exponential function to this inequality, we get that

$$M_{\frac{a+b}{2},\frac{a+b}{2}}(x,y) \underset{(\leq)}{\geq} M_{a,b}(x,y) \underset{(\leq)}{\geq} \sqrt{M_{a,a}(x,y)M_{b,b}(x,y)}$$

if $a + b \underset{(\leq)}{\geq} 0$. Hence the stated inequalities follow in the Gini and Stolarsky means setting, respectively. □

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