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UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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Abstract

In this paper we consider some integral operators and we determine conditions for the univalence of these integral operators.

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Contents

1. Introduction

Let $U = \{z \in C : |z| < 1\}$ be the unit disc in the complex plane. The class A and the class S are defined in [2]: let A be the class of functions $f(z) = z + a_2z^2 + \cdots$, which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$; let S the class of the functions $f \in A$ which are univalent in U.

In [7] is defined the class $S(\alpha)$. For $0 < \alpha \le 2$, let $S(\alpha)$ denote the class of functions $f \in A$ which satisfy the conditions:

$$f(z) \neq 0 \qquad \text{for } 0 < |z| < 1 \tag{1.1}$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \le \alpha \tag{1.2}$$

for all $z \in U$.

In [7] is proved the next result. For $0 < \alpha \le 2$, the functions $f \in S(\alpha)$ are univalent.

In this work, we consider the integral operators

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\, du \right]^{\frac{1}{\alpha}} \tag{1.3}$$

and

$$H_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{h(u)}{u} \right)^\gamma du \right]^{\frac{1}{\alpha}} \tag{1.4}$$

for $g(z) \in S$, $h(z) \in S$ and for some $\alpha, \gamma \in C$.

Kim - Merkes [1] studied the integral operator

$$F_\gamma(z) = \int_0^z \left(\frac{h(u)}{u}\right)^\gamma du \tag{1.5}$$

and obtained the following result

Theorem 1.1. *If the function $h(z)$ belongs to the class S, then for any complex number γ, $|\gamma| \leq \frac{1}{4}$, the function $F_\gamma(z)$ defined by (1.5) is in the class S.*

2. Preliminary Results

In order to prove our main results we will use the lemma due to N.N. Pascu [4] presented in this section.

Lemma 2.1. *Let the function* $f \in A$ *and* α *a complex number,* $\operatorname{Re}\alpha > 0$*. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in U$*, then for all complex numbers* β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is regular and univalent in U.

3. Main Results

Theorem 3.1. *Let* α *be a complex number,* $\operatorname{Re}\alpha \geq 0$ *and the function* $g \in S,\ \ g(z) = z + a_2z^2 + \cdots$ *. If*

$$(j_1) \qquad |\alpha - 1| \leq \frac{\operatorname{Re}\alpha}{4} \quad \textit{for } \operatorname{Re}\alpha \in (0,1)$$

or

$$(j_2) \qquad |\alpha - 1| \leq \frac{1}{4} \quad \textit{for } \operatorname{Re}\alpha \in [1,\infty),$$

then the function

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\,du\right]^{\frac{1}{\alpha}} \tag{3.1}$$

is in the class S.

Proof. From (3.1) we have

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{g(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{3.2}$$

The function $g(z)$ is regular and univalent, hence $\frac{g(z)}{z} \neq 0$ for all $z \in U$. We can choose the regular branch of the function $\left[\frac{g(z)}{z}\right]^{\alpha-1}$ to be equal to 1 at the origin.

Let us consider the regular function in U, given by

$$p(z)=\int_0^z\left(\frac{g(u)}{u}\right)^{\alpha-1}du. \tag{3.3}$$

Because $g\in S$, we obtain

$$\left|\frac{z\,g'(z)}{g(z)}\right|\le\frac{1+|z|}{1-|z|} \tag{3.4}$$

for all $z\in U$.

We have

$$\begin{aligned}\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,p''(z)}{p'(z)}\right|&=\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,g'(z)}{g(z)}-1\right|\\&\le\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right).\end{aligned} \tag{3.5}$$

From (3.5) and (3.4) we obtain

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{z\,p''(z)}{p'(z)}\right|\le\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}|\alpha-1|\frac{2}{1-|z|}. \tag{3.6}$$

Now, we consider the cases

$i_1)$ $\quad 0<\mathrm{Re}\,\alpha<1.$

The function

$$s:(0,1)\to\Re,\quad s(x)=1-a^{2x}\quad(0<a<1)$$

is a increasing function and for $a = |z|, \quad z \in U$, we obtain

$$1 - |z|^{2\,\mathrm{Re}\,\alpha} \leq 1 - |z|^2 \tag{3.7}$$

for all $z \in U$.

From (3.6) and (3.7), we have

$$\frac{1 - |z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left| \frac{zp''(z)}{p'(z)} \right| \leq \frac{4\,|\alpha - 1|}{\mathrm{Re}\,\alpha} \tag{3.8}$$

for all $z \in U$.

Using the condition (j_1) and (3.8) we get

$$\frac{1 - |z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left| \frac{zp''(z)}{p'(z)} \right| \leq 1 \tag{3.9}$$

for all $z \in U$.

i_2) $\quad \mathrm{Re}\,\alpha \geq 1.$

We observe that the function

$$q : [1, \infty) \to \Re, \qquad q(x) = \frac{1 - a^{2x}}{x} \quad (0 < a < 1)$$

is a decreasing function, and that, if we take $a = |z|$, $z \in U$, then

$$\frac{1 - |z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \leq 1 - |z|^2 \tag{3.10}$$

for all $z \in U$.

From (3.6) and (3.10) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \leq 4\,|\alpha-1|. \tag{3.11}$$

From (3.11) and (j_2), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1 \tag{3.12}$$

for all $z \in U$.

Using (3.9), (3.12) and because $p'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $G_\alpha(z)$ is in the class S. □

Theorem 3.2. *If α is a real number, $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$ and the function $g \in S(\alpha)$, then the function*

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u)\, du\right]^{\frac{1}{\alpha}} \tag{3.13}$$

is in the class S.

Proof. If $g \in S(\alpha)$, then $g \in S$ and by Theorem 3.1 for $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$, we obtain the function $G_\alpha(z)$ in the class S. □

Theorem 3.3. *Let α, γ be a complex numbers and the function $h \in S$, $h(z) = z + a_2 z^2 + \cdots$.*
If

$$|\gamma| \leq \frac{\operatorname{Re}\,\alpha}{4} \quad \textit{for} \quad \operatorname{Re}\alpha \in (0,1) \tag{p_1}$$

or

$$(p_2) \qquad |\gamma| \le \frac{1}{4} \quad \textit{for} \quad \operatorname{Re}\alpha \in [1,\infty)$$

then the function

$$H_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{\frac{1}{\alpha}} \tag{3.14}$$

is regular and univalent in U.

Proof. Let us consider the regular function in U, defined by

$$f(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\gamma} du. \tag{3.15}$$

For the function $h \in S$, we obtain

$$\left|\frac{z\,h'(z)}{h(z)}\right| \le \frac{1+|z|}{1-|z|} \tag{3.16}$$

for all $z \in U$.

We obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z\,f''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\gamma|\left(\left|\frac{zh'(z)}{h(z)}\right|+1\right). \tag{3.17}$$

From (3.17) and (3.16), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z\,f''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\gamma|\frac{2}{1-|z|} \tag{3.18}$$

We consider the cases

$j_1)$ $\quad 0 < \mathrm{Re}\ \alpha < 1.$

In this case we obtain

$$1 - |z|^{2\ \mathrm{Re}\ \alpha} \leq 1 - |z|^2 \tag{3.19}$$

for all $z \in U$.

From (3.18) and (3.19), we get

$$\frac{1 - |z|^{2\,\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{4\,|\gamma|}{\mathrm{Re}\ \alpha} \tag{3.20}$$

for all $z \in U$.

By (3.20) and (p_1) we have

$$\frac{1 - |z|^{2\,\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{3.21}$$

for all $z \in U$.

$j_2)$ $\quad \mathrm{Re}\ \alpha \geq 1.$

For this case we obtain

$$\frac{1 - |z|^{2\,\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \leq 1 - |z|^2 \tag{3.22}$$

for all $z \in U$.

From (3.18) and (3.22) we have

$$\frac{1 - |z|^{2\,\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 4\,|\gamma|. \tag{3.23}$$

From (3.23) and (p_2), we get

$$\frac{1-|z|^{2\,\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.24}$$

for all $z \in U$.

From (3.21), (3.24) and because $f'(z) = \left(\frac{h(z)}{z}\right)^{\gamma}$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $H_{\alpha,\gamma}(z)$ is in the class S. □

Remark 1. *For $\alpha = 1$, from Theorem 3.3 we obtain Theorem 1.1, the result due to Kim-Merkes.*

Theorem 3.4. *Let γ be a complex number and the function $h \in S(a)$.*
If

$$|\gamma| \leq \frac{\alpha}{4} \quad \textit{for} \quad \alpha \in (0,1) \tag{3.25}$$

or

$$|\gamma| \leq \frac{1}{4} \quad \textit{for} \quad \alpha \in [1,2] \tag{3.26}$$

then the function $H_{\alpha,\gamma}(z)$ defined by (3.14) is in the class S.

Proof. Because $h(z) \in S(\alpha)$, $0 < \alpha \leq 2$, then $h(z) \in S$ and by Theorem 3.3 the function $H_{\alpha,\gamma}(z)$ belongs to the class S. □

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