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UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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Abstract

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Abstract

In this paper we consider some integral operators and we determine conditions for the univalence of these integral operators.

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1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane. The class A and the class S are defined in [2]: let A be the class of functions $f(z) = z + a_2z^2 + \dots$, which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$; let S the class of the functions $f \in A$ which are univalent in U .

In [7] is defined the class $S(\alpha)$. For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f \in A$ which satisfy the conditions:

$$(1.1) \quad f(z) \neq 0 \quad \text{for } 0 < |z| < 1$$

and

$$(1.2) \quad \left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha$$

for all $z \in U$.

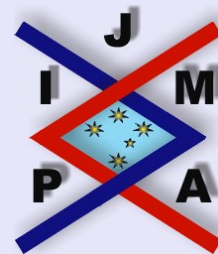
In [7] is proved the next result. For $0 < \alpha \leq 2$, the functions $f \in S(\alpha)$ are univalent.

In this work, we consider the integral operators

$$(1.3) \quad G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u) du \right]^{\frac{1}{\alpha}}$$

and

$$(1.4) \quad H_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{h(u)}{u} \right)^\gamma du \right]^{\frac{1}{\alpha}}$$



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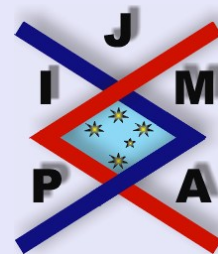
for $g(z) \in S$, $h(z) \in S$ and for some $\alpha, \gamma \in C$.

Kim - Merkes [1] studied the integral operator

$$(1.5) \quad F_\gamma(z) = \int_0^z \left(\frac{h(u)}{u} \right)^\gamma du$$

and obtained the following result

Theorem 1.1. *If the function $h(z)$ belongs to the class S , then for any complex number γ , $|\gamma| \leq \frac{1}{4}$, the function $F_\gamma(z)$ defined by (1.5) is in the class S .*



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2. Preliminary Results

In order to prove our main results we will use the lemma due to N.N. Pascu [4] presented in this section.

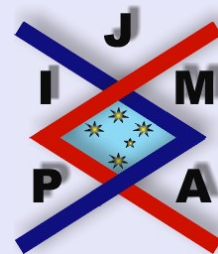
Lemma 2.1. *Let the function $f \in A$ and α a complex number, $\operatorname{Re} \alpha > 0$. If*

$$(2.1) \quad \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in U$, then for all complex numbers β , $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$ the function

$$(2.2) \quad F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}}$$

is regular and univalent in U .



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3. Main Results

Theorem 3.1. Let α be a complex number, $\operatorname{Re} \alpha \geq 0$ and the function $g \in S$, $g(z) = z + a_2 z^2 + \dots$. If

$$(j_1) \quad |\alpha - 1| \leq \frac{\operatorname{Re} \alpha}{4} \quad \text{for } \operatorname{Re} \alpha \in (0, 1)$$

or

$$(j_2) \quad |\alpha - 1| \leq \frac{1}{4} \quad \text{for } \operatorname{Re} \alpha \in [1, \infty),$$

then the function

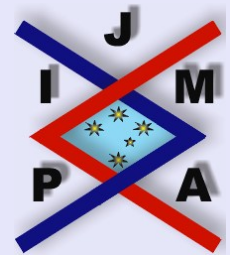
$$(3.1) \quad G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u) du \right]^{\frac{1}{\alpha}}$$

is in the class S .

Proof. From (3.1) we have

$$(3.2) \quad G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u} \right)^{\alpha-1} du \right]^{\frac{1}{\alpha}}.$$

The function $g(z)$ is regular and univalent, hence $\frac{g(z)}{z} \neq 0$ for all $z \in U$. We can choose the regular branch of the function $\left[\frac{g(z)}{z} \right]^{\alpha-1}$ to be equal to 1 at the origin.



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Let us consider the regular function in U , given by

$$(3.3) \quad p(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\alpha-1} du.$$

Because $g \in S$, we obtain

$$(3.4) \quad \left| \frac{z g'(z)}{g(z)} \right| \leq \frac{1 + |z|}{1 - |z|}$$

for all $z \in U$.

We have

$$(3.5) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z p''(z)}{p'(z)} \right| = \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z g'(z)}{g(z)} - 1 \right| \\ \leq \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha - 1| \left(\left| \frac{z g'(z)}{g(z)} \right| + 1 \right).$$

From (3.5) and (3.4) we obtain

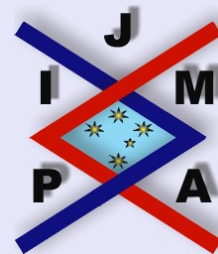
$$(3.6) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z p''(z)}{p'(z)} \right| \leq \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\alpha - 1| \frac{2}{1 - |z|}.$$

Now, we consider the cases

$i_1) \quad 0 < \operatorname{Re} \alpha < 1.$

The function

$$s : (0, 1) \rightarrow \mathfrak{R}, \quad s(x) = 1 - a^{2x} \quad (0 < a < 1)$$



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is an increasing function and for $a = |z|$, $z \in U$, we obtain

$$(3.7) \quad 1 - |z|^{2 \operatorname{Re} \alpha} \leq 1 - |z|^2$$

for all $z \in U$.

From (3.6) and (3.7), we have

$$(3.8) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z p''(z)}{p'(z)} \right| \leq \frac{4 |\alpha - 1|}{\operatorname{Re} \alpha}$$

for all $z \in U$.

Using the condition (j_1) and (3.8) we get

$$(3.9) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z p''(z)}{p'(z)} \right| \leq 1$$

for all $z \in U$.

$$i_2) \quad \operatorname{Re} \alpha \geq 1.$$

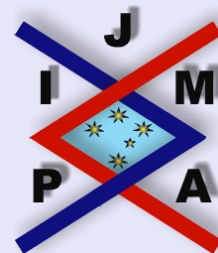
We observe that the function

$$q : [1, \infty) \rightarrow \mathfrak{R}, \quad q(x) = \frac{1 - a^{2x}}{x} \quad (0 < a < 1)$$

is a decreasing function, and that, if we take $a = |z|$, $z \in U$, then

$$(3.10) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \leq 1 - |z|^2$$

for all $z \in U$.



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From (3.6) and (3.10) we obtain

$$(3.11) \quad \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zp''(z)}{p'(z)} \right| \leq 4|\alpha - 1|.$$

From (3.11) and (j₂), we have

$$(3.12) \quad \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zp''(z)}{p'(z)} \right| \leq 1$$

for all $z \in U$.

Using (3.9), (3.12) and because $p'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $G_\alpha(z)$ is in the class S . \square

Theorem 3.2. *If α is a real number, $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$ and the function $g \in S(\alpha)$, then the function*

$$(3.13) \quad G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u) du \right]^{\frac{1}{\alpha}}$$

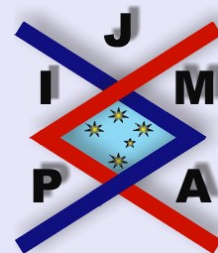
is in the class S .

Proof. If $g \in S(\alpha)$, then $g \in S$ and by Theorem 3.1 for $\alpha \in \left[\frac{4}{5}, \frac{5}{4}\right]$, we obtain the function $G_\alpha(z)$ in the class S . \square

Theorem 3.3. *Let α, γ be a complex numbers and the function $h \in S$, $h(z) = z + a_2z^2 + \dots$.*

If

$$(p_1) \quad |\gamma| \leq \frac{\operatorname{Re} \alpha}{4} \quad \text{for} \quad \operatorname{Re} \alpha \in (0, 1)$$



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or

$$(p_2) \quad |\gamma| \leq \frac{1}{4} \quad \text{for} \quad \operatorname{Re} \alpha \in [1, \infty)$$

then the function

$$(3.14) \quad H_{\alpha, \gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{h(u)}{u} \right)^\gamma du \right]^{\frac{1}{\alpha}}$$

is regular and univalent in U .

Proof. Let us consider the regular function in U , defined by

$$(3.15) \quad f(z) = \int_0^z \left(\frac{h(u)}{u} \right)^\gamma du.$$

For the function $h \in S$, we obtain

$$(3.16) \quad \left| \frac{z h'(z)}{h(z)} \right| \leq \frac{1 + |z|}{1 - |z|}$$

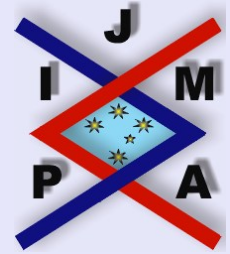
for all $z \in U$.

We obtain

$$(3.17) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\gamma| \left(\left| \frac{z h'(z)}{h(z)} \right| + 1 \right).$$

From (3.17) and (3.16), we have

$$(3.18) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} |\gamma| \frac{2}{1 - |z|}$$



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We consider the cases

$j_1)$ $0 < \operatorname{Re} \alpha < 1$.

In this case we obtain

$$(3.19) \quad 1 - |z|^{2 \operatorname{Re} \alpha} \leq 1 - |z|^2$$

for all $z \in U$.

From (3.18) and (3.19), we get

$$(3.20) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{4 |\gamma|}{\operatorname{Re} \alpha}$$

for all $z \in U$.

By (3.20) and (p_1) we have

$$(3.21) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1$$

for all $z \in U$.

$j_2)$ $\operatorname{Re} \alpha \geq 1$.

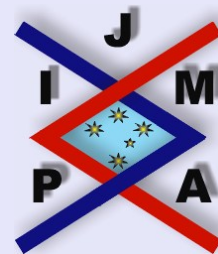
For this case we obtain

$$(3.22) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \leq 1 - |z|^2$$

for all $z \in U$.

From (3.18) and (3.22) we have

$$(3.23) \quad \frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 4 |\gamma|.$$



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From (3.23) and (p₂), we get

$$(3.24) \quad \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1$$

for all $z \in U$.

From (3.21), (3.24) and because $f'(z) = \left(\frac{h(z)}{z}\right)^\gamma$, from Lemma 2.1 for $\alpha = \beta$ it results that the function $H_{\alpha,\gamma}(z)$ is in the class S . \square

Remark 1. For $\alpha = 1$, from Theorem 3.3 we obtain Theorem 1.1, the result due to Kim-Merkes.

Theorem 3.4. Let γ be a complex number and the function $h \in S(a)$.

If

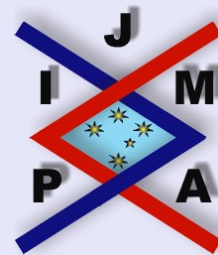
$$(3.25) \quad |\gamma| \leq \frac{\alpha}{4} \quad \text{for } \alpha \in (0, 1)$$

or

$$(3.26) \quad |\gamma| \leq \frac{1}{4} \quad \text{for } \alpha \in [1, 2]$$

then the function $H_{\alpha,\gamma}(z)$ defined by (3.14) is in the class S .

Proof. Because $h(z) \in S(\alpha)$, $0 < \alpha \leq 2$, then $h(z) \in S$ and by Theorem 3.3 the function $H_{\alpha,\gamma}(z)$ belongs to the class S . \square



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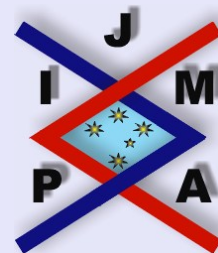
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