

Journal of Inequalities in Pure and Applied Mathematics

AN EXPLICIT MERTENS' TYPE INEQUALITY FOR ARITHMETIC PROGRESSIONS

OLIVIER BORDELLÈS

2 allée de la combe
La Boriette
43000 Aiguilhe, France.

EMail: borde43@wanadoo.fr

©2000 Victoria University
ISSN (electronic): 1443-5756
172-04



volume 6, issue 3, article 67,
2005.

*Received 27 September, 2004;
accepted 04 May, 2005.*

Communicated by: J. Sándor

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Abstract

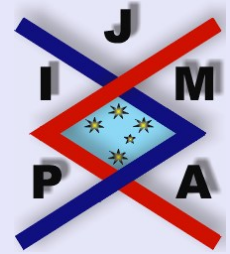
We give an explicit Mertens type formula for primes in arithmetic progressions using mean values of Dirichlet L -functions at $s = 1$.

2000 Mathematics Subject Classification: 11N13, 11M20

Key words: Mertens' formula, Arithmetic progressions, Mean values of Dirichlet L -functions.

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1. Introduction and Main Result

The very useful Mertens' formula states that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left\{1 + O\left(\frac{1}{\log x}\right)\right\}$$

for any real number $x \geq 2$, where $\gamma \approx 0.577215664\dots$ denotes the Euler constant. Some explicit inequalities have been given in [4] where it is showed for example that

$$(1.1) \quad \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} < e^{\gamma} \delta(x) \log x,$$

where

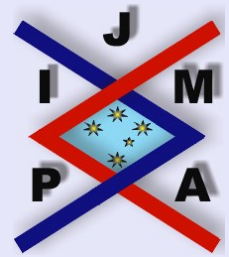
$$(1.2) \quad \delta(x) := 1 + \frac{1}{(\log x)^2}.$$

Let $1 \leq l \leq k$ be positive integers satisfying $(k, l) = 1$. The aim of this paper is to provide an explicit upper bound for the product

$$(1.3) \quad \prod_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \left(1 - \frac{1}{p}\right)^{-1}.$$

In [2, 5], the authors gave asymptotic formulas for (1.3) in the form

$$\prod_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \left(1 - \frac{1}{p}\right) \sim c(k, l) (\log x)^{-1/\varphi(k)},$$



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where φ is the Euler totient function and $c(k, l)$ is a constant depending on l and k . Nevertheless, because of the non-effectivity of the Siegel-Walfisz theorem, one cannot compute the implied constant in the error term. Moreover, the constant $c(k, l)$ is given only for some particular cases in [2], whereas K.S. Williams established a quite complicated expression of $c(k, l)$ involving a product of Dirichlet L -functions $L(s; \chi)$ and a function $K(s; \chi)$ at $s = 1$, where $K(s; \chi)$ is the generating Dirichlet series of the completely multiplicative function k_χ defined by

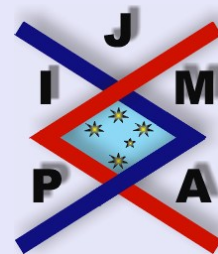
$$k_\chi(p) := p \left\{ 1 - \left(1 - \frac{\chi(p)}{p} \right) \left(1 - \frac{1}{p} \right)^{-\chi(p)} \right\}$$

for any prime number p and any Dirichlet character χ modulo k . The author then gave explicit expressions of $c(k, l)$ in the case $k = 24$.

It could be useful to have an explicit upper bound for (1.3) valid for a large range of k and x . Indeed, we shall see in a forthcoming paper that such a bound could be used to estimate class numbers of certain cyclic number fields. We prove the following result:

Theorem 1.1. *Let $1 \leq l \leq k$ be positive integers satisfying $(k, l) = 1$ and $k \geq 37$, and x be a positive real number such that $x > k$. We have:*

$$\prod_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \left(1 - \frac{1}{p} \right)^{-1}$$



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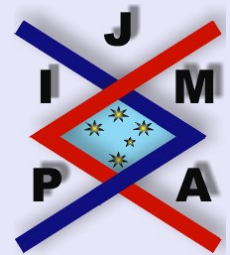
$$< e^{2(\gamma-B)} \sqrt{\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right)} \cdot \left(\frac{e^\gamma \varphi(k)}{k} \log x\right)^{\frac{1}{\varphi(k)}} \cdot \Phi(x, k),$$

where

$$\Phi(x, k) := \exp \left\{ \frac{2}{\log x} \left(\frac{2\sqrt{k} \log k}{\varphi(k)} \sum_{\chi \neq \chi_0} \left| \frac{L'}{L}(1; \chi) \right| + 2\sqrt{k} \log k + E - \gamma \right) \right\},$$

$B \approx 0.261497212847643 \dots$ and $E \approx 1.332582275733221 \dots$

The restriction $k \geq 37$ is given here just to use a simpler expression of the Polyá-Vinogradov inequality, but one can prove a similar result with $k \geq 9$ only, the constants in $\Phi(x, k)$ being slightly larger.



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2. Notation

p denotes always a prime, $1 \leq l \leq k$ are positive integers satisfying $(k, l) = 1$ and $k \geq 37$, $x > k$ is a real number,

$$\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \approx 0.5772156649015328 \dots$$

is the Euler constant and

$$\gamma_1 := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{\log k}{k} - \frac{(\log n)^2}{2} \right) \approx -0.07281584548367 \dots$$

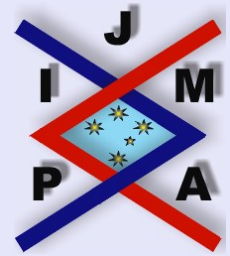
is the first Stieltjes constant. Similarly,

$$E := \lim_{n \rightarrow \infty} \left(\log n - \sum_{p \leq n} \frac{\log p}{p} \right) \approx 1.332582275733221 \dots$$

and

$$B := \lim_{n \rightarrow \infty} \left(\sum_{p \leq n} \frac{1}{p} - \log \log n \right) \approx 0.261497212847643 \dots$$

χ denotes a Dirichlet character modulo k and χ_0 is the principal character modulo k . For any Dirichlet character χ modulo k and any $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, $L(s; \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet L -function associated to χ . $\sum_{\chi \neq \chi_0}$ means that the sum is taken over all non-principal characters modulo k . Λ is the Von Mangoldt function and $f * g$ denotes the usual Dirichlet convolution product.



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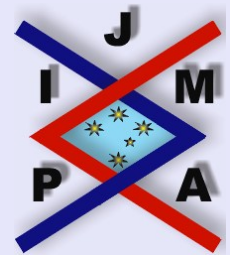
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3. Sums with Primes

From [4] we get the following estimates:

Lemma 3.1.

$$\sum_p \log p \sum_{\alpha=2}^{\infty} \frac{1}{p^\alpha} = E - \gamma \quad \text{and} \quad \sum_p \sum_{\alpha=2}^{\infty} \frac{1}{\alpha p^\alpha} = \gamma - B.$$



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4. The Polyá-Vinogradov Inequality and Character Sums with Primes

Lemma 4.1. Let χ be any non-principal Dirichlet character modulo $k \geq 37$.

(i) For any real number $x \geq 1$,

$$\left| \sum_{n \leq x} \chi(n) \right| < \frac{9}{10} \sqrt{k} \log k.$$

(ii) Let $F \in C^1([1; +\infty[, [0; +\infty[)$ such that $F(t) \searrow 0$ as $t \rightarrow \infty$. For any real number $x \geq 1$,

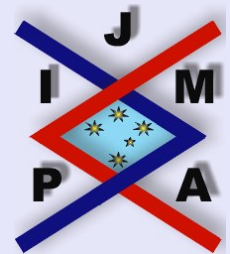
$$\left| \sum_{n > x} \chi(n) F(n) \right| \leq \frac{9}{5} F(x) \sqrt{k} \log k.$$

(iii) For any real number $x > k$,

$$\left| \sum_{p > x} \frac{\chi(p)}{p} \right| < \frac{2}{\log x} \left\{ 2\sqrt{k} \log k \left(\left| \frac{L'}{L}(1; \chi) \right| + 1 \right) + E - \gamma \right\}.$$

Proof.

- (i) The result follows from Qiu's improvement of the Polyá-Vinogradov inequality (see [3, p. 392]).
- (ii) Abel summation and (i).



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(iii) Let $\chi \neq \chi_0$ be a Dirichlet character modulo $k \geq 37$ and $x > k$ be any real number.

(a) Since $\chi(\mu * 1) = \varepsilon$ where $\varepsilon(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$ and $\mathbf{1}(n) = 1$, we get:

$$\sum_{d \leq x} \frac{\mu(d) \chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m)}{m} = 1$$

and hence, since $\chi \neq \chi_0$,

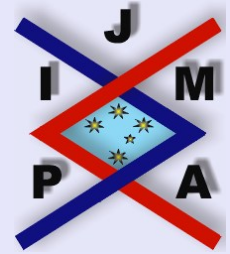
$$\sum_{d \leq x} \frac{\mu(d) \chi(d)}{d} = \frac{1}{L(1; \chi)} \left(\sum_{d \leq x} \frac{\mu(d) \chi(d)}{d} \sum_{m > x/d} \frac{\chi(m)}{m} + 1 \right)$$

and thus, using (ii),

$$(4.1) \quad \left| \sum_{d \leq x} \frac{\mu(d) \chi(d)}{d} \right| \leq \frac{\frac{9}{5} \sqrt{k} \log k + 1}{|L(1; \chi)|} < \frac{2\sqrt{k} \log k}{|L(1; \chi)|}.$$

(b) Since $\log = \Lambda * 1$, we get:

$$\begin{aligned} \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} &= \sum_{d \leq x} \frac{\mu(d) \chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m) \log m}{m} \\ &= \left(\sum_{d \leq x/e} + \sum_{x/e < d \leq x} \right) \sum_{m \leq x/d} \frac{\chi(m) \log m}{m} \end{aligned}$$



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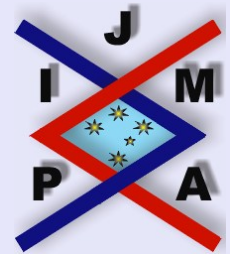
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$$\begin{aligned}
&= \sum_{d \leq x/e} \frac{\mu(d) \chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m) \log m}{m} + \frac{\chi(2) \log 2}{2} \sum_{x/e < d \leq x} \frac{\mu(d) \chi(d)}{d} \\
&= -L'(1; \chi) \sum_{d \leq x/e} \frac{\mu(d) \chi(d)}{d} - \sum_{d \leq x/e} \frac{\mu(d) \chi(d)}{d} \sum_{m > x/d} \frac{\chi(m) \log m}{m} \\
&\quad + \frac{\chi(2) \log 2}{2} \sum_{x/e < d \leq x} \frac{\mu(d) \chi(d)}{d}
\end{aligned}$$

and, by using (ii), (4.1) and the trivial bound for the third sum, we get:

$$\begin{aligned}
(4.2) \quad &\left| \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} \right| \\
&< \sqrt{k} \log k \left\{ 2 \left| \frac{L'}{L}(1; \chi) \right| \right. \\
&\quad \left. + \frac{9}{5x} \sum_{d \leq x/e} \log \left(\frac{x}{d} \right) + \frac{\log 2}{2} \left(1 + \frac{e}{x} \right) \right\} \\
&\leq \sqrt{k} \log k \left\{ 2 \left| \frac{L'}{L}(1; \chi) \right| + \frac{18}{5e} + \frac{\log 2}{2} \left(1 + \frac{e}{37} \right) \right\} \\
&< 2\sqrt{k} \log k \left\{ \left| \frac{L'}{L}(1; \chi) \right| + 1 \right\}
\end{aligned}$$

since $x > q \geq 37$.



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(c) By Abel summation, we get:

$$\left| \sum_{p>x} \frac{\chi(p)}{p} \right| \leq \frac{2}{\log x} \max_{t \geq x} \left| \sum_{p \leq t} \frac{\chi(p) \log p}{p} \right|.$$

Moreover,

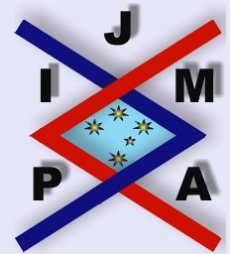
$$\sum_{p \leq t} \frac{\chi(p) \log p}{p} = \sum_{n \leq t} \frac{\chi(n) \Lambda(n)}{n} - \sum_p \sum_{\substack{\alpha=2 \\ p^\alpha \leq t}} \frac{\chi(p^\alpha) \log p}{p^\alpha}$$

and then:

$$\begin{aligned} \left| \sum_{p \leq t} \frac{\chi(p) \log p}{p} \right| &\leq \left| \sum_{n \leq t} \frac{\chi(n) \Lambda(n)}{n} \right| + \sum_p \log p \sum_{\alpha=2}^{\infty} \frac{1}{p^\alpha} \\ &= \left| \sum_{n \leq t} \frac{\chi(n) \Lambda(n)}{n} \right| + E - \gamma \\ &< 2\sqrt{k} \log k \left\{ \left| \frac{L'}{L} (1; \chi) \right| + 1 \right\} + E - \gamma \end{aligned}$$

by (4.2). This concludes the proof of Lemma 4.1.

□



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5. Mean Value Estimates of Dirichlet L -functions

Lemma 5.1.

(i) For any positive integers j, k ,

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{jk} \frac{1}{n} = \frac{\varphi(k)}{k} \left\{ \log(jk) + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right\} + \frac{c_0(j, k) 2^{\omega(k)}}{jk}$$

where $\omega(k) := \sum_{p|k} 1$ and $|c_0(j, k)| \leq 1$.

(ii) For any positive integer $k \geq 9$,

$$\left(\frac{k}{\varphi(k)} \right)^2 \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2} \right) + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log k + \sum_{p|k} \frac{\log p}{p-1} \right)^2 \leq 0.$$

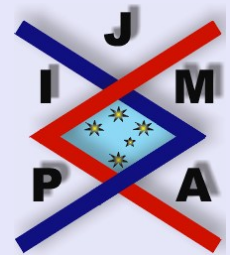
(iii) For any positive integer $k \geq 9$,

$$\prod_{\chi \neq \chi_0} |L(1; \chi)|^{1/\varphi(k)} \leq \sqrt{\zeta(2)} \prod_{p|k} \left(1 - \frac{1}{p^2} \right)^{\frac{1}{2}}.$$

Proof.

(i)

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{jk} \frac{1}{n} = \sum_{d|k} \frac{\mu(d)}{d} \sum_{n \leq jk/d} \frac{1}{n} = \sum_{d|k} \frac{\mu(d)}{d} \left(\log \left(\frac{jk}{d} \right) + \gamma + \frac{\varepsilon(d) d}{jk} \right)$$



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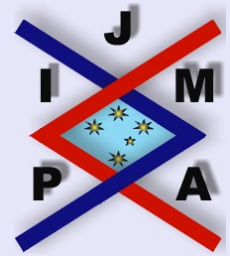


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where $|\varepsilon(d)| \leq 1$ and hence:

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{jk} \frac{1}{n} = \{\log(jk) + \gamma\} \sum_{d|k} \frac{\mu(d)}{d} - \sum_{d|k} \frac{\mu(d) \log d}{d} + \frac{1}{jk} \sum_{d|k} \varepsilon(d) \mu(d)$$

and we conclude by noting that

$$\sum_{d|k} \frac{\mu(d)}{d} = \frac{\varphi(k)}{k},$$

$$\sum_{d|k} \frac{\mu(d) \log d}{d} = -\frac{\varphi(k)}{k} \sum_{p|k} \frac{\log p}{p-1}$$

and

$$\left| \sum_{d|k} \varepsilon(d) \mu(d) \right| \leq \sum_{d|k} \mu^2(d) = 2^{\omega(k)}.$$

(ii) Define

$$A(k) := \left(\frac{k}{\varphi(k)} \right)^2 \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2} \right) + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log k + \sum_{p|k} \frac{\log p}{p-1} \right)^2.$$

Using [1] we check the inequality for $9 \leq k \leq 513$ and then suppose $k \geq 514$. Since

$$\frac{k}{\varphi(k)} = \prod_{p|k} \frac{p}{p-1} \leq \prod_{p|k} p^{\frac{1}{p-1}}$$

we have taking logarithms

$$\sum_{p|k} \frac{\log p}{p-1} \geq \log \left(\frac{k}{\varphi(k)} \right) \geq \log \left(\frac{k}{k-1} \right)$$

and from the inequality ([4])

$$\frac{k}{\varphi(k)} < e^\gamma \log \log k + \frac{2.50637}{\log \log k}$$

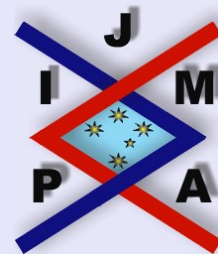
valid for any integer $k \geq 3$, we obtain

$$\begin{aligned} A(k) &\leq \zeta(2) \left(e^\gamma \log \log k + \frac{2.50637}{\log \log k} \right)^2 \\ &\quad + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log \left(\frac{k^2}{k-1} \right) \right)^2 < 0 \end{aligned}$$

if $k \geq 514$.

(iii) First,

$$\sum_{\chi \neq \chi_0} |L(1; \chi)|^2 = \lim_{N \rightarrow \infty} S(N)$$



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where

$$S(N) := \sum_{m,n=1}^{Nk} \frac{\chi(n) \bar{\chi}(m)}{nm} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2.$$

Following a standard argument, we have using (i):

$$S(N)$$

$$\begin{aligned} &= \varphi(k) \sum_{\substack{m \neq n=1 \\ m \equiv n \pmod{k} \\ (n,k)=(m,k)=1}}^{Nk} \frac{1}{mn} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\ &= \varphi(k) \sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n^2} + \varphi(k) \sum_{\substack{m \neq n=1 \\ m \equiv n \pmod{k} \\ (n,k)=(m,k)=1}}^{Nk} \frac{1}{mn} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\ &\leq \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2} \right) \\ &\quad + 2\varphi(k) \sum_{j=1}^N \sum_{\substack{n=1 \\ (n,k)=1}}^{(N-j)k} \frac{1}{n(n+jk)} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \end{aligned}$$



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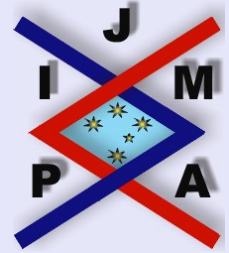


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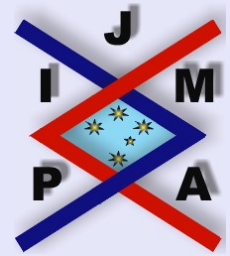
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$$\begin{aligned}
 &= \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\
 &\quad + \frac{2\varphi(k)}{k} \sum_{j=1}^N \frac{1}{j} \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{(N-j)k} \frac{1}{n} - \sum_{\substack{n=1+jk \\ (n,k)=1}}^{Nk} \frac{1}{n} \right) - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\
 &\leq \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) + \frac{2\varphi(k)}{k} \sum_{j=1}^N \frac{1}{j} \sum_{\substack{n=1 \\ (n,k)=1}}^{jk} \frac{1}{n} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\
 &= \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\
 &\quad + \frac{2\varphi(k)}{k} \sum_{j=1}^N \frac{1}{j} \left(\frac{\varphi(k)}{k} \left\{ \log(jk) + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right\} + \frac{c_0(j, k) 2^{\omega(k)}}{jk} \right) \\
 &\quad - \left(\frac{\varphi(k)}{k} \left\{ \log(Nk) + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right\} + \frac{c_0(N, k) 2^{\omega(k)}}{Nk} \right)^2.
 \end{aligned}$$

We now neglect the dependance of c_0 in k . Since

$$\sum_{m=1}^M \frac{1}{m} = \log M + \gamma + \frac{c_1(M)}{M}$$



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and

$$\sum_{m=1}^M \frac{\log m}{m} = \frac{(\log M)^2}{2} + \gamma_1 + \frac{c_2(M) \log M}{M},$$

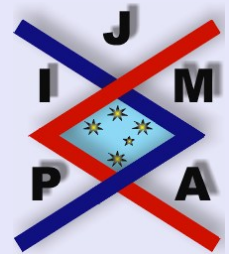
where $0 < c_1(M) \leq \frac{1}{2}$ and $|c_2(M)| \leq 1$, we get:

$$\begin{aligned} S(N) &\leq \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\ &\quad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{ (\log N)^2 + 2\gamma_1 + \frac{2c_2(N) \log N}{N} \right. \\ &\quad + 2 \left(\log k + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right) \left(\log N + \gamma + \frac{c_1(N)}{N} \right) \\ &\quad \left. - \left(\log(Nk) + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right)^2 \right\} \\ &\quad + \frac{2^{\omega(k)+1} \varphi(k)}{k^2} \left\{ \sum_{j=1}^N \frac{c_0(j)}{j^2} \right. \\ &\quad \left. - \frac{c_0(N)}{N} \left(\log(Nk) + \gamma + \sum_{p|k} \frac{\log p}{p-1} \right) \right\} - \frac{2^{2\omega(k)} c_0^2(N)}{N^2 k^2} \end{aligned}$$

$$\begin{aligned}
&= \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\
&\quad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{ 2\gamma_1 + \gamma - \left(\log k + \sum_{p|k} \frac{\log p}{p-1}\right)^2 \right. \\
&\quad + \left. \frac{2c_1(N)}{N} \left(\log k + \gamma + \sum_{p|k} \frac{\log p}{p-1}\right) + \frac{2c_2(N) \log N}{N} \right\} \\
&\quad + \frac{2^{\omega(k)+1} \varphi(k)}{k^2} \left\{ \sum_{j=1}^N \frac{c_0(j)}{j^2} \right. \\
&\quad \left. - \frac{c_0(N)}{N} \left(\log(Nk) + \gamma + \sum_{p|k} \frac{\log p}{p-1}\right) \right\} - \frac{2^{2\omega(k)} c_0^2(N)}{N^2 k^2}
\end{aligned}$$

and then

$$\begin{aligned}
\lim_{N \rightarrow \infty} S(N) &\leq \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\
&\quad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{ 2\gamma_1 + \gamma - \left(\log k + \sum_{p|k} \frac{\log p}{p-1}\right)^2 \right\} \\
&\quad + \frac{2^{\omega(k)} \varphi(k) \pi^2}{3k^2}
\end{aligned}$$



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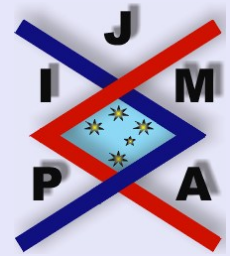
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and the inequality $2^{\omega(k)} \leq \varphi(k)$ (valid for any integer $k \geq 3$ and $\neq 6$) implies

$$\begin{aligned} \lim_{N \rightarrow \infty} S(N) &\leq \varphi(k) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\ &\quad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{ 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log k + \sum_{p|k} \frac{\log p}{p-1}\right)^2 \right\} \\ &= (\varphi(k) - 1) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\ &\quad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{ \left(\frac{k}{\varphi(k)}\right)^2 \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \right. \\ &\quad \left. + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log k + \sum_{p|k} \frac{\log p}{p-1}\right)^2 \right\} \\ &\leq (\varphi(k) - 1) \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \end{aligned}$$

if $k \geq 9$ by (ii). Hence

$$\frac{1}{\varphi(k) - 1} \sum_{\chi \neq \chi_0} |L(1; \chi)|^2 \leq \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right).$$



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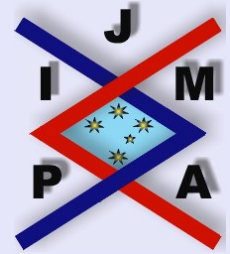
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Now the IAG inequality implies:

$$\begin{aligned}
 \prod_{\chi \neq \chi_0} |L(1; \chi)|^{\frac{1}{\varphi(k)}} &= \exp \left\{ \frac{1}{2\varphi(k)} \sum_{\chi \neq \chi_0} \log |L(1; \chi)|^2 \right\} \\
 &\leq \exp \left\{ \frac{\varphi(k) - 1}{2\varphi(k)} \log \left(\frac{1}{\varphi(k) - 1} \sum_{\chi \neq \chi_0} |L(1; \chi)|^2 \right) \right\} \\
 &\leq \left(\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2} \right) \right)^{\frac{\varphi(k)-1}{2\varphi(k)}} \\
 &\leq \left(\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2} \right) \right)^{\frac{1}{2}}.
 \end{aligned}$$

□



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6. Proof of the Theorem

Lemma 6.1. *If χ_0 is the principal character modulo k and if $x > k$, then:*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} < \frac{e^\gamma \varphi(k) \delta(x)}{k} \cdot \log x,$$

where δ is the function defined in (1.2).

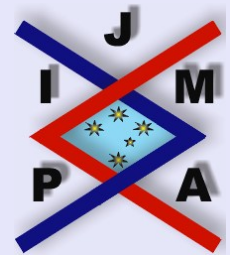
Proof. Since $x > k$,

$$\prod_{\substack{p \leq x \\ p|k}} \left(1 - \frac{1}{p}\right) = \prod_{p|k} \left(1 - \frac{1}{p}\right) = \frac{\varphi(k)}{k}$$

and then

$$\begin{aligned} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} &= \prod_{\substack{p \leq x \\ p \nmid k}} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p \leq x \\ p|k}} \left(1 - \frac{1}{p}\right) \\ &= \frac{\varphi(k)}{k} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \end{aligned}$$

and we use (1.1). □



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Proof of the theorem. Let $1 \leq l \leq k$ be positive integers satisfying $(k, l) = 1$ and $k \geq 37$, and x be a positive real number such that $x > k$. We have:

$$\prod_{\substack{p \leq x \\ p \equiv l \pmod{k}}} \left(1 - \frac{1}{p}\right)^{-\varphi(k)} = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} \cdot \prod_{\chi \neq \chi_0} \left(\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-\chi(p)} \right)^{\bar{\chi}(l)}$$

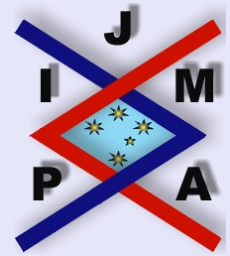
$$:= \Pi_1 \times \Pi_2$$

with $\Pi_1 < \frac{e^\gamma \varphi(k) \delta(x)}{k} \cdot \log x$ by Lemma 6.1. Moreover,

$$\begin{aligned} \Pi_2 &= \exp \left\{ \sum_{\chi \neq \chi_0} \bar{\chi}(l) \left(- \sum_{p \leq x} \chi(p) \log \left(1 - \frac{1}{p}\right) \right) \right\} \\ &= \exp \left(\sum_{\chi \neq \chi_0} \bar{\chi}(l) \sum_{p \leq x} \sum_{\alpha=1}^{\infty} \frac{\chi(p)}{\alpha p^\alpha} \right) \\ &= \exp \left\{ \sum_{\chi \neq \chi_0} \bar{\chi}(l) \left(\sum_{p \leq x} \frac{\chi(p)}{p} + \sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{\chi(p)}{\alpha p^\alpha} \right) \right\} \end{aligned}$$

and if $\chi \neq \chi_0$, we have

$$\begin{aligned} L(1; \chi) &= \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1} \\ &= \exp \left(\sum_{p \leq x} \frac{\chi(p)}{p} + \sum_{p > x} \frac{\chi(p)}{p} + \sum_p \sum_{\alpha=2}^{\infty} \frac{\chi(p^\alpha)}{\alpha p^\alpha} \right) \end{aligned}$$



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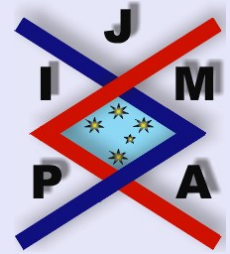
and thus

$$\begin{aligned} \Pi_2 &= \prod_{\chi \neq \chi_0} L(1; \chi)^{\bar{\chi}(l)} \\ &\times \exp \left\{ \sum_{\chi \neq \chi_0} \bar{\chi}(l) \left(- \sum_{p > x} \frac{\chi(p)}{p} + \sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{\chi(p)}{\alpha p^\alpha} - \sum_p \sum_{\alpha=2}^{\infty} \frac{\chi(p^\alpha)}{\alpha p^\alpha} \right) \right\} \end{aligned}$$

and hence

$$\begin{aligned} |\Pi_2| &\leq \prod_{\chi \neq \chi_0} |L(1; \chi)| \cdot \exp \left\{ \sum_{\chi \neq \chi_0} \left| \sum_{p > x} \frac{\chi(p)}{p} \right| + 2(\varphi(k) - 1) \sum_p \sum_{\alpha=2}^{\infty} \frac{1}{\alpha p^\alpha} \right\} \\ &= e^{2(\varphi(k)-1)(\gamma-B)} \prod_{\chi \neq \chi_0} |L(1; \chi)| \cdot \exp \left\{ \sum_{\chi \neq \chi_0} \left(\left| \sum_{p > x} \frac{\chi(p)}{p} \right| \right) \right\} \end{aligned}$$

and we use Lemma 4.1 (iii) and Lemma 5.1 (iii). We conclude the proof by noting that, if $x > 37$, $\frac{\delta(x)}{e^{2(\gamma-B)}} < 1$. \square



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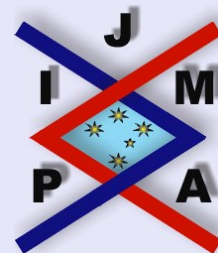
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