

ON CHAOTIC ORDER OF INDEFINITE TYPE

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Abstract: Let A, B be J -selfadjoint matrices with positive eigenvalues and $I \geq^J A, I \geq^J B$. Then it is proved as an application of Furuta inequality of indefinite type that

$$\text{Log } A \geq^J \text{Log } B$$

if and only if

$$A^r \geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

for all $p > 0$ and $r > 0$.



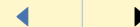
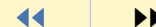
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In [2], T. Ando gave inequalities for matrices on an (indefinite) inner product space; for instance,

Proposition 1 ([2, Theorem 4]). *Let A, B be J -selfadjoint matrices with $\sigma(A), \sigma(B) \subseteq (\alpha, \beta)$. Then*

$$A \geq^J B \Rightarrow f(A) \geq^J f(B)$$

for any operator monotone function $f(t)$ on (α, β) .

Since the principal branch $\text{Log } x$ of the logarithm is operator monotone, as a corollary, we have

Corollary 2. *For J -selfadjoint matrices A, B with positive eigenvalues and $A \geq^J B$, we have*

$$\text{Log } A \geq^J \text{Log } B.$$

In this note, we give a characterization of this inequality relation, called a chaotic order, for J -selfadjoint matrices A, B with positive eigenvalues and $I \geq^J A, I \geq^J B$.

Before giving our theorem, we recall basic facts about matrices on an (indefinite) inner product space. We refer the reader to [3].

Let $M_n(\mathbb{C})$ be the set of all complex n -square matrices acting on \mathbb{C}^n and let $\langle \cdot, \cdot \rangle$ be the standard inner product on \mathbb{C}^n ; $\langle x, y \rangle := \sum_{i=1}^n x_i \bar{y}_i$ for $x = (x_i), y = (y_i) \in \mathbb{C}^n$. For a selfadjoint involution $J \in M_n(\mathbb{C})$; $J = J^*$ and $J^2 = I$, we consider the (indefinite) inner product $[\cdot, \cdot]$ on \mathbb{C}^n given by

$$[x, y] := \langle Jx, y \rangle \quad (x, y \in \mathbb{C}^n).$$



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The J -adjoint matrix A^\sharp of $A \in M_n(\mathbb{C})$ is defined as

$$[Ax, y] = [x, A^\sharp y] \quad (x, y \in \mathbb{C}^n).$$

In other words, $A^\sharp = JA^*J$. A matrix $A \in M_n(\mathbb{C})$ is said to be J -selfadjoint if $A^\sharp = A$ or $JA^*J = A$. And for J -selfadjoint matrices A and B , the J -order, denoted as $A \geq^J B$, is defined by

$$[Ax, x] \geq [Bx, x] \quad (x \in \mathbb{C}^n).$$

A matrix $A \in M_n(\mathbb{C})$ is called J -positive if $A \geq^J O$, or

$$[Ax, x] \geq 0 \quad (x \in \mathbb{C}^n).$$

A matrix $A \in M_n(\mathbb{C})$ is said to be a J -contraction if $I \geq^J A^\sharp A$ or $[x, x] \geq [Ax, Ax]$ ($x \in \mathbb{C}^n$). We remark that $I \geq^J A$ implies that all eigenvalues of A are real. Hence, for a J -contraction A all eigenvalues of $A^\sharp A$ are real. In fact, by a result of Potapov-Ginzburg (see [3, Chapter 2, Section 4]), all eigenvalues of $A^\sharp A$ are non-negative.

We also recall facts in [6]:

Proposition 3 ([6, Theorem 2.6]). *Let A, B be J -selfadjoint matrices with non-negative eigenvalues and $0 < \alpha < 1$. If*

$$I \geq^J A \geq^J B,$$

then J -selfadjoint powers A^α, B^α are well defined and

$$I \geq^J A^\alpha \geq^J B^\alpha.$$

Proposition 4 ([6, Lemma 3.1]). *Let A, B be J -selfadjoint matrices with non-negative eigenvalues and $I \geq^J A, I \geq^J B$. Then the eigenvalues of ABA are non-negative and*

$$I \geq^J A^\lambda$$

for all $\lambda > 0$.



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We also have a generalization; Furuta inequality of indefinite type:

Proposition 5 ([6, Theorem 3.4]). *Let A, B be J -selfadjoint matrices with non-negative eigenvalues and $I \geqslant^J A \geqslant^J B$. For each $r \geqslant 0$,*

$$\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

holds for all $p \geqslant 0, q \geqslant 1$ with $(1+r)q \geqslant p+r$.

Remark 1. Let $0 < \alpha < 1$. For J -selfadjoint matrices A, B with positive eigenvalues and $A \geqslant^J B$, we have

$$A^\alpha \geqslant^J B^\alpha,$$

by applying Proposition 1 to the operator monotone function x^α whose principal branch is considered. Hence,

$$\frac{A^\alpha - I}{\alpha} \geqslant^J \frac{B^\alpha - I}{\alpha}.$$

We remark that A^α is given by the Dunford integral and that

$$\frac{A^\alpha - I}{\alpha} = \frac{1}{2\pi i} \int_C \frac{\zeta^\alpha - 1}{\alpha} (\zeta I - A)^{-1} d\zeta,$$

where C is a closed rectifiable contour in the domain of ζ^α with positive direction surrounding all eigenvalues of A in its interior. Since

$$\frac{\zeta^\alpha - 1}{\alpha} \rightarrow \text{Log } \zeta \quad (\alpha \rightarrow 0)$$

uniformly for ζ , we also have Corollary 2.

Our theorem is as follows:



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Theorem 6. Let A, B be J -selfadjoint matrices with positive eigenvalues and $I \geq^J A, I \geq^J B$. Then the following statements are equivalent:

- (i) $\text{Log } A \geq^J \text{Log } B$.
- (ii) $A^r \geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p > 0$ and $r > 0$.

Here, principal branches of the functions are considered.

This theorem, as well as the corresponding result on a Hilbert space ([1, 4, 5, 7]), can be obtained and the similar approach in [7] also works. But careful arguments are necessary, and this is the reason for the present note.

Proof. (ii) \implies (i): Assume that

$$A^r \geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

for all $p > 0$ and $r > 0$. Then by Corollary 2, we have

$$r(p+r)\text{Log } A \geq^J r\text{Log } (A^{\frac{r}{2}} B^p A^{\frac{r}{2}}).$$

Dividing this inequality by $r > 0$ and taking p, r as $p = 1, r \rightarrow 0$, we have (i).

(i) \implies (ii): Since

$$I \geq^J A, B,$$

by assumption, it follows from Corollary 2 that

$$O = \text{Log } I \geq^J \text{Log } A, \text{Log } B.$$

Hence, for $n \in \mathbb{N}$

$$I \geq^J I + \frac{1}{n} \text{Log } A =: A_1, \quad I + \frac{1}{n} \text{Log } B =: B_1.$$



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For a sufficiently large n , all eigenvalues of A_1, B_1 are positive. Applying Proposition 5 to A_1, B_1 and $np, nr, \frac{nr+np}{nr}$ (resp.) as p, r, q (resp.), we get

$$(\sharp) \quad A_1^{nr} \geq^J \left(A_1^{\frac{nr}{2}} B_1^{np} A_1^{\frac{nr}{2}} \right)^{\frac{nr}{np+nr}}$$

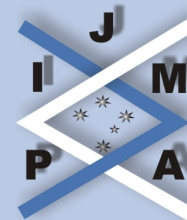
for all $p > 0, q > 0$. Recall that

$$\lim_{n \rightarrow \infty} \left(I + \frac{A}{n} \right)^n = e^A$$

for any matrix A and that $e^{\text{Log } X} = X$ for any matrix X with all eigenvalues positive. Therefore, taking n as $n \rightarrow \infty$ in the inequality (\sharp) , we obtain the conclusion. \square

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