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SEVERAL NEW PERTURBED OSTROWSKI-LIKE TYPE INEQUALITIES

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ABSTRACT. Several new perturbed Ostrowski-like type inequalities are established. Some recently results are generalized and other interesting inequalities are given as special cases. Furthermore, the first inequality we obtained is sharp.

Key words and phrases: Ostrowski's inequality, Ostrowski-like type inequality, Trapezoid type inequality, Sharp inequality, Mid-point-trapezoid type inequality.

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1. INTRODUCTION

In recent years a number of authors have considered error inequalities for some known and some new quadrature rules. Some have considered generalizations of these inequalities and estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae. For example, Ujević [7] obtained the following double integral inequality.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a, b)$. Then we have the double inequality:*

$$\frac{3S - \Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{3S-\gamma}{24}(b-a)^2, \tag{1.1}$$

where $S = (f'(b) - f'(a))/(b-a)$.

Ujević [8] derived the following perturbation of the trapezoid type inequality.

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Theorem 1.2. *If $f : [a,b] \to \mathbb{R}$ is such that f' is an absolutely continuous function and C is a constant, then*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} + \frac{C}{12}(b-a)^2 \right| \le \frac{\|f''-C\|_1}{8}(b-a). \tag{1.2}$$

Liu [6] established the following generalization of Ostrowski's inequality.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be (l,L)-Lipschitzian on $[a,b]$. Then for all $x \in [a,b]$, we have*

$$\begin{aligned} &\left| \frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a} \right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\qquad\qquad \le \frac{1}{2}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \min\{(S-l),(L-S)\}, \end{aligned} \tag{1.3}$$

where $S = (f(b)-f(a))/(b-a)$.

In this paper, we will derive several new perturbed Ostrowski-like type inequalities, which will not only provide generalizations of the above mentioned results, but also give some other interesting perturbed inequalities as special cases. Furthermore, the first inequality we obtain is sharp. Similar inequalities are also considered in [1] – [5] and [9] – [11].

2. MAIN RESULTS

Theorem 2.1. *Under the assumptions of Theorem 1.1, we have*

$$\begin{aligned} &\frac{\Gamma[(x-a)^3+(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^2 (S-\Gamma) \\ &\qquad \le \frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a} \right] - \frac{1}{b-a}\int_a^b f(t)dt \\ &\qquad \le \frac{\gamma[(x-a)^3+(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^2 (S-\gamma), \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$, where $S = \frac{f'(b)-f'(a)}{b-a}$. If γ, Γ are given by

$$\gamma = \min_{t\in[a,b]} f''(t), \qquad \Gamma = \max_{t\in[a,b]} f''(t)$$

then the inequality given by (2.1) is sharp in the usual sense.

Proof. Let $K(x,t) : [a,b]^2 \to \mathbb{R}$ be given by

$$K(x,t) = \begin{cases} \frac{1}{2}(x-t)(t-a), & t \in [a,x], \\ \frac{1}{2}(x-t)(t-b), & t \in (x,b]. \end{cases} \tag{2.2}$$

Then we have

$$\int_a^b K(x,t)dt = \frac{(x-a)^3+(b-x)^3}{12}. \tag{2.3}$$

Integrating by parts, we obtain (see [5])

$$\int_a^b K(x,t)f''(t)dt = \frac{1}{2}\{(b-a)f(x) + [(x-a)f(a)+(b-x)f(b)]\} - \int_a^b f(t)dt. \tag{2.4}$$

Then for any fixed $x \in [a,b]$ we can derive from (2.3) and (2.4) that

$$\int_a^b K(x,t)[f''(t)-\gamma]dt = -\int_a^b f(t)dt + \frac{1}{2}\{(b-a)f(x) + [(x-a)f(a)+(b-x)f(b)]\} - \frac{\gamma[(x-a)^3+(b-x)^3]}{12}. \tag{2.5}$$

We also have

$$\begin{aligned}\int_a^b K(x,t)[f''(t)-\gamma]dt &\le \max_{t\in[a,b]} |K(x,t)| \int_a^b |f''(t)-\gamma|dt \\ &= \frac{1}{8}\max\{(x-a)^2,(b-x)^2\}(S-\gamma)(b-a),\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\max\{(x-a)^2,(b-x)^2\} &= (\max\{x-a,b-x\})^2 \\ &= \frac{1}{4}\left[x-a+b-x+|x-a-b+x|\right]^2 \\ &= \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2.\end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7) we have

$$\begin{aligned}\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)dt \\ \le \frac{\gamma[(x-a)^3+(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2 (S-\gamma).\end{aligned} \tag{2.8}$$

On the other hand, we have

$$\int_a^b K(x,t)[\Gamma-f''(t)]dt = \int_a^b f(t)dt - \frac{1}{2}\{(b-a)f(x) + [(x-a)f(a)+(b-x)f(b)]\} + \frac{\Gamma[(x-a)^3+(b-x)^3]}{12} \tag{2.9}$$

and

$$\begin{aligned}\int_a^b K(x,t)[\Gamma-f''(t)]dt &\le \max_{t\in[a,b]} |K(x,t)| \int_a^b |\Gamma-f''(t)|dt \\ &= \frac{1}{8}\max\{(x-a)^2,(b-x)^2\}(\Gamma-S)(b-a).\end{aligned} \tag{2.10}$$

From (2.7), (2.9) and (2.10) we have

$$\begin{aligned}\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)dt \\ \ge \frac{\Gamma[(x-a)^3+(b-x)^3]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2 (S-\Gamma).\end{aligned} \tag{2.11}$$

From (2.8) and (2.11), we see that (2.1) holds.

If we now substitute $f(t)=(t-a)^2$ in the inequalities (2.1) then we find that the left-hand side, middle term and right-hand side are all equal to $\frac{(x-a)^3+(b-x)^3}{6(b-a)}$. Thus, the inequality (2.1) is sharp in the usual sense. □

Remark 2.2. We note that in the special cases, if we take $x = a$ or $x = b$ in (2.1), we get (1.1). Therefore Theorem 2.1 is a generalization of Theorem 1.1.

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $x = \frac{a+b}{2}$, we have the following sharp averaged mid-point-trapezoid type inequality*

$$\begin{aligned}\frac{3S-\Gamma}{96}(b-a)^2 &\le \frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{2}\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \\ &\le \frac{3S-\gamma}{96}(b-a)^2.\end{aligned} \tag{2.12}$$

Theorem 2.4. *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b f(t)dt &- \frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] \\ &+\frac{C[(x-a)^3+(b-x)^3]}{12(b-a)}\Bigg| \le \frac{1}{8(b-a)}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2\|f''-C\|_1\end{aligned} \tag{2.13}$$

for all $x \in [a,b]$.

Proof. Let $K(x,t)$ be given by (2.2). From (2.3) and (2.4), it follows that

$$\begin{aligned}\int_a^b K(x,t)[f''(t)-C]dt &= -\int_a^b f(t)dt + \frac{1}{2}\{(b-a)f(x) \\ &+ [(x-a)f(a)+(b-x)f(b)]\} - \frac{C[(x-a)^3+(b-x)^3]}{12}.\end{aligned} \tag{2.14}$$

We also have

$$\begin{aligned}\int_a^b K(x,t)[f''(t)-C]dt &\le \max_{t\in[a,b]}|K(x,t)|\int_a^b |f''(t)-C|dt \\ &= \frac{1}{8}\max\{(x-a)^2,(b-x)^2\}\|f''-C\|_1.\end{aligned} \tag{2.15}$$

From (2.7), (2.14) and (2.15), we easily obtain (2.13). □

Remark 2.5. We note that in the special cases, if we take $x = a$ or $x = b$ in (2.13), we get (1.2). Therefore Theorem 2.4 is a generalization of Theorem 1.2.

Corollary 2.6. *Under the assumptions of Theorem 2.4 with $x = \frac{a+b}{2}$, we have the following perturbed averaged mid-point-trapezoid type inequality*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{2}\frac{f(a)+f(b)}{2}+\frac{C}{48}(b-a)^2\right| \\ \le \frac{\|f''-C\|_1}{32}(b-a).\end{aligned} \tag{2.16}$$

Theorem 2.7. *Let the assumptions of Theorem 2.1 hold. Then we have the following perturbed Ostrowski type inequality*

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b f(t)dt &- \frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] \\ &+\frac{(\Gamma+\gamma)}{24}\frac{(x-a)^3+(b-x)^3}{b-a}\Bigg| \le \frac{\Gamma-\gamma}{8}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12}\right]\end{aligned} \tag{2.17}$$

for all $x \in [a,b]$.

Proof. Let $K(x,t) : [a,b]^2 \to \mathbb{R}$ be given by (2.2) and $C = (\Gamma + \gamma)/2$. From (2.3) and (2.4), it follows that

$$\int_a^b K(x,t)[f''(t) - C]dt = -\int_a^b f(t)dt + \frac{1}{2}\{(b-a)f(x) + [(x-a)f(a) + (b-x)f(b)]\} - \frac{C[(x-a)^3 + (b-x)^3]}{12}. \tag{2.18}$$

We also have

$$\left|\int_a^b K(x,t)[f''(t) - C]dt\right| \le \max_{t\in[a,b]} |f''(t) - \gamma| \int_a^b |K(x,t)|dt \le \frac{\Gamma - \gamma}{8}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right](b-a). \tag{2.19}$$

From (2.18) and (2.19), we easily obtain (2.17). □

Corollary 2.8. *Under the assumptions of Theorem 2.7 with $x = a$ or $x = b$ we have the following perturbed trapezoid type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2} + \frac{\Gamma+\gamma}{24}(b-a)^2\right| \le \frac{\Gamma-\gamma}{24}(b-a)^2. \tag{2.20}$$

Corollary 2.9. *Under the assumptions of Theorem 2.7 with $x = \frac{a+b}{2}$ we have the following perturbed averaged mid-point-trapezoid type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}f\left(\frac{a+b}{2}\right) - \frac{1}{2}\frac{f(a)+f(b)}{2} + \frac{\Gamma+\gamma}{96}(b-a)^2\right| \le \frac{\Gamma-\gamma}{96}(b-a)^2. \tag{2.21}$$

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