SOME INEQUALITIES FOR THE q-DIGAMMA FUNCTION

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Abstract: For the q-digamma function and it's derivatives are established the functional inequalities of the types:

$$f^2(x \cdot y) \lessapprox f(x) \cdot f(y),$$

$$f(x + y) \lessapprox f(x) + f(y).$$

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1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{1.1}$$

where $\gamma = 0.57721\ldots$ denotes Euler's constant. Another interesting series representation for ψ, which is "more rapidly convergent" than the one given in (1.1), was discovered by Ramanujan [3, page 374].

Jackson (see [5, 6, 7, 8]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1, \tag{1.2}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x} q^{\binom{x}{2}}, \quad q > 1, \tag{1.3}$$

where $(a;q)_\infty = \prod_{j \geq 0}(1 - aq^j)$.

The q-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the q-gamma function, that is,

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x).$$

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Many properties of the q-gamma function were derived by Askey [2]. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. From (1.2), for $0 < q < 1$ and $x > 0$ we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}} \tag{1.4}$$
$$= -\log(1-q) + \log q \sum_{n\geq 1} \frac{q^{nx}}{1-q^n}$$

and from (1.3) for $q > 1$ and $x > 0$ we obtain

$$\psi_q(x) = -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 0} \frac{q^{-n-x}}{1-q^{-n-x}} \right) \tag{1.5}$$
$$= -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 1} \frac{q^{-nx}}{1-q^{-n}} \right).$$

A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [4]. It is well-known that ψ' is strictly completely monotonic on $(0, \infty)$, that is,

$$(-1)^n (\psi'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0,$$

see [1, Page 260]. From (1.4) and (1.5) we conclude that ψ_q' has the same property for any $q > 0$

$$(-1)^n (\psi_q'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0.$$

If $q \in (0,1)$, using the second representation of $\psi_q(x)$ given in (1.4), it can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k \cdot q^{nx}}{1-q^n} \tag{1.6}$$

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and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in (1.5), we obtain

$$\psi_q'(x) = \log q \left(1 + \sum_{n\geq 1} \frac{nq^{-nx}}{1-q^{-nx}}\right) \tag{1.7}$$

and for $k \geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{-nx}}{1-q^{-nx}} \tag{1.8}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

In this paper we derive several inequalities for $\psi^{(k)}(x)$, where $k \geq 0$.

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2. Inequalities of the type $f^2(x \cdot y) \lessgtr f(x) \cdot f(y)$

We start with the following lemma.

Lemma 2.1. *For* $0 < q < \frac{1}{2}$ *and* $0 < x < 1$ *we have that* $\psi_q(x) < 0$.

Proof. At first let us prove that $\psi_q(x) < 0$ for all $x > 0$. From (1.4) we get that

$$\psi_q(x) = \frac{q^x}{1-q} \log q - \log(1-q) + \log q \sum_{n \geq 2} \frac{q^{nx}}{1-q^n}.$$

In order to see that $\psi_q(x) < 0$, we need to show that the function

$$g(x) = \frac{q^x}{1-q} \log q - \log(1-q)$$

is a negative for all $0 < x < 1$ and $0 < q < \frac{1}{2}$. Indeed $g'(x) = \frac{q^x}{1-q} \log^2 q > 0$, which implies that $g(x)$ is an increasing function on $0 < x < 1$, hence

$$\begin{aligned} g(x) < g(1) &= \frac{q}{1-q} \log q - \log(1-q) \\ &= \frac{1}{1-q} \log \frac{q^q}{(1-q)^{1-q}} < 0, \end{aligned}$$

for all $0 < q < \frac{1}{2}$. □

Theorem 2.2. *Let* $0 < q < \frac{1}{2}$ *and* $0 < x, y < 1$. *Let* $k \geq 0$ *be an integer. Then*

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) < (\psi_q^{(k)}(xy))^2.$$

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Proof. We will consider two different cases: (1) $k = 0$ and (2) $k \geq 1$.
(1) Let $f(x) = \psi_q^2(x)$ defined on $0 < x < 1$. By Lemma 2.1 we have that

$$f'(x) = 2\psi_q(x)\psi_q'(x) < 0$$

for all $0 < x < 1$, which gives that $f(x)$ is a decreasing function on $0 < x < 1$. Hence, for all $0 < x, y < 1$ we have

$$\psi_q^2(xy) > \psi_q^2(x) \quad \text{and} \quad \psi_q^2(xy) > \psi_q^2(y),$$

which gives that

$$\psi_q^4(xy) > \psi_q^2(x)\psi_q^2(y).$$

Since $\psi_q(x)\psi_q(y) > 0$ for all $0 < x, y < 1$, see Lemma 2.1, we obtain that

$$\psi_q^2(xy) > \psi_q(x)\psi_q(y),$$

as claimed.

(2) From (1.6) we have that

$$\begin{aligned}
&\psi_q^{(k)}(x)\psi_q^{(k)}(y) - (\psi_q^{(k)}(xy))^2 \\
&\quad = \left(\log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{nx}}{1-q^n}\right)\left(\log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{ny}}{1-q^n}\right) - \left(\log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{nxy}}{1-q^n}\right)^2 \\
&\quad = (\log^{k+1} q)^2 \sum_{n,m\geq 1} \frac{n^k q^{nx}}{1-q^n} \cdot \frac{m^k q^{my}}{1-q^m} - (\log^{k+1} q)^2 \sum_{n,m\geq 1} \frac{(nm)^k q^{(n+m)xy}}{(1-q^n)(1-q^m)} \\
&\quad = (\log^{k+1} q)^2 \sum_{n,m\geq 1} \frac{(nm)^k (q^{nx+my} - q^{(n+m)xy})}{(1-q^n)(1-q^m)}.
\end{aligned}$$

For $0 < x, y < 1$, $q^{nx+my} - q^{(n+m)xy} < 0$ and for $x, y > 1$, $q^{nx+my} - q^{(n+m)xy} > 0$ and the results follow. $\square$

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Note that the above theorem for $k \geq 1$ remains true also for $q \in \left[\frac{1}{2}, 1\right]$. Also, if $x, y > 1$, $k \geq 1$ and $0 < q < 1$ then

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) > \left(\psi_q^{(k)}(xy)\right)^2.$$

Now we extend Lemma 2.1 to the case $q > 1$. In order to do that we denote the zero of the function $f(q) = \frac{q-3}{2(q-1)} \log(q) - \log(q-1)$, $q > 1$, by q^*. The numerical solution shows that $q^* \approx 1.56683201\ldots$ as shown on Figure 1.

Lemma 2.3. *For $q > q^*$ and $0 < x < 1$ we have that $\psi_q(x) < 0$.*

Proof. From (1.5) we get that

$$\psi_q(x) = -\frac{q^{-x}}{1-q^{-1}} \log q - \log(q-1) + \log q \left(x - \frac{1}{2}\right) - \log q \sum_{n \geq 2} \frac{q^{-nx}}{1-q^{-n}}.$$

In order to show our claim, we need to prove that

$$g(x) = -\frac{q^{-x}}{1-q^{-1}} \log q - \log(q-1) + \log q \left(x - \frac{1}{2}\right) < 0$$

on $0 < x < 1$. Since $g'(x) = \frac{q^{-x}}{1-q^{-1}} \log^2 q + \log q > 0$, it implies that $g(x)$ is an increasing function on $0 < x < 1$. Hence

$$g(x) < g(1) = \frac{q-3}{2(q-1)} \log q - \log(q-1) < 0,$$

for all $q > q^*$, see Figure 1. □

Theorem 2.4. *Let $q > 2$ and $0 < x, y < 1$. Let $k \geq 0$ be an integer. Then*

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) < \left(\psi_q^{(k)}(xy)\right)^2.$$

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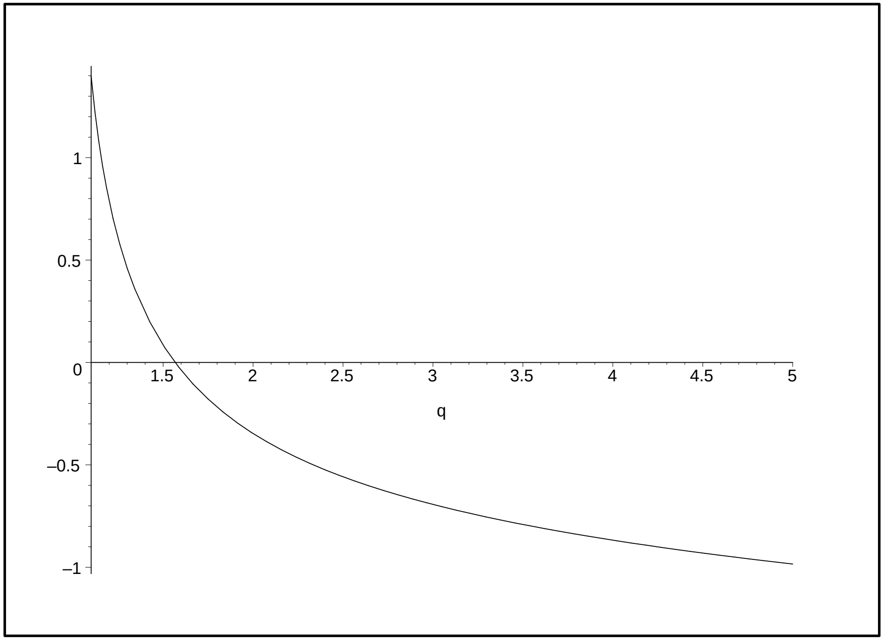

Figure 1: Graph of the function $\frac{q-3}{2(q-1)} \log q - \log(q-1)$.

Proof. As in the previous theorem we will consider two different cases: (1) $k = 0$ and (2) $k \geq 1$.

(1) As shown in the introduction the function $\psi_q'(x)$ is an increasing function on $0 < x < 1$. Therefore, for all $0 < x, y < 1$ we have that

$$\psi_q(xy) < \psi_q(x) \quad \text{and} \quad \psi_q(xy) < \psi_q(y).$$

Hence, Lemma 2.3 gives that $\psi_q^2(xy) > \psi_q(x)\psi_q(y)$, as claimed.

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(2) Analogous to the second case of Theorem 2.2. $\square$

Note that Theorem 2.4 for $k \geq 1$ remains true also for $q > 1$. Also, if $x, y > 1$, $k \geq 1$ and $q > 1$ then

$$\psi_q^{(k)}(x)\psi_q^{(k)}(y) > \left(\psi_q^{(k)}(xy)\right)^2.$$

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3. Inequalities of the Type $f(x+y) \lessgtr f(x)+f(y)$

The main goal of this section is to show that $\psi_q(x+y) \geq \psi_q(x)+\psi_q(y)$, for all $0<x,y<1$ and $0<q<1$. In order to do that we define

$$\rho(q) = \log(1-q) + \log q \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j}.$$

Lemma 3.1. *For all* $0<q<1$, $\rho(q)>0$.

Proof. Let $0<q<1$ and let $g_m(q) = c + \sum_{j=1}^{m-1} \frac{q^j(q^j-2)}{1-q^j}$ with constant $c>0$ for $m \geq 2$. Then $g_m(0)=c$, $\lim_{q\to 1^-} g_m(q) < 0$ and $g_m(q)$ is a decreasing function since

$$g'_m(q) = -\sum_{j=1}^{m-1} \frac{jq^{j-1}(1+(1-q^j)^2)}{(1-q^j)^2} < 0.$$

On the other hand

$$g_{m+1}(q) - g_m(q) = \frac{q^m(q^m-2)}{1-q^m} < 0$$

for all $0<q<1$. Hence, for all $m\geq 2$ we have that

$$g_{m+1}(q) < g_m(q), \quad 0<q<1.$$

Thus, if b_m is the positive zero of the function $g_m(q)$ (because $g_n(q)$ is decreasing) on $0<q<1$ (by Maple or any mathematical programming we can see that $b_1 = 0.38196601\ldots$, $b_2 = 0.3184588966$ and $b_3 = 0.3055970874$), then $g_m(q)>0$ for all $0<q<b_m$ and $g_m(q)<0$ for all $b_m<q<1$. Furthermore, the sequence $\{b_m\}_{m\geq 0}$ is a strictly decreasing sequence of positive real numbers, that is $0<b_{m+1}<b_m$,

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and bounded by zero, which implies that

$$\lim_{m\to\infty} g_m(q) = c + \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j} < 0,$$

for all $0<q<1$. Hence, if we choose $c=\frac{2\log(1-q)}{\log q}$ (c is positive since $0<q<1$), then we have that

$$\sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j} < -\frac{2\log(1-q)}{\log q},$$

which implies that

$$\begin{aligned}\rho(q) &= \log(1-q) + \log q \sum_{j\geq 1} \frac{q^j(q^j-2)}{1-q^j}\\ &> -2\log(1-q) + \log(1-q)\\ &= -\log(1-q) > 0,\end{aligned}$$

as requested. □

Theorem 3.2. *For all* $0<q<1$ *and* $0<x,y<1$,

$$\psi_q(x+y) > \psi_q(x) + \psi_q(y).$$

Proof. From the definitions we have that

$$\psi_q(x+y) - \psi_q(x) - \psi_q(y) = \log(1-q) + \log q \sum_{n\geq 1} \frac{q^{n(x+y)} - q^{nx} - q^{ny}}{1-q^n}.$$

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Since $0 < x, y, q < 1$, we have that

$$\begin{aligned} q^{n(x+y)} - q^{nx} - q^{ny} &= (1 - q^{nx})(1 - q^{ny}) - 1 \\ &< (1 - q^n)^2 - 1 \\ &= q^n(q^n - 2). \end{aligned}$$

Hence, by Lemma 3.1

$$\psi_q(x + y) - \psi_q(x) - \psi_q(y) > \rho(q) > 0,$$

which completes the proof. $\square$

The above theorem is not true for $x, y > 1$, for example

$$\begin{aligned} &\psi_{1/10}(4) = 0.1051046497\dots, \quad \psi_{1/10}(5) = 0.1053349312\dots, \\ &\psi_{1/10}(9) = 0.1053605131\dots. \end{aligned}$$

Theorem 3.3. *For all* $q > 1$ *and* $0 < x, y < 1$,

$$\psi_q(x + y) > \psi_q(x) + \psi_q(y).$$

Proof. From the definitions we have that

$$\psi_q(x+y)-\psi_q(x)-\psi_q(y) = \log(q-1)+\frac{1}{2}\log q+\log Q\sum_{n\geq 1}\frac{Q^{n(x+y)} - Q^{nx} - Q^{ny}}{1 - Q^n},$$

where $Q = 1/q$. Thus

$$\begin{aligned} &\psi_q(x + y) - \psi_q(x) - \psi_q(y) \\ &\quad = \log(q - 1) + \frac{1}{2}\log q + \psi_Q(x + y) - \psi_Q(x) - \psi_Q(y) - \log(1 - Q). \end{aligned}$$

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Using Theorem 3.2 we get that

$$\psi_q(x+y)-\psi_q(x)-\psi_q(y)>\log(q-1)+\frac{1}{2}\log q-\log(q-1)+\log q>0,$$

which completes the proof. □

Note that the above theorem holds for $q>2$ and $x,y>1$, since

$$\psi_q(x+y)-\psi_q(x)-\psi_q(y) \\ =\log(q-1)+\frac{1}{2}\log q+\log q\sum_{n\geq1}\frac{q^{-nx}(1-q^{-ny})+q^{-ny}}{1-q^{-n}}>0.$$

The above theorem is not true for $x,y>1$ when $1<q<2$, for example

$$\psi_{3/2}(4)=1.83813910\ldots,\quad \psi_{3/2}(5)=2.34341101\ldots, \\ \psi_{3/2}(9)=4.10745515\ldots.$$

Theorem 3.4. *Let $q\in(0,1)$. Let $k\geq1$ be an integer.*

(1) If k is even then
$$\psi_q^{(k)}(x+y)\geq\psi_q^{(k)}(x)+\psi_q^{(k)}(y).$$

(2) If k is odd then
$$\psi_q^{(k)}(x+y)\leq\psi_q^{(k)}(x)+\psi_q^{(k)}(y).$$

Proof. From (1.6) we have

$$\psi_q^k(x+y)-\psi_q^k(x)-\psi_q^k(x)=\log^{k+1}q\sum_{n\geq1}\frac{n^k}{1-q^n}(q^{n(x+y)}-q^{nx}-q^{ny}).$$

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Since the function $f(z) = q^{nz}$ is convex from

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)),$$

we obtain that

$$2 \cdot q^{n\frac{x+y}{2}} \leq q^{nx} + q^{ny}. \tag{3.1}$$

On the other hand it is clear that

$$2 \cdot q^{n\frac{x+y}{2}} > q^{n(x+y)}. \tag{3.2}$$

From (3.1) and (3.2) we have that

$$q^{n(x+y)} - q^{nx} - q^{ny} < 0.$$

(1) Since for $q \in (0,1)$ and k even we have $\log^{k+1} q < 0$, hence

$$\psi_q^{(k)}(x+y) - \psi_q^{(k)}(x) - \psi_q^{(k)}(x) \geq 0.$$

(2) The other case can be proved in a similar manner. $\square$

Using a similar approach one may prove analogue results for $q > 1$.

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