Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 3, Article 84, 2005

LITTLEWOOD-PALEY g-FUNCTION IN THE DUNKL ANALYSIS ON $\mathbb{R}^d$

FETHI SOLTANI

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES OF TUNIS
UNIVERSITY OF EL-MANAR TUNIS 2092 TUNIS, TUNISIA
Fethi.Soltani@fst.rnu.tn

Received 11 October, 2004; accepted 22 July, 2005
Communicated by S.S. Dragomir

ABSTRACT. We prove L^p-inequality for the Littlewood-Paley g-function in the Dunkl case on $\mathbb{R}^d$.

Key words and phrases: Dunkl operators, Generalized Poisson integral, g-function.

2000 *Mathematics Subject Classification.* 42B15, 42B25.

1. INTRODUCTION

In the Euclidean case, the Littlewood-Paley g-function is given by

$$g(f)(x) := \left[\int_0^\infty \left(\left|\frac{\partial}{\partial t}u(x,t)\right|^2 + |\nabla_x u(x,t)|^2\right) t\,dt\right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where u is the Poisson integral of f and ∇ is the usual gradient. The L^p-norm of this operator is comparable with the L^p-norm of f for $p \in]1,\infty[$ (see [19]). Next, this operator plays an important role in questions related to multipliers, Sobolev spaces and Hardy spaces (see [19]).

Over the past twenty years considerable effort has been made to extend the Littlewood-Paley g-function on generalized hypergroups [20, 1, 2], and complete Riemannian manifolds [4].

In this paper we consider the differential-difference operators T_j; $j = 1,\dots,d$, on $\mathbb{R}^d$ introduced by Dunkl in [5] and aptly called Dunkl operators in the literature. These operators extend the usual partial derivatives by additional reflection terms and give generalizations of many multi-variable analytic structures like the exponential function, the Fourier transform, the convolution product and the Poisson integral (see [12, 23, 16] and [13]).

During the last years, these operators have gained considerable interest in various fields of mathematics and in certain parts of quantum mechanics; one expects that the results in this paper will be useful when discussing the boundedness property of the Littlewood-Paley g-function in

ISSN (electronic): 1443-5756

The author is very grateful to the referee for many comments on this paper.
183-04

the Dunkl analysis on $\mathbb{R}^d$. Moreover they are naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many body systems [3, 9].

The main purpose of this paper is to give the L^p-inequality for the Littlewood-Paley g-function in the Dunkl case on $\mathbb{R}^d$ by using continuity properties of the Dunkl transform $\mathcal{F}_k$, the Dunkl translation operators of radial functions and the generalized convolution product $*_k$. We will adapt to this case techniques Stein used in [18, 19].

The paper is organized as follows. In Section 2 we recall some basic harmonic analysis results related to the Dunkl operators on $\mathbb{R}^d$. In particular, we list some basic properties of the Dunkl transform $\mathcal{F}_k$ and the generalized convolution product $*_k$ (see [8, 23, 15]).

In Section 3 we study the Littlewood-Paley g-function:

$$g(f)(x) := \left[\int_0^\infty \left(\left|\frac{\partial}{\partial t}u_k(x,t)\right|^2 + |\nabla_x u_k(x,t)|^2\right) t\,dt\right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where $u_k(\cdot, t)$ is the generalized Poisson integral of f.

We prove that g is L^p-boundedness for $p \in]1, 2]$.

Throughout the paper c denotes a positive constant whose value may vary from line to line.

2. THE DUNKL ANALYSIS ON $\mathbb{R}^d$

We consider $\mathbb{R}^d$ with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and norm $\|x\| = \sqrt{\langle x, x\rangle}$.

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha x := x - \left(\frac{2\langle\alpha, x\rangle}{\|\alpha\|^2}\right)\alpha.$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $R \cap \mathbb{R}, \alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $\|\alpha\|^2 = 2$ for all $\alpha \in R$.

For a root system R, the reflections σ_α, $\alpha \in R$ generate a finite group $G \subset O(d)$, the reflection group associated with R. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in H := \mathbb{R}^d \backslash \bigcup_{\alpha\in R} H_\alpha$, we fix the positive subsystem:

$$R_+ := \{\alpha \in R \;/\; \langle\alpha, \beta\rangle > 0\}.$$

Then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

Let $k : R \to \mathbb{C}$ be a multiplicity function on R (i.e. a function which is constant on the orbits under the action of G). For brevity, we introduce the index:

$$\gamma = \gamma(k) := \sum_{\alpha\in R_+} k(\alpha).$$

Moreover, let w_k denote the weight function:

$$w_k(x) := \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

We introduce the Mehta-type constant c_k, by

$$(2.1) \qquad c_k := \left(\int_{\mathbb{R}^d} e^{-\|x\|^2} d\mu_k(x)\right)^{-1}, \quad \text{where} \quad d\mu_k(x) := w_k(x)dx.$$

The Dunkl operators T_j; $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given for a function f of class C^1 on $\mathbb{R}^d$, by

$$T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

The generalized Laplacian Δ_k associated with G and k, is defined by $\Delta_k := \sum_{j=1}^d T_j^2$. It is given explicitly by

$$\Delta_k f(x) := L_k f(x) - 2 \sum_{\alpha \in R_+} k(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}, \tag{2.2}$$

with the singular elliptic operator:

$$L_k f(x) := \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle}, \tag{2.3}$$

where Δ denotes the usual Laplacian.

The operator L_k can also be written in divergence form:

$$L_k f(x) = \frac{1}{w_k(x)} \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(w_k(x) \frac{\partial}{\partial x_i} \right). \tag{2.4}$$

This is a canonical multi-variable generalization of the Sturm-Liouville operator for the classical spherical Bessel function [1, 2, 20].

For $y \in \mathbb{R}^d$, the initial value problem $T_j u(x, \cdot)(y) = x_j\, u(x, y)$; $j = 1, \dots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called a Dunkl kernel [6, 14, 16, 23].

This kernel has the Bochner-type representation (see [12]):

$$E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z\rangle} d\Gamma_x(y); \quad x \in \mathbb{R}^d,\ z \in \mathbb{C}^d, \tag{2.5}$$

where $\langle y, z\rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B_d(o, \|x\|)$ of center o and radius $\|x\|$.

Example 2.1 (see [23, p. 21])**.** If $G = \mathbb{Z}_2$, the Dunkl kernel is given by

$$E_\gamma(x, z) = \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\gamma)} \cdot \frac{\operatorname{sgn}(x)}{|x|^{2\gamma}} \int_{-|x|}^{|x|} e^{yz}(x^2 - y^2)^{\gamma - 1}(x + y)dy.$$

Notation. We denote by $\mathcal{D}(\mathbb{R}^d)$ the space of $C^\infty-$functions on $\mathbb{R}^d$ with compact support.

The Dunkl kernel gives an integral transform, called the Dunkl transform on $\mathbb{R}^d$, which was studied by de Jeu in [8]. The Dunkl transform of a function f in $\mathcal{D}(\mathbb{R}^d)$ is given by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$

Note that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$:

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} e^{-i\langle x, y\rangle} f(y) dy, \quad x \in \mathbb{R}^d.$$

The Dunkl transform of a function $f \in \mathcal{D}(\mathbb{R}^d)$ which is radial is again radial, and could be computed via the associated Fourier-Bessel transform $\mathcal{F}^B_{\gamma+d/2-1}$ [11, p. 586] that is:

$$\mathcal{F}_k(f)(x) = 2^{\gamma + d/2} c_k^{-1}\, \mathcal{F}^B_{\gamma+d/2-1}(F)(\|x\|),$$

where $f(x) = F(\|x\|)$, and

$$\mathcal{F}^B_{\gamma+d/2-1}(F)(\|x\|) := \int_0^\infty F(r) \frac{j_{\gamma+d/2-1}(\|x\|r)}{2^{\gamma+d/2-1}\Gamma\left(\gamma+\frac{d}{2}\right)} r^{2\gamma+d-1} dr.$$

Here j_γ is the spherical Bessel function [24].

Notations. We denote by $L^p_k(\mathbb{R}^d)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p_k} := \left[\int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x)\right]^{\frac{1}{p}} < \infty, \quad p \in [1, \infty[,$$

$$\|f\|_{L^\infty_k} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty,$$

where μ_k is the measure given by (2.1).

Theorem 2.1 (see [7]).

i) ***Plancherel theorem:*** *the normalized Dunkl transform $2^{-\gamma-d/2}c_k\,\mathcal{F}_k$ is an isometric automorphism on $L^2_k(\mathbb{R}^d)$. In particular,*

$$\|f\|_{L^2_k} = 2^{-\gamma-d/2} c_k \,\|\mathcal{F}_k(f)\|_{L^2_k}.$$

ii) ***Inversion formula:*** *let f be a function in $L^1_k(\mathbb{R}^d)$, such that $\mathcal{F}_k(f) \in L^1_k(\mathbb{R}^d)$. Then*

$$\mathcal{F}_k^{-1}(f)(x) = 2^{-2\gamma-d} c_k^2 \,\mathcal{F}_k(f)(-x), \quad \textit{a.e. } x \in \mathbb{R}^d.$$

In [6], Dunkl defines the intertwining operator V_k on $\mathcal{P} := \mathbb{C}[\mathbb{R}^d]$ (the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^d$), by

$$V_k(p)(x) := \int_{\mathbb{R}^d} p(y) d\Gamma_x(y), \quad x \in \mathbb{R}^d,$$

where Γ_x is the representing measure on $\mathbb{R}^d$ given by (2.5).

Next, Rösler proved the positivity properties of this operator (see [12]).

Notation. We denote by $\mathcal{E}(\mathbb{R}^d)$ and by $\mathcal{E}'(\mathbb{R}^d)$ the spaces of $C^\infty-$functions on $\mathbb{R}^d$ and of distributions on $\mathbb{R}^d$ with compact support respectively.

In [22, Theorem 6.3], Trimèche has proved the following results:

Proposition 2.2.

i) *The operator V_k can be extended to a topological automorphism on $\mathcal{E}(\mathbb{R}^d)$.*

ii) *For all $x \in \mathbb{R}^d$, there exists a unique distribution $\eta_{k,x}$ in $\mathcal{E}'(\mathbb{R}^d)$ with supp$(\eta_{k,x}) \subset \{y \in \mathbb{R}^d \ / \ \|y\| \le \|x\|\}$, such that*

$$(V_k)^{-1}(f)(x) = \langle \eta_{k,x}, f\rangle, \quad f \in \mathcal{E}(\mathbb{R}^d).$$

Next in [23], the author defines:

- The Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$, by

$$\tau_x f(y) := (V_k)_x \otimes (V_k)_y[(V_k)^{-1}(f)(x+y)], \quad y \in \mathbb{R}^d.$$

These operators satisfy for x, y and $z \in \mathbb{R}^d$ the following properties:

$$\tau_0 f = f, \quad \tau_x f(y) = \tau_y f(x), \tag{2.6}$$

$$E_k(x,z)E_k(y,z) = \tau_x(E_k(\cdot,z))(x),$$

and

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix,y)\mathcal{F}_k(f)(y), \quad f \in \mathcal{D}(\mathbb{R}^d). \tag{2.7}$$

Thus by (2.7), the Dunkl translation operators can be extended on $L^2_k(\mathbb{R}^d)$, and for $x \in \mathbb{R}^d$ we have

$$\|\tau_x f\|_{L^2_k} \le \|f\|_{L^2_k}, \quad f \in L^2_k(\mathbb{R}^d).$$

- The generalized convolution product $*_k$ of two functions f and g in $L^2_k(\mathbb{R}^d)$, by

$$f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y) g(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$

Note that $*_0$ agrees with the standard convolution $*$ on $\mathbb{R}^d$:

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy, \quad x \in \mathbb{R}^d.$$

The generalized convolution $*_k$ satisfies the following properties:

Proposition 2.3.

i) *Let* $f, g \in \mathcal{D}(\mathbb{R}^d)$. *Then*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g).$$

ii) *Let* $f, g \in L^2_k(\mathbb{R}^d)$. *Then* $f *_k g$ *belongs to* $L^2_k(\mathbb{R}^d)$ *if and only if* $\mathcal{F}_k(f)\mathcal{F}_k(g)$ *belongs to* $L^2_k(\mathbb{R}^d)$ *and we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g), \quad \text{in the } L^2_k - \text{case}.$$

Proof. The assertion i) is shown in [23, Theorem 7.2]. We can prove ii) in the same manner demonstrated in [21, p. 101–103]. □

Theorem 2.4. *Let* $p, q, r \in [1, \infty]$ *satisfy the Young's condition:* $1/p + 1/q = 1 + 1/r$. *Assume that* $f \in L^p_k(\mathbb{R}^d)$ *and* $g \in L^q_k(\mathbb{R}^d)$. *If* $\|\tau_x f\|_{L^q_k} \le c\,\|f\|_{L^q_k}$ *for all* $x \in \mathbb{R}^d$, *then*

$$\|f *_k g\|_{L^r_k} \le c\,\|f\|_{L^p_k} \|g\|_{L^q_k}.$$

Proof. The assumption that τ_x is a bounded operator on $L^p_k(\mathbb{R}^d)$ ensures that the usual proof of Young's inequality (see [25, p. 37]) works. □

Proposition 2.5.

i) *If* $f(x) = F(\|x\|)$ *in* $\mathcal{E}(\mathbb{R}^d)$, *then we have*

$$\tau_x f(y) = \int_{\mathcal{A}_{x,y}} F\left(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle y, \xi\rangle}\right) d\Gamma_x(\xi); \quad x, y \in \mathbb{R}^d,$$

where

$$\mathcal{A}_{x,y} = \left\{\xi \in \mathbb{R}^d \,/\, \min_{g \in G} \|x + gy\| \le \|\xi\| \le \max_{g \in G} \|x + gy\|\right\},$$

and Γ_x *the representing measure given by* (2.5).

ii) *For all* $x \in \mathbb{R}^d$ *and for* $f \in L^p_k(\mathbb{R}^d)$, *radial,* $p \in [1, \infty]$,

$$\|\tau_x f\|_{L^p_k} \le \|f\|_{L^p_k}.$$

iii) *Let* $p, q, r \in [1, \infty]$ *satisfy the Young's condition:* $1/p + 1/q = 1 + 1/r$. *Assume that* $f \in L^p_k(\mathbb{R}^d)$, *radial, and* $g \in L^q_k(\mathbb{R}^d)$, *then*

$$\|f *_k g\|_{L^r_k} \le \|f\|_{L^p_k} \|g\|_{L^q_k}.$$

Proof. The assertion i) is shown by Rösler in [13, Theorem 5.1].
ii) Since f is a radial function, the explicit formula of $\tau_x f$ shows that

$$|\tau_x f(y)| \leq \tau_x(|f|)(y).$$

Hence, it follows readily from (2.6) that

$$\|\tau_x f\|_{L^1_k} \leq \|f\|_{L^1_k}.$$

By duality the same inequality holds for $p = \infty$.

Thus by interpolation we obtain the result for $p \in]1, \infty[$.
iii) follows directly from Theorem 2.4. □

Notation. For all $x, y, z \in \mathbb{R}$, we put

$$W_\gamma(x, y, z) := [1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}]\, B_\gamma(|x|, |y|, |z|),$$

where

$$\sigma_{x,y,z} := \begin{cases} \dfrac{x^2 + y^2 - z^2}{2xy}, & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ 0, & \text{otherwise} \end{cases}$$

and B_γ is the Bessel kernel given by

$$B_\gamma(|x|, |y|, |z|) := \begin{cases} d_\gamma \dfrac{[((|x| + |y|)^2 - z^2)\,(z^2 - (|x| - |y|)^2)]^{\gamma-1}}{|xyz|^{2\gamma-1}}, & \text{if } |z| \in A_{x,y} \\ \\ 0, & \text{otherwise,} \end{cases}$$

$$d_\gamma = \frac{2^{-2\gamma+1}\Gamma\left(\gamma + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\gamma)}, \quad A_{x,y} = \Big[\Big||x| - |y|\Big|, |x| + |y|\Big].$$

Remark 2.6 (see [10])**.** The signed kernel W_γ is even and satisfies:

$$W_\gamma(x, y, z) = W_\gamma(y, x, z) = W_\gamma(-x, z, y),$$

$$W_\gamma(x, y, z) = W_\gamma(-z, y, -x) = W_\gamma(-x, -y, -z),$$

and

$$\int_{\mathbb{R}} |W_\gamma(x, y, z)|\, dz \leq 4.$$

We consider the signed measures $\nu_{x,y}$ (see [10]) defined by

$$d\nu_{x,y}(z) := \begin{cases} W_\gamma(x, y, z)|z|^{2\gamma} dz, & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0. \end{cases}$$

The measures $\nu_{x,y}$ have the following properties:

$$\text{supp }(\nu_{x,y}) = A_{x,y} \cup (-A_{x,y}), \quad \|\nu_{x,y}\| := \int_{\mathbb{R}} d|\nu_{x,y}| \leq 4.$$

Proposition 2.7 (see [10, 15])**.** *If $d = 1$ and $G = \mathbb{Z}_2$, then*

i) *For all $x, y \in \mathbb{R}$ and for f a continuous function on $\mathbb{R}$, we have*

$$\tau_x f(y) = \int_{A_{x,y}} f(\xi) d\nu_{x,y}(\xi) + \int_{(-A_{x,y})} f(\xi) d\nu_{x,y}(\xi).$$

ii) *For all* $x \in \mathbb{R}$ *and for* $f \in L^p_\gamma(\mathbb{R})$, $p \in [1, \infty]$,

$$\|\tau_x f\|_{L^p_\gamma} \leq 4 \|f\|_{L^p_\gamma}.$$

iii) *Assume that* $p, q, r \in [1, \infty]$ *satisfy the Young's condition:* $1/p + 1/q = 1 + 1/r$. *Then the map* $(f, g) \to f *_\gamma g$ *extends to a continuous map from* $L^p_\gamma(\mathbb{R}) \times L^q_\gamma(\mathbb{R})$ *to* $L^r_\gamma(\mathbb{R})$ *and we have*

$$\|f *_\gamma g\|_{L^r_\gamma} \leq 4 \|f\|_{L^p_\gamma} \|g\|_{L^q_\gamma}.$$

3. THE LITTLEWOOD-PALEY g-FUNCTION

By analogy with the case of Euclidean space [19, p. 61] we define, for $t > 0$, the functions W_t and P_t on $\mathbb{R}^d$, by

$$W_t(x) := 2^{-2\gamma-d} c_k^2 \int_{\mathbb{R}^d} e^{-t\|\xi\|^2} E_k(ix, \xi) d\mu_k(\xi), \quad x \in \mathbb{R}^d,$$

and

$$P_t(x) := 2^{-2\gamma-d} c_k^2 \int_{\mathbb{R}^d} e^{-t\|\xi\|} E_k(ix, \xi) d\mu_k(\xi), \quad x \in \mathbb{R}^d.$$

The function W_t, may be called the generalized heat kernel and the function P_t, the generalized Poisson kernel respectively.

From [23, p. 37] we have

$$W_t(x) = \frac{c_k}{(4t)^{\gamma+d/2}} e^{-\|x\|^2/4t}, \quad x \in \mathbb{R}^d.$$

Writing

$$P_t(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} W_{t^2/4s}(x) ds, \quad x \in \mathbb{R}^d, \tag{3.1}$$

we obtain

$$P_t(x) = \frac{a_k\, t}{(t^2 + \|x\|^2)^{\gamma+(d+1)/2}}, \quad a_k := \frac{c_k \Gamma\left(\gamma + \frac{d+1}{2}\right)}{\sqrt{\pi}}. \tag{3.2}$$

However, for $t > 0$ and for all $f \in L^p_k(\mathbb{R}^d)$, $p \in [1, \infty]$, we put:

$$u_k(x, t) := P_t *_k f(x), \quad x \in \mathbb{R}^d.$$

The function u_k is called the generalized Poisson integral of f, which was studied by Rösler in [11, 13].

Let us consider the Littlewood-Paley g-function (in the Dunkl case). This auxiliary operator is defined initially for $f \in \mathcal{D}(\mathbb{R}^d)$, by

$$g(f)(x) := \left[\int_0^\infty \left(\left|\frac{\partial}{\partial t} u_k(x, t)\right|^2 + |\nabla_x u_k(x, t)|^2\right) t\, dt\right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,$$

where u_k is the generalized Poisson integral.

The main result of the paper is:

Theorem 3.1. *For* $p \in]1, 2]$, *there exists a constant* $A_p > 0$ *such that, for* $f \in L^p_k(\mathbb{R}^d)$,

$$\|g(f)\|_{L^p_k} \leq A_p \|f\|_{L^p}.$$

For the proof of this theorem we need the following lemmas:

Lemma 3.2. *Let* $f \in \mathcal{D}(\mathbb{R}^d)$ *be a positive function.*

i) $u_k(x, t) \geq 0$ *and* $\left|\frac{\partial^N u_k}{\partial t^N}(x, t)\right| \leq \frac{c}{t^{2\gamma+d+N}}$; $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^d$.

ii) *For* $\|x\|$ *large we have*

$$u_k(x,t) \le \frac{c}{(t^2+\|x\|^2)^{\gamma+d/2}} \quad \textit{and} \quad \left|\frac{\partial u_k}{\partial x_i}(x,t)\right| \le \frac{c}{(t^2+\|x\|^2)^{\gamma+(d+1)/2}}.$$

Proof. i) If the generalized Poisson kernel P_t is a positive radial function, then from Proposition 2.5 i) we obtain $u_k(x,t) \ge 0$.

On the other hand from Proposition 2.5 iii) we have

$$\left|\frac{\partial^N u_k}{\partial t^N}(x,t)\right| \le \|f\|_{L^1_k} \left\|\frac{\partial^N P_t}{\partial t^N}\right\|_{L^\infty_k}.$$

Then we obtain the result from the fact that

$$\left\|\frac{\partial^N P_t}{\partial t^N}\right\|_{L^\infty_k} \le \frac{c}{t^{2\gamma+d+N}}.$$

ii) From Proposition 2.5 i) we can write

$$\tau_x P_t(-y) = a_k \int_{\mathbb{R}^d} \frac{t\, d\Gamma_x(\xi)}{[t^2+\|x\|^2+\|y\|^2-2\langle y,\xi\rangle]^{\gamma+(d+1)/2}}; \quad x,y\in\mathbb{R},$$

where a_k is the constant given by (3.2).

Since $f \in \mathcal{D}(\mathbb{R}^d)$, there exists $a>0$, such that $\operatorname{supp}(f) \subset B_d(o,a)$. Then

$$u_k(x,t) = a_k \int_{B_d(o,a)} \int_{\mathcal{A}_{x,y}} \frac{t\, f(y) d\Gamma_x(\xi) d\mu_k(y)}{[t^2+\|x\|^2+\|y\|^2-2\langle y,\xi\rangle]^{\gamma+(d+1)/2}}.$$

It is easily verified for $\|x\|$ large and $y \in B_d(o,a)$ that

$$\frac{1}{[t^2+\|x\|^2+\|y\|^2-2\langle y,\xi\rangle]^{\gamma+(d+1)/2}} \le \frac{c}{(t^2+\|x\|^2)^{\gamma+(d+1)/2}}.$$

Therefore and using the fact that $t \le (t^2+\|x\|^2)^{1/2}$, we obtain

$$u_k(x,t) \le \frac{c\,t}{(t^2+\|x\|^2)^{\gamma+(d+1)/2}} \le \frac{c}{(t^2+\|x\|^2)^{\gamma+d/2}}.$$

Thus the first inequality is proven.

From (2.6) we can write

$$u_k(x,t) = a_k \int_{B_d(o,a)} \int_{\mathcal{A}_{x,y}} \frac{t\, f(-y) d\Gamma_y(\xi) d\mu_k(y)}{[t^2+\|x\|^2+\|y\|^2+2\langle x,\xi\rangle]^{\gamma+(d+1)/2}}.$$

By derivation under the integral sign we obtain

$$\frac{\partial u_k}{\partial x_i}(x,t) = a_k \int_{B_d(o,a)} \int_{\mathcal{A}_{x,y}} \frac{-t(2x_i+\xi_i) f(-y) d\Gamma_y(\xi) d\mu_k(y)}{[t^2+\|x\|^2+\|y\|^2+2\langle x,\xi\rangle]^{\gamma+(d+3)/2}}.$$

But for $\|x\|$ large and $y \in B_d(o,a)$ we have

$$\frac{t|2x_i+\xi_i|}{[t^2+\|x\|^2+\|y\|^2+2\langle x,\xi\rangle]^{\gamma+(d+3)/2}} \le \frac{t(2|x_i|+|\xi_i|)}{(t^2+\|x\|^2)^{\gamma+(d+3)/2}}.$$

Using the fact that $t(2|x_i|+|\xi_i|) \le (1+|\xi_i|)(t^2+\|x\|^2)$ when $\|x\|$ large, we obtain

$$\left|\frac{\partial u_k}{\partial x_i}(x,t)\right| \le \frac{c}{(t^2+\|x\|^2)^{\gamma+(d+1)/2}},$$

which proves the second inequality. □

Lemma 3.3. *Let* $f \in \mathcal{D}(\mathbb{R}^d)$ *be a positive function and* $p \in]1,\infty[$.

i) $\lim\limits_{N\to\infty} \int_{B_d(o,N)} \int_0^N \frac{\partial^2 u_k^p}{\partial t^2}(x,t)\, tdtd\mu_k(x) = \int_{\mathbb{R}^d} f^p(x)d\mu_k(x)$.

ii) $\lim\limits_{N\to\infty} \int_0^N \int_{B_d(o,N)} L_k u_k^p(\cdot,t)(x)\, d\mu_k(x)tdt = 0$,

where L_k is the singular elliptic operator given by (2.4).

Proof. i) Integrating by parts, we obtain

$$\begin{aligned}\int_{B_d(o,N)} \int_0^N \frac{\partial^2 u_k^p}{\partial t^2}(x,t)\, tdtd\mu_k(x) &= \int_{B_d(o,N)} f^p(x)d\mu_k(x) - \int_{B_d(o,N)} u_k^p(x,N)d\mu_k(x) \\ &\quad + pN \int_{B_d(o,N)} u_k^{p-1}(x,N)\frac{\partial u_k}{\partial t}(x,N)d\mu_k(x).\end{aligned}$$

From Lemma 3.2 i), we easily get

$$\int_{B_d(o,N)} u_k^p(x,N)d\mu_k(x) \le c\, N^{-(p-1)(2\gamma+d)},$$

and

$$N\int_{B_d(o,N)} u_k^{p-1}(x,N)\frac{\partial u_k}{\partial t}(x,N)d\mu_k(x) \le c\, N^{-(p-1)(2\gamma+d)},$$

which gives i).

ii) We have

$$\int_0^N \int_{B_d(o,N)} L_k u_k^p(\cdot,t)(x)\, d\mu_k(x)tdt = \sum_{i=1}^d I_{i,N},$$

where

$$I_{i,N} = \int_0^N \int_{B_d(o,N)} \frac{\partial}{\partial x_i}\left(w_k(x)\frac{\partial u_k^p}{\partial x_i}(x,t)\right) dxtdt, \quad i = 1,\dots,d.$$

Let us study $I_{1,N}$:

$$\begin{aligned}I_{1,N} = p\int_0^N \int_{B_{d-1}(o,N)} w_k(x^{(N)}) \Bigg[& u_k^{p-1}(x^{(N)},t)\frac{\partial u_k}{\partial x_1}(x^{(N)},t) \\ & - u_k^{p-1}(-x^{(N)},t)\frac{\partial u_k}{\partial x_1}(-x^{(N)},t)\Bigg] dx_2\dots dx_d\, tdt,\end{aligned}$$

where $x^{(N)} = \left(\sqrt{N^2 - \sum_{i=2}^d x_i^2}\,,\, x_2,\dots,x_d\right)$.

Then, by using Lemma 3.2 ii) and the fact that $w_k(x^{(N)}) \le 2^\gamma N^{2\gamma}$ we obtain for N large,

$$\begin{aligned}I_{1,N} &\le c\, N^{2\gamma}\int_0^N \int_{B_{d-1}(o,N)} \frac{dx_2\dots dx_d\, tdt}{(t^2+N^2)^{(\gamma+d/2)p+1/2}} \\ &\le c\, N^{-p(2\gamma+d)+2\gamma-1}\int_0^N \int_{B_{d-1}(o,N)} dx_2\dots dx_d\, tdt \\ &\le c\, N^{-(p-1)(2\gamma+d)-(d-1)/2}.\end{aligned}$$

The same result holds for $I_{i,N}$, $i = 2,\dots,d$, which proves ii). □

Lemma 3.4. *Let $f \in \mathcal{D}(\mathbb{R}^d)$ be a positive function. Define the maximal function $\mathcal{M}_k(f)$, by*

$$\mathcal{M}_k(f)(x) := \sup_{t>0} \left(u_k(x,t)\right), \quad x \in \mathbb{R}^d. \tag{3.3}$$

Then for $p \in]1, \infty[$, there exists a constant $C_p > 0$ such that, for $f \in L^p_k(\mathbb{R}^d)$,

$$\|\mathcal{M}_k(f)\|_{L^p_k} \leq C_p \|f\|_{L^p_k},$$

moreover the operator $\mathcal{M}_k$ is of weak type $(1,1)$.

Proof. From (3.1) it follows that

$$u_k(x,t) = \frac{t}{8\sqrt{\pi}} \int_0^\infty W_s *_k f(x) e^{-t^2/4s} s^{-3/2} ds,$$

which implies, as in [18, p. 49] that

$$\mathcal{M}_k(f)(x) \leq c \sup_{y>0} \left(\frac{1}{y} \int_0^y Q_s f(x) ds \right), \quad x \in \mathbb{R}^d,$$

where $Q_s f(x) = W_s *_k f(x)$, which is a semigroup of operators on $L^p_k(\mathbb{R}^d)$. Hence using the Hopf-Dunford-Schwartz ergodic theorem as in [18, p. 48], we get the boundedness of $\mathcal{M}_k$ on $L^p_k(\mathbb{R}^d)$ for $p \in]1, \infty]$ and weak type $(1,1)$. ☐

Proof of Theorem 3.1. Let $f \in \mathcal{D}(\mathbb{R}^d)$ be a positive function. From Lemma 3.2 i) the generalized Poisson integral u_k of f is positive.

First step: Estimate of the quantity $\left|\frac{\partial}{\partial t} u_k(x,t)\right|^2 + |\nabla_x u_k(x,t)|^2$.

Let $\mathcal{H}_k$ be the operator:

$$\mathcal{H}_k := L_k + \frac{\partial^2}{\partial t^2},$$

where L_k is the singular elliptic operator given by (2.3).

Using the fact that

$$\Delta_k u_k(\cdot, t)(x) + \frac{\partial^2}{\partial t^2} u_k(x,t) = 0,$$

we obtain for $p \in]1, \infty[$,

$$\mathcal{H}_k u_k^p(x,t) = p(p-1) u_k^{p-2}(x,t) \left[\left| \frac{\partial}{\partial t} u_k(x,t) \right|^2 + |\nabla_x u_k(x,t)|^2 \right] + p \sum_{\alpha \in R_+} k(\alpha) \frac{U_\alpha(x,t)}{\langle \alpha, x \rangle^2},$$

where

$$U_\alpha(x,t) := 2 u_k^{p-1}(x,t) \left[u_k(x,t) - u_k(\sigma_\alpha x, t)\right], \quad \alpha \in R_+.$$

Let $A,\ B \geq 0$, then the inequality

$$2A^{p-1}(A - B) \geq (A^{p-1} + B^{p-1})(A - B)$$

is equivalent to

$$(A^{p-1} - B^{p-1})(A - B) \geq 0,$$

which holds if $A \geq B$ or $A < B$. Thus we deduce that

$$U_\alpha(x,t) \geq \left[u_k^{p-1}(x,t) + u_k^{p-1}(\sigma_\alpha x, t)\right] \left[u_k(x,t) - u_k(\sigma_\alpha x, t)\right],$$

and therefore we get

$$\left| \frac{\partial}{\partial t} u_k(x,t) \right|^2 + |\nabla_x u_k(x,t)|^2 \leq \frac{1}{p(p-1)} u_k^{2-p}(x,t) \left[v_k(x,t) + \mathcal{H}_k u_k^p(x,t)\right], \tag{3.4}$$

where

$$v_k(x,t) = p \sum_{\alpha \in R_+} \frac{k(\alpha)}{\langle \alpha, x\rangle^2} \left[u_k^{p-1}(\sigma_\alpha x, t) + u_k^{p-1}(x,t)\right] \left[u_k(\sigma_\alpha x, t) - u_k(x,t)\right].$$

Second step: The inequality $\|g(f)\|_{L_k^p} \le A_p \|f\|_{L_k^p}$, for $p \in]1,2[$.

From (3.4), we have

$$\begin{aligned}[g(f)(x)]^2 &\le \frac{1}{p(p-1)} \int_0^\infty u_k^{2-p}(x,t)\left[v_k(x,t) + \mathcal{H}_k u_k^p(x,t)\right] tdt \\ &\le \frac{1}{p(p-1)} \mathcal{I}_k(f)(x) \left[\mathcal{M}_k(f)(x)\right]^{2-p}, \quad x \in \mathbb{R}^d,\end{aligned}$$

where

$$\mathcal{I}_k(f)(x) := \int_0^\infty \left[v_k(x,t) + \mathcal{H}_k u_k^p(x,t)\right] tdt,$$

and $\mathcal{M}_k(f)$ the maximal function given by (3.3).

Thus it is proven that

$$\|g(f)\|_{L_k^p}^p \le \left(\frac{1}{p(p-1)}\right)^{\frac{p}{2}} \int_{\mathbb{R}^d} [\mathcal{I}_k(f)(x)]^{p/2} \left[\mathcal{M}_k(f)(x)\right]^{(2-p)p/2} d\mu_k(x).$$

By applying Hölder's inequality, we obtain

$$\|g(f)\|_{L_k^p}^p \le \left(\frac{1}{p(p-1)}\right)^{\frac{p}{2}} \|\mathcal{I}_k(f)\|_{L_k^1}^{p/2} \|\mathcal{M}_k(f)\|_{L_k^p}^{(2-p)p/2}. \tag{3.5}$$

Since $v_k(x,t) + \mathcal{H}_k u_k^p(x,t) \ge 0$, we can apply Fubini-Tonnelli's Theorem to obtain

$$\|\mathcal{I}_k(f)\|_{L_k^1} = \lim_{N\to\infty} \int_0^N \int_{B_d(o,N)} \left[v_k(x,t) + \mathcal{H}_k u_k^p(x,t)\right] d\mu_k(x) tdt.$$

Putting $y = \sigma_\alpha x$ and using the fact that $\sigma_\alpha^2 = id$; $\langle \sigma_\alpha y, \alpha\rangle = -\langle y, \alpha\rangle$, then as in the argument of [16, p. 390] we obtain

$$\int_{B_d(o,N)} v_k(x,t) d\mu_k(x) = -\int_{B_d(o,N)} v_k(y,t) d\mu_k(y).$$

Thus

$$\int_{B_d(o,N)} v_k(x,t) d\mu_k(x) = 0.$$

Hence from Lemma 3.3, we deduce that

$$\|\mathcal{I}_\alpha(f)\|_{L_k^1} = \lim_{N\to\infty} \int_{B_d(o,N)} \int_0^N \mathcal{H}_k u_k^p(x,t) tdt d\mu_k(x) = \|f\|_{L_k^p}^p. \tag{3.6}$$

On the other hand from Lemma 3.4 we have

$$\|\mathcal{M}_k(f)\|_{L_k^p} \le C_p \|f\|_{L_k^p}. \tag{3.7}$$

Finally, from (3.5), (3.6) and (3.7), we obtain

$$\|g(f)\|_{L_k^p} \le A_p \|f\|_{L_k^p}, \quad A_p = \left(\frac{1}{p(p-1)}\right)^{\frac{1}{2}} C_p^{(2-p)/2}.$$

Since the operator g is sub-linear, we obtain the inequality for $f \in \mathcal{D}(\mathbb{R}^d)$. And by an easy limiting argument one shows that the result is also true for any $f \in L_k^p(\mathbb{R}^d)$, $p \in]1,2[$.

For the case $p = 2$, using (3.4) and (3.6) we get

$$\|g(f)\|_{L_k^2}^2 \leq \frac{1}{2}\int_{\mathbb{R}^d}\int_0^\infty \left[v_k(x,t) + \mathcal{H}_k u_k^2(x,t)\right] tdtd\mu_k(x) = \frac{1}{2}\|f\|_{L_k^2}^2,$$

which completes the proof of the theorem. □

REFERENCES

[1] A. ACHOUR AND K. TRIMÈCHE, La g-fonction de Littlewood-Paley associée à un opérateur différentiel singulier sur $(0,\infty)$, *Ann. Inst. Fourier, Grenoble,* **33** (1983), 203–226.

[2] H. ANNABI AND A. FITOUHI, La g-fonction de Littlewood-Paley associée à une classe d'opérateurs différentiels sur $]0,\infty[$ contenant l'opérateur de Bessel, *C.R. Acad. Sc. Paris*, **303** (1986), 411–413.

[3] T.H. BAKER AND P.J. FORRESTER, The Calogero-Sutherland model and generalized classical polynomials, *Comm. Math. Phys*., **188** (1997), 175–216.

[4] T. COULHON, X.T. DUONG AND X.D. LI, Littlewood-Paley-Stein functions on manifolds $1 \leq p \leq 2$, *Studia Math*., **154** (2003), 37–57.

[5] C.F. DUNKL, Differential-difference operators associated with reflection groups, *Trans. Amer. Math. Soc*., **311** (1989), 167–183.

[6] C.F. DUNKL, Integral kernels with reflection group invariance, *Can. J. Math*., **43** (1991), 1213–1227.

[7] C.F. DUNKL, Hankel transforms associated to finite reflection groups, *Contemp. Math*., **138** (1992) 123–138.

[8] M.F.E. de JEU, The Dunkl transform, *Inv. Math*., **113** (1993), 147–162.

[9] L. LAPOINTE AND L. VINET, Exact operator solution of the Calogero- Sutherland model, *Comm. Math. Phys*., **178** (1996), 425–452.

[10] M. RÖSLER, Bessel-type signed hypergroups on $\mathbb{R}$, In : H.Heyer, A.Mukherjea (Eds.), *Proc XI, Probability measures on groups and related structures*, Oberwolfach, 1994, World Scientific, Singapore, (1995), p. 292–304.

[11] M. RÖSLER AND M. VOIT, Markov processes related with Dunkl operator, *Adv. Appl. Math*., **21** (1998), 575–643.

[12] M. RÖSLER, Positivity of Dunkl's intertwining operator, *Duke Math. J.*, **98** (1999), 445–463.

[13] M. RÖSLER, A positive radial product formula for the Dunkl kernel, *Trans. Amer. Math. Soc.*, **355** (2003) 2413–2438.

[14] M. SIFI AND F. SOLTANI, Generalized Fock spaces and Weyl relations for the Dunkl kernel on the real line, *J. Math. Anal. Appl*., **270** (2002), 92–106.

[15] F. SOLTANI, L^p-Fourier multipliers for the Dunkl operator on the real line, *J. Functional Analysis*, **209** (2004), 16–35.

[16] F. SOLTANI, Generalized Fock spaces and Weyl commutation relations for the Dunkl kernel, *Pacific J. Math*., **214** (2004), 379–397.

[17] F. SOLTANI, Inversion formulas in the Dunkl-type heat conduction on $\mathbb{R}^d$, *Applicable Analysis,* **84** (2005), 541–553.

[18] E.M. STEIN, Topics in Harmonic Analysis related to the Littlewood-Paley theory, *Annals of Mathematical Studies,* **63** (1970), Princeton Univ. Press, Princeton, New Jersey.

[19] E.M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, New Jersey, 1971.

[20] K. STEMPAK, La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel, *C.R.A.S. Paris, Série I, Math.*, **303** (1986), 15–18.

[21] K. TRIMÈCHE, Inversion of the Lions transmutation operators using generalized wavelets, *App. Comput. Harm. Anal.*, **4** (1997), 97–112.

[22] K. TRIMÈCHE, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of dual, *Int. Trans. Spec. Funct.*, **12** (2001), 349–374.

[23] K. TRIMÈCHE, Paley-Wiener Theorems for the Dunkl transform and Dunkl translation operators, *Int. Trans. Spec. Funct.*, **13** (2002), 17–38.

[24] G.N. WATSON, *A Treatise on Theory of Bessel Functions*, Cambridge University Press, 1966.

[25] A. ZYGMUND, *Trigonometric Series*, Cambridge University Press, 1959.