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A VARIANT OF A GENERAL INEQUALITY OF THE HARDY-KNOPP TYPE

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ABSTRACT. In this paper, we prove a variant of a general Hardy-Knopp type inequality. We also formulate a convolution inequality in the language of topological groups. By our main results we obtain a general form of multidimensional strengthened Hardy and Pólya-Knopp-type inequalities.

Key words and phrases: Inequalities, Hardy's inequality, Pólya-Knopp's inequality, Multidimensional inequalities, Convolution inequalities.

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1. INTRODUCTION

The well-known Hardy's inequality is stated below (cf. [5, Theorem 327]):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \quad p>1, f\ge 0. \tag{1.1}$$

By replacing f with $f^{\frac{1}{p}}$ in (1.1) and letting $p\to\infty$, we have the Pólya-Knopp inequality (cf. [5, Theorem 335]):

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right) dx \le e\int_0^\infty f(x)dx. \tag{1.2}$$

The constants $(p/(p-1))^p$ and e in (1.1) and (1.2), respectively, are the best possible. On the other hand, the following Hardy-Knopp type inequality (1.3) was proved (cf. [1, Eq.(4.3)] and [7, Theorem 4.1]):

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^\infty \phi(f(x))\frac{dx}{x}, \tag{1.3}$$

where ϕ is a convex function on $(0,\infty)$. In [7], S. Kaijser et al. also pointed out that (1.1) and (1.2) can be obtained from (1.3). Furthermore, in [2] and [3], Čižmešija and Pečarić proved the so-called strengthened Hardy and Pólya-Knopp-type inequalities and their multidimensional

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forms. In [4, Theorem 1 & Theorem 2], Čižmešija et al. obtained a strengthened Hardy-Knopp type inequality and its dual result. With suitable substitutions, they also showed that the strengthened Hardy and Pólya-Knopp-type inequalities given in the paper [2] are special cases of their results. In the paper [6], Kaijser et al. proved some multidimensional Hardy-type inequalities. They also proved the following generalization of the Hardy and Pólya-Knopp-type inequality:

$$\int_0^b \phi\left(\frac{1}{K(x)}\int_0^x k(x,t)f(t)dt\right)u(x)\frac{dx}{x} \le \int_0^b \phi(f(x))v(x)\frac{dx}{x}, \tag{1.4}$$

where $0 < b \le \infty$, $k(x,t) \ge 0$, $K(x) = \int_0^x k(x,t)dt$, $u(x) \ge 0$, and

$$v(x) = x\int_x^b \frac{k(z,x)}{K(z)}u(z)\frac{dz}{z}.$$

A dual inequality to (1.4) was also given. Inequality (1.4) can be obtained by using Jensen's inequality and the Fubini theorem. It is elementary but powerful. On the other hand, in the proof of [8, Lemma 3.1], for proving a variant of Schur's lemma, Sinnamon obtained an inequality of the form

$$\left\{\int_X |T_k f(x)|^q dx\right\}^{\frac{1}{q}} \le \left\{\int_T |f(t)|^p (Hw(t))^{\frac{p}{q}} w(t)^{1-p} dt\right\}^{\frac{1}{p}}, \tag{1.5}$$

where $1 < p \le q < \infty$, X and T are measure spaces, $T_k f(x) = \int_T k(x,t)f(t)dt$, w is a positive measurable function on T, and

$$Hw(t) = \int_X k(x,t)^m \left(\int_T k(x,y)^m w(y)dy\right)^{q-\frac{q}{p}} dx, \quad m = \frac{pq}{pq+p-q}. \tag{1.6}$$

In this paper, let (X,μ) and (T,λ) be two σ-finite measure spaces. Let k be a nonnegative measurable function on $X \times T$ such that

$$\int_T k(x,t)d\lambda(t) = 1 \quad \text{for } \mu\text{−a.e. } x \in X. \tag{1.7}$$

For a nonnegative measurable function f on (T,λ), define

$$T_k f(x) = \int_T k(x,t)f(t)d\lambda(t), \quad x \in X. \tag{1.8}$$

The purpose of this paper is to establish a modular inequality of the form

$$\left\{\int_X \phi^q(T_k f(x))d\mu(x)\right\}^{\frac{1}{q}} \le \left\{\int_T \phi^p(f(t))(H_s w(t))^{\frac{p}{q}} w(t)^{1-sp} d\lambda(t)\right\}^{\frac{1}{p}} \tag{1.9}$$

for $0 < p \le q < \infty$, $\phi \in \Phi_s^+(I)$, $s \ge 1/p$, and $H_s w(t)$ is defined by (2.1). As applications, we prove a convolution inequality in the language of integration on a locally compact Abelian group. We also show that by suitable choices of w, we can obtain many forms of strengthened Hardy and Pólya-Knopp-type inequalities. Here $\Phi_s^+(I)$ denotes the class of all nonnegative functions ϕ on $I \subseteq (0,\infty)$ such that $\phi^{1/s}$ is convex on I and ϕ takes its limiting values, finite or infinite, at the ends of I. Note that $\Phi_s^+(I) \subset \Phi_r^+(I)$ for $0 < r < s$ and we denote $\Phi_\infty^+(I) = \bigcap_{s>0}\Phi_s^+(I)$.

The functions involved in this paper are all measurable on their domains. We work under the convention that $0^0 = \infty^0 = 1$ and $\infty/\infty = 0 \cdot \infty = 0$.

2. MAIN RESULTS

The following theorem is based on Jensen's inequality and [8, Lemma 3.1]. For the convenience of readers, we give a complete proof here.

Theorem 2.1. *Let $0 < p \leq q < \infty$, $1/p \leq s < \infty$, and $\phi \in \Phi_s^+(I)$. Let f be a nonnegative function on (T, λ) and the range of values of f lie in the closure of I. Suppose that w is a positive function on (T, λ) such that the function*

$$H_s w(t) = \int_X k(x,t)^m \left(\int_T k(x,y)^m w(y) d\lambda(y) \right)^{sq - \frac{q}{p}} d\mu(x), \tag{2.1}$$

where $m = spq/(spq + p - q)$, is finite for $\lambda-$a.e. $t \in T$. Then we have

$$\left\{ \int_X \phi^q(T_k f(x)) d\mu(x) \right\}^{\frac{1}{q}} \leq \left\{ \int_T \phi^p(f(t)) (H_s w(t))^{\frac{p}{q}} w(t)^{1-sp} d\lambda(t) \right\}^{\frac{1}{p}}. \tag{2.2}$$

Proof. Since $\phi^{1/s}$ is convex, $\phi(T_k f(x)) \leq \{T_k(\phi^{1/s}(f))(x)\}^s$ for $\mu-$a.e. $x \in X$ and hence

$$\int_X \phi^q(T_k f(x)) d\mu(x) \leq \int_X \left(\int_T k(x,t) \phi^{1/s}(f(t)) d\lambda(t) \right)^{sq} d\mu(x). \tag{2.3}$$

Let $m = spq/(spq + p - q)$ and w be a positive function on (T, λ) such that $H_s w(t)$ defined by (2.1) is finite for $\lambda-$a.e. $t \in T$. By Hölder's inequality with indices sp and $(sp)^*$, we have

$$\begin{aligned} &\int_T k(x,t) \phi^{1/s}(f(t)) d\lambda(t) \\ &= \int_T k(x,t)^{1-m/(sp)^* + m/(sp)^*} \phi^{1/s}(f(t)) w(t)^{\frac{1}{(sp)^*} - \frac{1}{(sp)^*}} d\lambda(t) \\ &\leq \left(\int_T k(x,y)^m w(y) d\lambda(y) \right)^{\frac{1}{(sp)^*}} \\ &\quad \times \left(\int_T k(x,t)^{(1-m/(sp)^*)sp} \phi^p(f(t)) w(t)^{-sp/(sp)^*} d\lambda(t) \right)^{\frac{1}{(sp)}} \end{aligned} \tag{2.4}$$

and this implies

$$\begin{aligned} &\left\{ \int_X \phi^q(T_k f(x)) d\mu(x) \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_X \left(\int_T k(x,t)^{(1-m/(sp)^*)sp} \phi^p(f(t)) w(t)^{-sp/(sp)^*} d\lambda(t) \right)^{\frac{q}{p}} \right. \\ &\qquad \left. \times \left(\int_T k(x,y)^m w(y) d\lambda(y) \right)^{(sp-1)\frac{q}{p}} d\mu(x) \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_T \phi^p(f(t)) (H_s w(t))^{\frac{p}{q}} w(t)^{1-sp} d\lambda(t) \right\}^{\frac{1}{p}}. \end{aligned} \tag{2.5}$$

The last inequality is based on the Minkowski's integral inequality with index $\frac{q}{p}$. This completes the proof. □

We can apply Theorem 2.1 to obtain some multidimensional strengthened Hardy and Pólya-Knopp-type inequalities. These are discussed in Section 3. In the following corollary, we consider the norm inequality

$$\left\{\int_X \phi^q(T_k f(x))d\mu(x)\right\}^{\frac{1}{q}} \le C\left\{\int_T \phi^p(f(t))d\lambda(t)\right\}^{\frac{1}{p}}. \tag{2.6}$$

The results of Corollary 2.2 can be obtained by Theorem 2.1 and the fact that $\Phi_s^+(I) \subset \Phi_r^+(I)$ for $0 < r < s$.

Corollary 2.2. *Let $0 < p \le q < \infty$, $1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let f be given as in Theorem 2.1.*

(i) *If there exists a positive function w on (T, λ) such that the following condition (2.7) holds for some $1/p \le r \le s$ and for some positive constant A_r:*

$$H_r w(t) \le A_r w(t)^{(r-1/p)q} \quad \text{for } \lambda\text{-a.e. } t \in T, \tag{2.7}$$

then we have (2.6) where the best constant C satisfies

$$C \le A_r^{\frac{1}{q}}. \tag{2.8}$$

(ii) *If w satisfies (2.7) for each $1/p \le r \le s$, then we have (2.6) with*

$$C \le \inf_{1/p \le r \le s} A_r^{\frac{1}{q}}. \tag{2.9}$$

(iii) *If $\phi \in \Phi_\infty^+(I)$ and w satisfies (2.7) for each $1/p \le r < \infty$, then we have (2.6) with*

$$C \le \inf_{1/p \le r < \infty} A_r^{\frac{1}{q}}. \tag{2.10}$$

In the case $1 < p \le q < \infty$ and $\phi(x) = x$, choose $s = r = 1$ and then Corollary 2.2 can be reduced to [8, Lemma 3.1].

In the following, we consider the particular case $X = T = G$, where G is a locally compact Abelian group (written multiplicatively), with Haar measure μ. Let h be a nonnegative function on G such that $\int_G h d\mu = 1$. For a nonnegative function f on G, define the convolution operator

$$h * f(x) = \int_G h(xt^{-1})f(t)d\mu(t), \quad x \in G. \tag{2.11}$$

Moreover, if $\int_G h^m d\mu$ is also finite, where m is given in Theorem 2.1, then by (2.1) with $k(x,y) = h(xy^{-1})$ and $w \equiv 1$, we have

$$\begin{aligned} H_s w(t) &= \int_G h(xt^{-1})^m \left(\int_G h(xy^{-1})^m d\mu(y)\right)^{sq-\frac{q}{p}} d\mu(x) \\ &= \left(\int_G h(x)^m d\mu(x)\right)^{\frac{sq}{m}}. \end{aligned} \tag{2.12}$$

We then obtain the following result:

Corollary 2.3. *Let $0 < p \le q < \infty$, $1/p \le s < \infty$, and $\phi \in \Phi_s^+(I)$. Let h be a nonnegative function on G such that $\int_G h d\mu = 1$ and $\int_G h^m d\mu < \infty$, where $m = spq/(spq + p - q)$. Let f be given as in Theorem 2.1. Then we have*

$$\left\{\int_G \phi^q(h * f(x))d\mu(x)\right\}^{\frac{1}{q}} \le \left\{\int_G h(x)^m d\mu(x)\right\}^{\frac{s}{m}} \left\{\int_G \phi^p(f(t))d\mu(t)\right\}^{\frac{1}{p}}. \tag{2.13}$$

Moreover, if $p < q$, $\phi \in \Phi_\infty^+(I)$ *and* $\int_G h^r d\mu < \infty$ *for some* $r > 1$, *then*

$$\left\{\int_G \phi^q(h * f(x))d\mu(x)\right\}^{\frac{1}{q}} \leq \left\{\exp\left(\int_G h(x)\log h(x)d\mu(x)\right)\right\}^{\frac{1}{p}-\frac{1}{q}} \left\{\int_G \phi^p(f(t))d\mu(t)\right\}^{\frac{1}{p}}. \tag{2.14}$$

Inequality (2.14) can be obtained by letting $s \to \infty$ in (2.13). In the case $\phi(x) = x$ and $s = 1$ in (2.13), the condition $\int_G hd\mu = 1$ is not necessary and (2.13) can be reduced to Young's inequality:

$$\left\{\int_G (h * f(x))^q d\mu(x)\right\}^{\frac{1}{q}} \leq \left\{\int_G h(x)^m d\mu(x)\right\}^{\frac{1}{m}} \left\{\int_G f(t)^p d\mu(t)\right\}^{\frac{1}{p}}, \tag{2.15}$$

where $1 \leq p \leq q < \infty$ and $m = pq/(pq + p - q)$. If $\phi(x) = e^x$ and f is replaced by $\log f$ in (2.14), then for $0 < p < q < \infty$,

$$\left\{\int_G \left\{\exp\left(\int_G h(xt^{-1})\log f(t)d\mu(t)\right)\right\}^q d\mu(x)\right\}^{\frac{1}{q}} \leq \left\{\exp\left(\int_G h(x)\log h(x)d\mu(x)\right)\right\}^{\frac{1}{p}-\frac{1}{q}} \left\{\int_G f(t)^p d\mu(t)\right\}^{\frac{1}{p}}. \tag{2.16}$$

Let $G = \mathbb{R}^n$ under addition and μ be the Lebesgue measure. Then (2.15) can be reduced to

$$\left\{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} h(x-t)f(t)dt\right)^q dx\right\}^{\frac{1}{q}} \leq \left\{\int_{\mathbb{R}^n} h(x)^m dx\right\}^{\frac{1}{m}} \left\{\int_{\mathbb{R}^n} f(t)^p dt\right\}^{\frac{1}{p}}. \tag{2.17}$$

Moreover, if $\int_{\mathbb{R}^n} h(x)dx = 1$ and $\int_{\mathbb{R}^n} h(x)^r dx < \infty$ for some $r > 1$, then by (2.16),

$$\left\{\int_{\mathbb{R}^n} \left\{\exp\left(\int_{\mathbb{R}^n} h(x-t)\log f(t)dt\right)\right\}^q dx\right\}^{\frac{1}{q}} \leq \left\{\exp\left(\int_{\mathbb{R}^n} h(x)\log h(x)dx\right)\right\}^{\frac{1}{p}-\frac{1}{q}} \left\{\int_{\mathbb{R}^n} f(t)^p dt\right\}^{\frac{1}{p}}. \tag{2.18}$$

Let $G = (0, \infty)$ under multiplication and $d\mu = x^{-1}dx$. Then by (2.15),

$$\left\{\int_0^\infty \left(\int_0^\infty h(x/t)f(t)\frac{dt}{t}\right)^q \frac{dx}{x}\right\}^{\frac{1}{q}} \leq \left\{\int_0^\infty h(x)^m \frac{dx}{x}\right\}^{\frac{1}{m}} \left\{\int_0^\infty f(t)^p \frac{dt}{t}\right\}^{\frac{1}{p}}. \tag{2.19}$$

Moreover, if $\int_0^\infty h(x)x^{-1}dx = 1$ and $\int_0^\infty h(x)^r x^{-1}dx < \infty$ for some $r > 1$, then (2.16) can be reduced to

$$\left\{\int_0^\infty \left\{\exp\left(\int_0^\infty h(x/t)\log f(t)\frac{dt}{t}\right)\right\}^q \frac{dx}{x}\right\}^{\frac{1}{q}} \leq \left\{\exp\left(\int_0^\infty h(x)\log h(x)\frac{dx}{x}\right)\right\}^{\frac{1}{p}-\frac{1}{q}} \left\{\int_0^\infty f(t)^p \frac{dt}{t}\right\}^{\frac{1}{p}}. \tag{2.20}$$

There are multidimensional cases corresponding to (2.19) and (2.20). For example, the 2-dimensional analogue of (2.19) is

$$\left\{\int_0^\infty\int_0^\infty\left(\int_0^\infty\int_0^\infty h\left(\frac{x}{s},\frac{y}{t}\right)f(s,t)\frac{ds}{s}\frac{dt}{t}\right)^q\frac{dx}{x}\frac{dy}{y}\right\}^{\frac{1}{q}} \leq\left\{\int_0^\infty\int_0^\infty h(x,y)^m\frac{dx}{x}\frac{dy}{y}\right\}^{\frac{1}{m}}\left\{\int_0^\infty\int_0^\infty f(s,t)^p\frac{ds}{s}\frac{dt}{t}\right\}^{\frac{1}{p}}, \tag{2.21}$$

and we can also obtain similar results to (2.20).

3. Multidimensional Hardy and Pólya-Knopp-Type Inequalities

In this section, we apply our main results to the case $X = T = \mathbb{R}^N$ and obtain some multi-dimensional forms of the strengthened Hardy and Pólya-Knopp-type inequalities. Let Σ^{N-1} be the unit sphere in $\mathbb{R}^N$, that is, $\Sigma^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$, where $|x|$ denotes the Euclidean norm of x. Let A be a Lebesgue measurable subset of Σ^{N-1}, $0 < b \leq \infty$, and define

$$E = \{x \in \mathbb{R}^N : x = s\rho,\ 0 \leq s < b,\ \rho \in A\}.$$

For $x \in E$, we define

$$S_x = \{y \in \mathbb{R}^N : y = s\rho,\ 0 \leq s \leq |x|,\ \rho \in A\},$$

and denote by $|S_x|$ the Lebesgue measure of S_x. We have the following result:

Theorem 3.1. *Let $0 < p \leq q < \infty$, $1/p \leq s < \infty$, and $\phi \in \Phi_s^+(I)$. Let g be a nonnegative function on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\int_{S_x} g(x,t)dt = 1$ for almost all $x \in E$ and let f be a nonnegative function on $\mathbb{R}^N$ and the range of values of f lie in the closure of I. Suppose that u is a nonnegative function on $\mathbb{R}^N$ and w is a positive function on E such that the function*

$$H_s w(t) = \int_E g(x,t)^m\left(\int_{S_x} g(x,y)^m w(y)dy\right)^{sq-\frac{q}{p}} u(x)\chi_{S_x}(t)dx, \tag{3.1}$$

where $m = spq/(spq + p - q)$, is finite for almost all $t \in E$. Then we have

$$\left\{\int_E \phi^q\left(\int_{S_x} g(x,t)f(t)dt\right)u(x)dx\right\}^{\frac{1}{q}} \leq \left\{\int_E \phi^p(f(t))(H_s w(t))^{\frac{p}{q}} w(t)^{1-sp}dt\right\}^{\frac{1}{p}}. \tag{3.2}$$

Proof. Let $X = T = \mathbb{R}^N$, $d\mu = u(x)\chi_E(x)dx$, $d\lambda = \chi_E(x)dx$, and $k(x,t) = g(x,t)\chi_{S_x}(t)$ in Theorem 2.1. Then $H_s w$ defined by (2.1) can be reduced to (3.1) and we have (3.2) by Theorem 2.1. □

In the case $p = q = s = 1$, then $m = 1$ and we have

$$\int_E \phi\left(\int_{S_x} g(x,t)f(t)dt\right)u(x)dx \leq \int_E \phi(f(t))\left(\int_E g(x,t)u(x)\chi_{S_x}(t)dx\right)dt. \tag{3.3}$$

In particular, if $N = 1$, $E = [0,b)$, $S_x = [0,x)$, and $u(x)$ is replaced by $u(x)/x$, then (3.3) can be reduced to

$$\int_0^b \phi\left(\int_0^x g(x,t)f(t)dt\right)\frac{u(x)}{x}dx \leq \int_0^b \phi(f(t))\left(\int_t^b g(x,t)\frac{u(x)}{x}dx\right)dt. \tag{3.4}$$

Inequality (3.4) was also obtained in [6, Theorem 4.1].

Now we consider (3.2) with $u(x) = |S_x|^a$ and $g(x,t) = |S_x|^{-1}h(|S_t|/|S_x|)$, where $a \in \mathbb{R}$, h is a nonnegative function defined on $[0,1)$ and $\int_0^1 h(x)dx = 1$. By (3.1) with $w(y) = |S_y|^{m(\frac{q}{p}-a-2)/(sq)}$, we have

$$H_s w(t) = \left(\int_0^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi\right)^{sq-\frac{q}{p}} |S_t|^{-1+m(a+2-q/p)/(sq)} \times \int_{(|t|/b)^N}^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi. \tag{3.5}$$

As a consequence of Theorem 3.1, we have the following result:

Corollary 3.2. *Let* $0 < p \le q < \infty$, $1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, *and* f *be given as in Theorem 3.1. Let* $a \in \mathbb{R}$, h *be given as above, and* $\int_0^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi < \infty$, *where* $m = spq/(spq+p-q)$. *Then we have*

$$\left\{\int_E \phi^q \left(\frac{1}{|S_x|}\int_{S_x} h\left(\frac{|S_t|}{|S_x|}\right) f(t)dt\right) |S_x|^a dx\right\}^{\frac{1}{q}} \le \left(\int_0^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi\right)^{s-\frac{1}{p}} \left\{\int_E \phi^p(f(t)) |S_t|^{(a+1)\frac{p}{q}-1} v(t)^{\frac{p}{q}} dt\right\}^{\frac{1}{p}}, \tag{3.6}$$

where

$$v(t) = \int_{(|t|/b)^N}^1 h(\xi)^m \xi^{m(q/p-a-2)/(sq)} d\xi.$$

By (3.6), we see that

$$\left\{\int_E \phi^q \left(\frac{1}{|S_x|}\int_{S_x} h\left(\frac{|S_t|}{|S_x|}\right) f(t)dt\right) |S_x|^a dx\right\}^{\frac{1}{q}} \le C\left\{\int_E \phi^p(f(t)) |S_t|^{(a+1)\frac{p}{q}-1} dt\right\}^{\frac{1}{p}}, \tag{3.7}$$

where C satisfies

$$C \le \left(\int_0^1 h(\xi)^m \xi^{m(\frac{q}{p}-a-2)/(sq)} d\xi\right)^{s-\frac{1}{p}+\frac{1}{q}}. \tag{3.8}$$

Moreover, if $\phi \in \Phi_\infty^+(I)$ and $p < q$, then the estimation given in (3.8) can be replaced by

$$C \le \left\{\exp\left(\int_0^1 h(\xi) \log[h(\xi)\xi^{(q-(a+2)p)/(q-p)}] d\xi\right)\right\}^{\frac{1}{p}-\frac{1}{q}}. \tag{3.9}$$

In the following, we consider the particular case $p = q$. In this case, $m = 1$ and (3.6) can be reduced to

$$\int_E \phi^p \left(\frac{1}{|S_x|}\int_{S_x} h\left(\frac{|S_t|}{|S_x|}\right) f(t)dt\right) |S_x|^a dx \le \left(\int_0^1 h(\xi)\xi^{(-a-1)/(sp)} d\xi\right)^{sp-1} \int_E \phi^p(f(t)) \left(\int_{(|t|/b)^N}^1 h(\xi)\xi^{(-a-1)/(sp)} d\xi\right) |S_t|^a dt. \tag{3.10}$$

In the case $\phi \in \Phi_\infty^+(I)$, by letting $s \to \infty$ in (3.10), we have

$$\int_E \phi^p \left(\frac{1}{|S_x|}\int_{S_x} h\left(\frac{|S_t|}{|S_x|}\right) f(t)dt\right) |S_x|^a dx \le \left\{\exp\left(\int_0^1 h(\xi)\log\xi d\xi\right)\right\}^{-a-1} \int_E \phi^p(f(t)) \left(\int_{(|t|/b)^N}^1 h(\xi) d\xi\right) |S_t|^a dt. \tag{3.11}$$

If $h(\xi) = \alpha\xi^{\alpha-1}$, $\alpha > 0$, then we have the following corollary.

Corollary 3.3. *Let* $0 < p < \infty$, $1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, $\alpha > 0$, $a+1 < \alpha sp$, *and* f *be given as in Theorem 3.1. Then we have*

$$\int_E \phi^p \left(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} |S_t|^{\alpha-1} f(t)dt \right) |S_x|^a dx \leq \left(\frac{\alpha sp}{\alpha sp - a - 1} \right)^{sp} \int_E \phi^p(f(t)) \left(1 - \left(\frac{|t|}{b} \right)^{N(\alpha sp-a-1)/(sp)} \right) |S_t|^a dt. \tag{3.12}$$

Moreover, if $\phi \in \Phi_\infty^+(I)$*, then for* $a \in \mathbb{R}$*, we have*

$$\int_E \phi^p \left(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} |S_t|^{\alpha-1} f(t)dt \right) |S_x|^a dx \leq e^{(a+1)/\alpha} \int_E \phi^p(f(t)) \left(1 - \left(\frac{|t|}{b} \right)^{N\alpha} \right) |S_t|^a dt. \tag{3.13}$$

Inequality (3.12) was obtained in [3, Theorem 1(i)] for the case $\phi(x) = x$, $p > 1$, $s = 1$, $a < p-1$, $\alpha = 1$, and E is the ball in $\mathbb{R}^N$ centered at the origin and of radius b. If $\phi(x) = e^x$, $p = 1$, and f is replaced by $\log f$ in (3.13), then we have [3, Theorem 2(i)]. If $h(\xi) = \alpha(1-\xi)^{\alpha-1}$, $\alpha > 0$, then we have the following corollary.

Corollary 3.4. *Let* $0 < p < \infty$, $1/p \leq s < \infty$, $\phi \in \Phi_s^+(I)$, $\alpha > 0$, $a + 1 < sp$*, and* f *be given as in Theorem 3.1. Then we have*

$$\int_E \phi^p \left(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} (|S_x| - |S_t|)^{\alpha-1} f(t)dt \right) |S_x|^a dx \leq \left\{ \alpha B \left(\frac{sp-a-1}{sp}, \alpha \right) \right\}^{sp-1} \int_E \phi^p(f(t)) |S_t|^a v(t)dt, \tag{3.14}$$

where $B(\delta, \eta)$ *is the Beta function and*

$$v(t) = \int_{(|t|/b)^N}^1 \alpha(1-\xi)^{\alpha-1} \xi^{(-a-1)/(sp)} d\xi.$$

Moreover, if $\phi \in \Phi_\infty^+(I)$*, then for* $a \in \mathbb{R}$ *we have*

$$\int_E \phi^p \left(\frac{\alpha}{|S_x|^\alpha} \int_{S_x} (|S_x| - |S_t|)^{\alpha-1} f(t)dt \right) |S_x|^a dx \leq \left\{ \exp \left(\int_0^1 \alpha(1-\xi)^{\alpha-1} \log \xi d\xi \right) \right\}^{-a-1} \int_E \phi^p(f(t)) \left(1 - \left(\frac{|t|}{b} \right)^N \right)^\alpha |S_t|^a dt. \tag{3.15}$$

In the following, we consider the dual result of Theorem 3.1. Let $0 \leq b < \infty$ and

$$\tilde{E} = \{x \in \mathbb{R}^N : x = s\rho, b \leq s < \infty, \rho \in A\}.$$

For $x \in \tilde{E}$, we define

$$\tilde{S}_x = \{y \in \mathbb{R}^N : y = s\rho, |x| \leq s < \infty, \rho \in A\}.$$

Let u be a nonnegative function on $\mathbb{R}^N$, $d\mu = u(x)\chi_{\tilde{E}}(x)dx$, $d\lambda = \chi_{\tilde{E}}(t)dt$, and $k(x,t) = g(x,t)\chi_{\tilde{S}_x}(t)$, where g is a nonnegative function on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\int_{\tilde{S}_x} g(x,t)dt = 1$ for almost all $x \in \tilde{E}$. Suppose that w is a positive function on $\tilde{E}$. Then $H_s w$ defined by (2.1) can be reduced to

$$H_s w(t) = \int_{\tilde{E}} g(x,t)^m \left(\int_{\tilde{S}_x} g(x,y)^m w(y)dy \right)^{sq-\frac{q}{p}} u(x)\chi_{\tilde{S}_x}(t)dx. \tag{3.16}$$

We have the following theorem.

Theorem 3.5. *Let* $0 < p \le q < \infty$, $1/p \le s < \infty$, $\phi \in \Phi_s^+(I)$, *and* g, u, w *be given as above. Let* f *be given as in Theorem 3.1. Suppose that* $H_s w(t)$ *given in (3.16) is finite for almost all* $t \in \tilde{E}$. *Then we have*

$$\left\{\int_{\tilde{E}} \phi^q \left(\int_{\tilde{S}_x} g(x,t)f(t)dt\right) u(x)dx\right\}^{\frac{1}{q}} \le \left\{\int_{\tilde{E}} \phi^p(f(t))(H_s w(t))^{\frac{p}{q}} w(t)^{1-sp}dt\right\}^{\frac{1}{p}}. \tag{3.17}$$

In the case $p = q = s = 1$, then $m = 1$ and we have

$$\int_{\tilde{E}} \phi \left(\int_{\tilde{S}_x} g(x,t)f(t)dt\right) u(x)dx \le \int_{\tilde{E}} \phi(f(t)) \left(\int_{\tilde{E}} g(x,t)u(x)\chi_{\tilde{S}_x}(t)dx\right) dt. \tag{3.18}$$

In particular, if $N = 1$, $\tilde{E} = [b, \infty)$, $\tilde{S}_x = [x, \infty)$, and $u(x)$ is replaced by $u(x)/x$, then by (3.18) we have

$$\int_b^\infty \phi \left(\int_x^\infty g(x,t)f(t)dt\right) \frac{u(x)}{x}dx \le \int_b^\infty \phi(f(t)) \left(\int_b^t g(x,t)\frac{u(x)}{x}dx\right) dt. \tag{3.19}$$

Inequality (3.19) was also obtained in [6, Theorem 4.3]. Using a similar method, we can also obtain companion results of (3.6) – (3.15). We omit the details.

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