

SOME INEQUALITIES OF PERTURBED TRAPEZOID TYPE

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

A new generalized perturbed trapezoid type inequality is established by Peano kernel approach. Some related results are also given.

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Key words: Perturbed trapezoid inequality; n -times continuously differentiable mapping; Absolutely continuous.

Contents

1	Introduction	3
2	For Differentiable Mappings With Bounded Derivatives	5
3	Bounds In Terms of Some Lebesgue Norms	10
4	Non-Symmetric Bounds	12
References		



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 2 of 18

1. Introduction

In recent years, some authors have considered the perturbed trapezoid inequality

$$\left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq C(\Gamma_2 - \gamma_2)(b-a)^3,$$

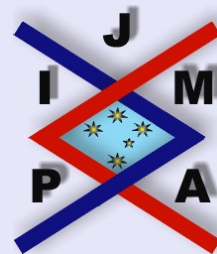
where $f : [a, b] \rightarrow \mathbb{R}$ is a twice differentiable mapping on (a, b) with $\gamma_2 = \inf_{x \in [a, b]} f''(x) > -\infty$ and $\Gamma_2 = \sup_{x \in [a, b]} f''(x) < +\infty$ while C is a constant.

(e.g. see [1] – [8]) It seems that the best result $C = \frac{\sqrt{3}}{108}$ was separately and independently discovered by the authors of [5] and [8]. The perturbed trapezoid inequality has been established as

$$(1.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq \frac{\sqrt{3}}{108}(\Gamma_2 - \gamma_2)(b-a)^3.$$

Moreover, we can also find in [5] the following two perturbed trapezoid inequalities as

$$(1.2) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq \frac{1}{384}(\Gamma_3 - \gamma_3)(b-a)^4,$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

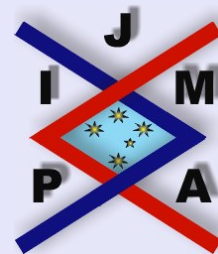
Page 3 of 18

where $f : [a, b] \rightarrow \mathbb{R}$ is a third-order differentiable mapping on (a, b) with $\gamma_3 = \inf_{x \in [a, b]} f'''(x) > -\infty$ and $\Gamma_3 = \sup_{x \in [a, b]} f'''(x) < +\infty$, and

$$(1.3) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right| \leq \frac{1}{720} M_4 (b-a)^5,$$

where $f : [a, b] \rightarrow \mathbb{R}$ is a fourth-order differentiable mapping on (a, b) with $M_4 = \sup_{x \in [a, b]} |f^{(4)}(x)| < +\infty$.

The purpose of this paper is to extend these above results to a more general version by choosing appropriate harmonic polynomials such as the Peano kernel. A new generalized perturbed trapezoid type inequality is established and some related results are also given.



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 4 of 18

2. For Differentiable Mappings With Bounded Derivatives

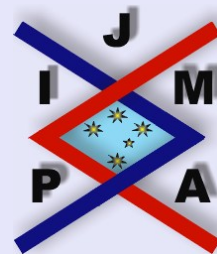
Theorem 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an n -times continuously differentiable mapping, $n \geq 2$ and such that $M_n := \sup_{x \in [a, b]} |f^{(n)}(x)| < \infty$. Then

$$(2.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq M_n \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54} & \text{if } n = 2; \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n} & \text{if } n \geq 3, \end{cases}$$

where $\lfloor \frac{n-1}{2} \rfloor$ denotes the integer part of $\frac{n-1}{2}$.

Proof. It is not difficult to find the identity

$$(2.2) \quad (-1)^n \int_a^b T_n(x) f^{(n)}(x) dx = \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right),$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 5 of 18

where $T_n(x)$ is the kernel given by

$$(2.3) \quad T_n(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!} & \text{if } x \in \left[a, \frac{a+b}{2} \right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!} & \text{if } x \in \left(\frac{a+b}{2}, b \right]. \end{cases}$$

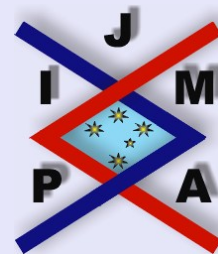
Using the identity (2.2), we get

$$(2.4) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| = \left| \int_a^b T_n(x) f^{(n)}(x) dx \right| \leq M_n \int_a^b |T_n(x)| dx.$$

For brevity, we put

$$P_n(x) := \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!} = \frac{(x-a)^{n-2}}{n!} \left[(x-a)^2 - \frac{n(b-a)(x-a)}{2} + \frac{n(n-1)(b-a)^2}{12} \right],$$

$$x \in \left[a, \frac{a+b}{2} \right]$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 6 of 18

and

$$\begin{aligned}
 Q_n(x) &:= \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!} \\
 &= \frac{(x-b)^{n-2}}{n!} \left[(x-b)^2 + \frac{n(b-a)(x-b)}{2} + \frac{n(n-1)(b-a)^2}{12} \right], \\
 x &\in \left[\frac{a+b}{2}, b \right].
 \end{aligned}$$

It is clear that $P_n(x)$ and $Q_n(x)$ are symmetric with respect to the line $x = \frac{a+b}{2}$ for n even, and symmetric with respect to the point $(\frac{a+b}{2}, 0)$ for n odd. Therefore,

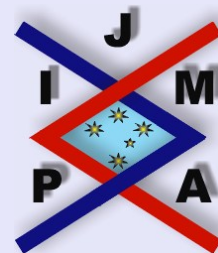
$$\begin{aligned}
 \int_a^b |T_n(x)| dx &= 2 \int_a^{\frac{a+b}{2}} |P_n(x)| dx \\
 &= \frac{(b-a)^{n+1}}{n!2^n} \int_0^1 \left| t^{n-2} \left[t^2 - nt + \frac{n(n-1)}{3} \right] \right| dt
 \end{aligned}$$

by substitution $x = a + \frac{b-a}{2}t$, and it is easy to find that

$$r_n(t) := t^{n-2} \left[t^2 - nt + \frac{n(n-1)}{3} \right]$$

is always nonnegative on $[0, 1]$ for $n \geq 3$. Thus we have

$$\int_0^1 |r_n(t)| dt = \int_0^1 t^{n-2} \left[t^2 - nt + \frac{n(n-1)}{3} \right] dt = \frac{n(n-2)}{3(n+1)}$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 7 of 18

for $n \geq 3$, and

$$\begin{aligned} \int_0^1 |r_2(t)| dt &= \int_0^1 \left| t^2 - 2t + \frac{2}{3} \right| dt \\ &= \int_0^{t_0} \left(t^2 - 2t + \frac{2}{3} \right) dt - \int_{t_0}^1 \left(t^2 - 2t + \frac{2}{3} \right) dt, \end{aligned}$$

where $t_0 = 1 - \frac{\sqrt{3}}{3}$ is the unique zero of $r_2(t)$ in $(0, 1)$. Hence,

$$(2.5) \quad \int_a^b |T_n(x)| dx = \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3. \end{cases}$$

Consequently, the inequality (2.1) follows from (2.4) and (2.5). □

Remark 1. If in the inequality (2.1) we choose $n = 2, 3, 4$, then we get

$$\begin{aligned} \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right| &\leq \frac{\sqrt{3}}{54} M_2 (b-a)^3, \\ \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right| &\leq \frac{1}{192} M_3 (b-a)^4 \end{aligned}$$

and the inequality (1.3), respectively.

For convenience in further discussions, we will now collect some technical results related to (2.3) which are not difficult to obtain by elementary calculus



**Some Inequalities of Perturbed
Trapezoid Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

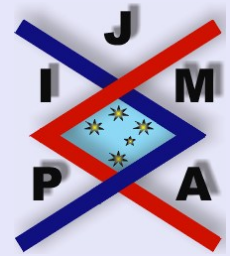
Page 8 of 18

as:

$$(2.6) \quad \int_a^b T_n(x) dx = \begin{cases} 0, & n \text{ odd,} \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \text{ even.} \end{cases}$$

$$(2.7) \quad \max_{x \in [a,b]} |T_n(x)| = \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4. \end{cases}$$

$$(2.8) \quad \max_{x \in [a,b]} \left| T_{2m}(x) - \frac{1}{b-a} \int_a^b T_{2m}(x) dx \right| = \begin{cases} \frac{(b-a)^4}{720}, & m = 2, \\ \frac{(8m^3 - 16m^2 + 2m + 3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases}$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 9 of 18

3. Bounds In Terms of Some Lebesgue Norms

Theorem 3.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 2$) is absolutely continuous on $[a, b]$. If $f^{(n)} \in L_\infty[a, b]$, then we have

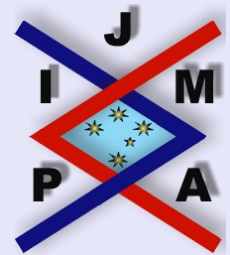
$$(3.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3, \end{cases}$$

where $\lfloor \frac{n-1}{2} \rfloor$ denotes the integer part of $\frac{n-1}{2}$ and $\|f^{(n)}\|_\infty := \text{ess sup}_{x \in [a, b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a, b]$.

The proof of inequality (3.1) is similar to the proof of inequality (2.1) and so is omitted.

Theorem 3.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 2$) is absolutely continuous on $[a, b]$. If $f^{(n)} \in L_1[a, b]$, then we have

$$(3.2) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right|$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents

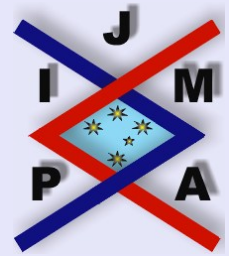


Go Back

Close

Quit

Page 10 of 18



Title Page

Contents



Go Back

Close

Quit

Page 11 of 18

$$\begin{aligned}
 & - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg| \\
 & \leq \|f^{(n)}\|_1 \times \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4, \end{cases}
 \end{aligned}$$

where $\|f^{(n)}\|_1 := \int_a^b |f^{(n)}(x)| dx$ is the usual Lebesgue norm on $L_1[a, b]$.

Proof. By using the identity (2.2), we get

$$\begin{aligned}
 & \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right. \\
 & \quad \left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\
 & = \left| \int_a^b T_n(x) f^{(n)}(x) dx \right| \leq \max_{x \in [a, b]} |T_n(x)| \int_a^b |f^{(n)}(x)| dx.
 \end{aligned}$$

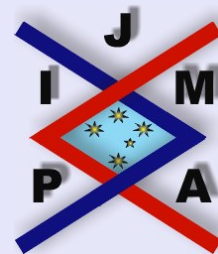
Then the inequality (3.2) follows from (2.7). □

4. Non-Symmetric Bounds

Theorem 4.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n)} (n \geq 2)$ is integrable with $\gamma_n = \inf_{x \in [a, b]} f^{(n)}(x) > -\infty$ and $\Gamma_n = \sup_{x \in [a, b]} f^{(n)}(x) < +\infty$. Then we have

$$(4.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{\Gamma_n - \gamma_n}{2} \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3 \text{ and odd,} \end{cases}$$

$$(4.2) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq [f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)]$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents

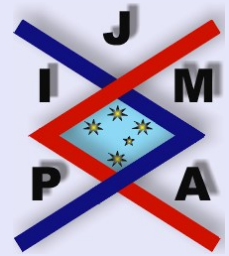


Go Back

Close

Quit

Page 12 of 18



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 13 of 18

$$\times \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 5 \text{ and odd,} \end{cases}$$

$$(4.3) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right.$$

$$\left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)} \left(\frac{a+b}{2} \right) \right|$$

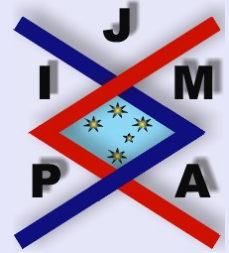
$$\leq [\Gamma_n(b-a) - f^{(n-1)}(b) + f^{(n-1)}(a)]$$

$$\times \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 5 \text{ and odd,} \end{cases}$$

$$(4.4) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right.$$

$$\left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)} \left(\frac{a+b}{2} \right) \right|$$

$$\left. - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] \right|$$



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 14 of 18

$$\leq [f^{(2m-1)}(b) - f^{(2m-1)}(a) - \gamma_{2m}(b-a)]$$

$$\times \begin{cases} \frac{(b-a)^4}{720}, & m = 2, \\ \frac{(8m^3 - 16m^2 + 2m + 3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3, \end{cases}$$

$$(4.5) \quad \left| \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right.$$

$$\left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right.$$

$$\left. - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] \right|$$

$$\leq [\Gamma_{2m}(b-a) - f^{(2m-1)}(b) + f^{(2m-1)}(a)] \begin{cases} \frac{(b-a)^4}{720}, & m = 2, \\ \frac{(8m^3 - 16m^2 + 2m + 3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases}$$

Proof. For n odd and $n = 2$, by (2.2) and (2.6) we get

$$(-1)^n \int_a^b T_n(x)[f^{(n)}(x) - C] dx$$

$$= \int_a^b f(x) dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)]$$

$$- \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right).$$

where $C \in \mathbb{R}$ is a constant.

If we choose $C = \frac{\gamma_n + \Gamma_n}{2}$, then we have

$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right. \\ & \quad \left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{\Gamma_n - \gamma_n}{2} \int_a^b |T_n(x)| dx. \end{aligned}$$

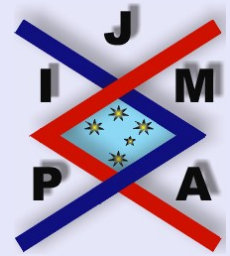
and hence the inequality (4.1) follows from (2.5).

If we choose $C = \gamma_n$, then we have

$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right. \\ & \quad \left. - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ & \leq \max_{x \in [a,b]} |T_n(x)| \int_a^b |f^{(n)}(x) - \gamma_n| dx, \end{aligned}$$

and hence the inequality (4.2) follows from (2.7).

Similarly we can prove that the inequality (4.3) holds.



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 15 of 18

By (2.2) and (2.6) we can also get

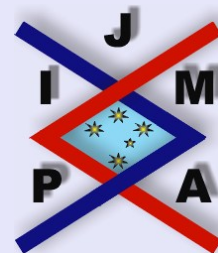
$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right. \\ & \quad - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\ & \quad \left. - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] \right| \\ & = \left| \int_a^b \left[T_{2m}(x) - \frac{1}{b-a} \int_a^b T_{2m}(x) dx \right] [f^{2m}(x) - C] dx \right|, \end{aligned}$$

where $C \in \mathbb{R}$ is a constant.

If we choose $C = \gamma_{2m}$, then we have

$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(b) - f'(a)] \right. \\ & \quad - \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\ & \quad \left. - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] \right| \\ & \leq \max_{x \in [a,b]} \left| T_{2m}(x) - \frac{1}{b-a} \int_a^b T_{2m}(x) dx \right| \int_a^b |f^{(2m)}(x) - \gamma_{2m}| dx \end{aligned}$$

and hence the inequality (4.4) follows from (2.8). \square



Some Inequalities of Perturbed
Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 16 of 18

Similarly we can prove that the inequality (4.5) holds.

Remark 2. *It is not difficult to find that the inequality (4.1) is sharp in the sense that we can choose f to attain the equality in (4.1). Indeed, for $n = 2$, we construct the function $f(x) = \int_a^x \left(\int_a^y j(z) dz \right) dy$, where*

$$j(x) = \begin{cases} \Gamma_2, & a \leq x < \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6}, \\ \gamma_2, & \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6} \leq x < \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6}, \\ \Gamma_2, & \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6} \leq x \leq b, \end{cases}$$

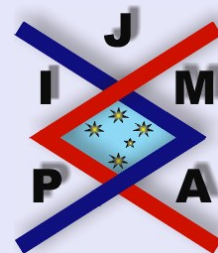
and for $n \geq 3$ and odd, we construct the function

$$f(x) = \int_a^x \left(\int_a^{y_n} \left(\dots \int_a^{y_2} j(y_1) dy_1 \dots \right) dy_{n-1} \right) dy_n,$$

where

$$j(x) = \begin{cases} \Gamma_n, & a \leq x < \frac{a+b}{2}, \\ \gamma_n, & \frac{a+b}{2} \leq x \leq b. \end{cases}$$

Remark 3. *If in the inequality (4.1) we choose $n = 2, 3$, then we recapture the inequalities (1.1) and (1.2), respectively.*



Some Inequalities of Perturbed
Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

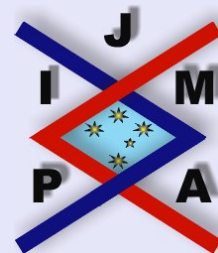
Close

Quit

Page 17 of 18

References

- [1] P. CERONE, On perturbed trapezoidal and midpoint rules, *Korean J. Comput. Appl. Math.*, **2** (2002), 423–435.
- [2] P. CERONE AND S.S. DRAGOMIR, Trapezoidal type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press N.Y.(2000), 65–134.
- [3] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration, *Kyungpook Math. J.*, **39** (1999), 331–341.
- [4] X.L. CHENG, Improvement of some Ostrowski-Grüss type inequalities, *Comput. Math. Appl.*, **42** (2001), 109–114.
- [5] X.L. CHENG AND J. SUN, A note on the perturbed trapezoid inequality, *J. Inequal. in Pure and Appl. Math.*, **3**(2) (2002), Art. 29. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=181>].
- [6] S.S. DRAGOMIR, P. CERONE AND A. SOFO, Some remarks on the trapezoid rule in numerical integration, *Indian J. of Pure and Appl. Math.*, **31**(5) (2000), 489–501.
- [7] M. MATIĆ, J. PEČARIĆ AND N. UJEVIĆ, Improvement and further generalization of inequalities of Ostrowski-Grüss type, *Computer Math. Appl.*, **39** (2000), 161–175.
- [8] N. UJEVIĆ, On perturbed mid-point and trapezoid inequalities and applications, *Kyungpook Math. J.*, **43** (2003), 327–334.



Some Inequalities of Perturbed Trapezoid Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 18 of 18