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# SOME INEQUALITIES OF PERTURBED TRAPEZOID TYPE

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## Abstract

A new generalized perturbed trapezoid type inequality is established by Peano kernel approach. Some related results are also given.

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*Key words:* Perturbed trapezoid inequality; $n$-times continuously differentiable mapping; Absolutely continuous.

## Contents

## 1. Introduction

In recent years, some authors have considered the perturbed trapezoid inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le C(\Gamma_2-\gamma_2)(b-a)^3,$$

where $f : [a,b] \to \mathbb{R}$ is a twice differentiable mapping on $(a,b)$ with $\gamma_2 = \inf_{x\in[a,b]} f''(x) > -\infty$ and $\Gamma_2 = \sup_{x\in[a,b]} f''(x) < +\infty$ while $C$ is a constant. (e.g. see [1] – [8]) It seems that the best result $C = \frac{\sqrt{3}}{108}$ was separately and independently discovered by the authors of [5] and [8]. The perturbed trapezoid inequality has been established as

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{\sqrt{3}}{108}(\Gamma_2-\gamma_2)(b-a)^3. \tag{1.1}$$

Moreover, we can also find in [5] the following two perturbed trapezoid inequalities as

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{1}{384}(\Gamma_3-\gamma_3)(b-a)^4, \tag{1.2}$$

where $f : [a,b] \to \mathbb{R}$ is a third-order differentiable mapping on $(a,b)$ with $\gamma_3 = \inf_{x\in[a,b]} f'''(x) > -\infty$ and $\Gamma_3 = \sup_{x\in[a,b]} f'''(x) < +\infty$, and

$$(1.3)\quad \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \le \frac{1}{720} M_4 (b-a)^5,$$

where $f : [a,b] \to \mathbb{R}$ is a fourth-order differentiable mapping on $(a,b)$ with $M_4 = \sup_{x\in[a,b]} |f^{(4)}(x)| < +\infty$.

The purpose of this paper is to extend these above results to a more general version by choosing appropriate harmonic polynomials such as the Peano kernel. A new generalized perturbed trapezoid type inequality is established and some related results are also given.

## 2. For Differentiable Mappings With Bounded Derivatives

**Theorem 2.1.** *Let $f : [a, b] \to \mathbb{R}$ be an $n$-times continuously differentiable mapping, $n \geq 2$ and such that $M_n := \sup_{x\in[a,b]} |f^{(n)}(x)| < \infty$. Then*

$$(2.1)\quad \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq M_n \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54} & \text{if } n = 2; \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n} & \text{if } n \geq 3, \end{cases}$$

*where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$.*

*Proof.* It is not difficult to find the identity

$$(2.2)\quad (-1)^n \int_a^b T_n(x) f^{(n)}(x)\,dx = \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right),$$

where $T_n(x)$ is the kernel given by

$$(2.3)\quad T_n(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Using the identity (2.2), we get

$$(2.4)\quad \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right.$$
$$\left. - \sum_{k=2}^{\left[\frac{n-1}{2}\right]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right|$$
$$= \left| \int_a^b T_n(x) f^{(n)}(x)\,dx \right| \le M_n \int_a^b |T_n(x)|\,dx.$$

For brevity, we put

$$P_n(x) := \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-a)^{n-2}}{12(n-2)!}$$
$$= \frac{(x-a)^{n-2}}{n!}\left[(x-a)^2 - \frac{n(b-a)(x-a)}{2} + \frac{n(n-1)(b-a)^2}{12}\right],$$
$$x \in \left[a, \frac{a+b}{2}\right]$$

and

$$Q_n(x) := \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{2(n-1)!} + \frac{(b-a)^2(x-b)^{n-2}}{12(n-2)!}$$
$$= \frac{(x-b)^{n-2}}{n!}\left[(x-b)^2 + \frac{n(b-a)(x-b)}{2} + \frac{n(n-1)(b-a)^2}{12}\right],$$
$$x \in \left[\frac{a+b}{2}, b\right].$$

It is clear that $P_n(x)$ and $Q_n(x)$ are symmetric with respect to the line $x = \frac{a+b}{2}$ for $n$ even, and symmetric with respect to the point $(\frac{a+b}{2}, 0)$ for $n$ odd. Therefore,

$$\int_a^b |T_n(x)|\,dx = 2\int_a^{\frac{a+b}{2}} |P_n(x)|\,dx$$
$$= \frac{(b-a)^{n+1}}{n!2^n}\int_0^1 \left|t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right]\right| dt$$

by substitution $x = a + \frac{b-a}{2}t$, and it is easy to find that

$$r_n(t) := t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right]$$

is always nonnegative on $[0,1]$ for $n \geq 3$. Thus we have

$$\int_0^1 |r_n(t)|\,dt = \int_0^1 t^{n-2}\left[t^2 - nt + \frac{n(n-1)}{3}\right] dt = \frac{n(n-2)}{3(n+1)}$$

for $n \geq 3$, and

$$\int_0^1 |r_2(t)|\, dt = \int_0^1 \left| t^2 - 2t + \frac{2}{3} \right| dt$$
$$= \int_0^{t_0} \left( t^2 - 2t + \frac{2}{3} \right) dt - \int_{t_0}^1 \left( t^2 - 2t + \frac{2}{3} \right) dt,$$

where $t_0 = 1 - \frac{\sqrt{3}}{3}$ is the unique zero of $r_2(t)$ in $(0, 1)$. Hence,

$$\int_a^b |T_n(x)|\, dx = \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n = 2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3. \end{cases} \tag{2.5}$$

Consequently, the inequality (2.1) follows from (2.4) and (2.5). □

**Remark 1.** *If in the inequality (2.1) we choose $n = 2, 3, 4$, then we get*

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq \frac{\sqrt{3}}{54} M_2 (b-a)^3,$$
$$\left| \int_a^b f(x)\, dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq \frac{1}{192} M_3 (b-a)^4$$

*and the inequality (1.3), respectively.*

For convenience in further discussions, we will now collect some technical results related to (2.3) which are not difficult to obtain by elementary calculus as:

$$\int_a^b T_n(x)\,dx = \begin{cases} 0, & n \text{ odd,} \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \text{ even.} \end{cases} \tag{2.6}$$

$$\max_{x\in[a,b]} |T_n(x)| = \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4. \end{cases} \tag{2.7}$$

$$\max_{x\in[a,b]} \left| T_{2m}(x) - \frac{1}{b-a}\int_a^b T_{2m}(x)\,dx \right| = \begin{cases} \frac{(b-a)^4}{720}, & m = 2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases} \tag{2.8}$$

## 3. Bounds In Terms of Some Lebesgue Norms

**Theorem 3.1.** *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_\infty[a,b]$, then we have*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right.$$
$$\left. - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right|$$
$$\leq \|f^{(n)}\|_\infty \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3, \end{cases} \tag{3.1}$$

*where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$ and $\|f^{(n)}\|_\infty := ess\sup_{x\in[a,b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a,b]$.*

The proof of inequality (3.1) is similar to the proof of inequality (2.1) and so is omitted.

**Theorem 3.2.** *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n \geq 2)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_1[a,b]$, then we have*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right. \tag{3.2}$$

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$$
- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg|
$$

$$
\leq \|f^{(n)}\|_1 \times \begin{cases} \frac{(b-a)^2}{12}, & n=2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n=3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 4, \end{cases}
$$

*where* $\|f^{(n)}\|_1 := \int_a^b |f(x)|\, dx$ *is the usual Lebesgue norm on* $L_1[a,b]$.

*Proof.* By using the identity (2.2), we get

$$
\Bigg| \int_a^b f(x)\, dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]
$$

$$
- \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \Bigg|
$$

$$
= \left| \int_a^b T_n(x) f^{(n)}(x)\, dx \right| \leq \max_{x\in[a,b]} |T_n(x)| \int_a^b |f^{(n)}(x)|\, dx.
$$

Then the inequality (3.2) follows from (2.7). □

# 4. Non-Symmetric Bounds

**Theorem 4.1.** *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping such that the derivative* $f^{(n)}(n \geq 2)$ *is integrable with* $\gamma_n = \inf_{x\in[a,b]} f^{(n)}(x) > -\infty$ *and* $\Gamma_n = \sup_{x\in[a,b]} f^{(n)}(x) < +\infty$*. Then we have*

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{\left[\frac{n-1}{2}\right]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \frac{\Gamma_n - \gamma_n}{2} \times \begin{cases} \frac{\sqrt{3}(b-a)^3}{54}, & n=2, \\ \frac{n(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n \geq 3 \text{ and odd}, \end{cases} \tag{4.1}$$

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{\left[\frac{n-1}{2}\right]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)\right] \tag{4.2}$$

$$\times \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 5 \textit{ and odd}, \end{cases}$$

$$\begin{aligned} \text{(4.3)} \quad & \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right. \\ & \left. - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ & \leq [\Gamma_n(b-a) - f^{(n-1)}(b) + f^{(n-1)}(a)] \\ & \times \begin{cases} \frac{(b-a)^2}{12}, & n = 2, \\ \frac{\sqrt{3}(b-a)^3}{216}, & n = 3, \\ \frac{(n-1)(n-3)(b-a)^n}{3(n!)2^n}, & n \geq 5 \textit{ and odd}, \end{cases} \end{aligned}$$

$$\begin{aligned} \text{(4.4)} \quad & \left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right. \\ & - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \\ & \left. - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)] \right| \end{aligned}$$

$$\leq [f^{(2m-1)}(b) - f^{(2m-1)}(a) - \gamma_{2m}(b-a)] \times \begin{cases} \frac{(b-a)^4}{720}, & m=2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3, \end{cases}$$

$$\text{(4.5)}\quad \Bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) - \frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b) - f^{(2m-1)}(a)] \Bigg|$$

$$\leq [\Gamma_{2m}(b-a) - f^{(2m-1)}(b) + f^{(2m-1)}(a)] \begin{cases} \frac{(b-a)^4}{720}, & m=2, \\ \frac{(8m^3-16m^2+2m+3)(b-a)^{2m}}{3(2m+1)!2^{2m}}, & m \geq 3. \end{cases}$$

*Proof.* For $n$ odd and $n=2$, by (2.2) and (2.6) we get

$$(-1)^n \int_a^b T_n(x)[f^{(n)}(x) - C]\,dx$$
$$= \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] - \sum_{k=2}^{[\frac{n-1}{2}]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right).$$

where $C \in \mathbb{R}$ is a constant.

If we choose $C = \frac{\gamma_n + \Gamma_n}{2}$, then we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right. \\ \left. - \sum_{k=2}^{\left[\frac{n-1}{2}\right]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{\Gamma_n - \gamma_n}{2} \int_a^b |T_n(x)|\,dx.$$

and hence the inequality (4.1) follows from (2.5).

If we choose $C = \gamma_n$, then we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right. \\ \left. - \sum_{k=2}^{\left[\frac{n-1}{2}\right]} \frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\ \leq \max_{x\in[a,b]} |T_n(x)| \int_a^b |f^{(n)}(x) - \gamma_n|\,dx,$$

and hence the inequality (4.2) follows from (2.7).

Similarly we can prove that the inequality (4.3) holds.

By (2.2) and (2.6) we can also get

$$\begin{aligned}&\Bigg|\int_a^b f(x)\,dx-\frac{b-a}{2}[f(a)+f(b)]+\frac{(b-a)^2}{12}[f'(b)-f'(a)]\\&\qquad-\sum_{k=2}^{[\frac{n-1}{2}]}\frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}}f^{(2k)}\left(\frac{a+b}{2}\right)\\&\qquad\quad-\frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b)-f^{(2m-1)}(a)]\Bigg|\\&\qquad\quad=\left|\int_a^b\left[T_{2m}(x)-\frac{1}{b-a}\int_a^b T_{2m}(x)\,dx\right][f^{2m}(x)-C]\,dx\right|,\end{aligned}$$

where $C\in\mathbb{R}$ is a constant.

If we choose $C=\gamma_{2m}$, then we have

$$\begin{aligned}&\Bigg|\int_a^b f(x)\,dx-\frac{b-a}{2}[f(a)+f(b)]+\frac{(b-a)^2}{12}[f'(b)-f'(a)]\\&\qquad-\sum_{k=2}^{[\frac{n-1}{2}]}\frac{k(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-2}}f^{(2k)}\left(\frac{a+b}{2}\right)\\&\qquad\quad-\frac{m(m-1)(b-a)^{2m}}{3(2m+1)!2^{2m-2}}[f^{(2m-1)}(b)-f^{(2m-1)}(a)]\Bigg|\\&\qquad\le\max_{x\in[a,b]}\left|T_{2m}(x)-\frac{1}{b-a}\int_a^b T_{2m}(x)\,dx\right|\int_a^b|f^{(2m)}(x)-\gamma_{2m}|\,dx\end{aligned}$$

and hence the inequality (4.4) follows from (2.8). □

Similarly we can prove that the inequality (4.5) holds.

**Remark 2.** *It is not difficult to find that the inequality (4.1) is sharp in the sense that we can choose $f$ to attain the equality in (4.1). Indeed, for $n = 2$, we construct the function $f(x) = \int_a^x \left(\int_a^y j(z)\,dz\right) dy$, where*

$$j(x) = \begin{cases} \Gamma_2, & a \le x < \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6}, \\ \gamma_2, & \frac{(3+\sqrt{3})a+(3-\sqrt{3})b}{6} \le x < \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6}, \\ \Gamma_2, & \frac{(3-\sqrt{3})a+(3+\sqrt{3})b}{6} \le x \le b, \end{cases}$$

*and for $n \ge 3$ and odd, we construct the function*

$$f(x) = \int_a^x \left( \int_a^{y_n} \left( \cdots \int_a^{y_2} j(y_1)\,dy_1 \cdots \right) dy_{n-1} \right) dy_n,$$

*where*

$$j(x) = \begin{cases} \Gamma_n, & a \le x < \frac{a+b}{2}, \\ \gamma_n, & \frac{a+b}{2} \le x \le b. \end{cases}$$

**Remark 3.** *If in the inequality (4.1) we choose $n = 2, 3$, then we recapture the inequalities (1.1) and (1.2), respectively.*

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