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TIME SCALE INTEGRAL INEQUALITIES SIMILAR TO QI'S INEQUALITY

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ABSTRACT. In this study, some integral inequalities and Qi's inequalities of which is proved by the Bougoffa [5] – [7] are extended to the general time scale.

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1. INTRODUCTION

The unification and extension of continuous calculus, discret calculus, $q-$calculus, and indeed arbitrary real-number calculus to time scale calculus was first accomplished by Hilger in his PhD. thesis [8]. The purpose of this work is to extend some integral inequalities and Qi inequalities proved by Bougoffa [5] – [7]. The following definitions will serve as a short primer on time scale calculus; they can be found in [1] – [4]. A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Within that set, define the jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$

where $\inf \phi := \sup \mathbb{T}$ and $\sup \phi := \inf \mathbb{T}$. If $\rho(t) = t$ and $\rho(t) < t$, then the point $t \in \mathbb{T}$ is left-dense, left-scattered. If $\sigma(t) = t$ and $\sigma(t) > t$, then the point $t \in \mathbb{T}$ is right-dense, right-scattered. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The so-called graininess functions are $\mu(t) := \sigma(t) - t$ and $v(t) := t - \rho(t)$.

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For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, the delta derivative in [3, 4] of f at t, denoted $f^{\Delta}(t)$, is the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right|\leq\varepsilon\left|\sigma(t)-s\right|$$

for all $s\in U$. For $\mathbb{T}=\mathbb{R},\ f^{\Delta}=f'$, the usual derivative; for $\mathbb{T}=\mathbb{Z}$ the delta derivative is the forward difference operator, $f^{\Delta}(t)=f(t+1)-f(t)$; in the case of $q-$difference equations with $q>1$,

$$f^{\Delta}(t)=\frac{f(qt)-f(t)}{(q-1)t},\qquad f^{\Delta}(0)=\lim_{s\to0}\frac{f(s)-f(0)}{s}.$$

A function $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous or rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. If $\mathbb{T}=\mathbb{R}$, then f is rd-continuous if and only if f is continuous. It is known from Theorem 1.74 in [3] that if f is right-dense continuous, there is a function F such that $F^{\Delta}(t)=f(t)$ and

$$\int_a^b f(t)\Delta t=F(b)-F(a).$$

Note that we have

$$\sigma(t)=t,\quad \mu(t)\equiv0,\quad f^{\Delta}=f',\quad \int_a^b f(t)\Delta t=\int_a^b f(t)dt,\quad \text{when } \mathbb{T}=\mathbb{R}$$

while

$$\sigma(t)=t+1,\quad \mu(t)\equiv1,\quad f^{\Delta}=\Delta f,\quad \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1}f(t),\quad \text{when } \mathbb{T}=\mathbb{Z}.$$

Much more information concerning time scales and dynamic equations on time scales can be found in the books [3, 4].

Theorem 1.1 (Hölder's inequality on time scales [3])**.** *Let* $a,b\in\mathbb{T}$. *For rd-continuous functions* $f,g:[a,b]\to\mathbb{R}$ *we have*

$$\int_a^b\left|f(x)g(x)\right|\Delta x\leq\left(\int_a^b\left|f(x)\right|^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b\left|g(x)\right|^q\Delta x\right)^{\frac{1}{q}},$$

where $p>1$ *and* $q=\frac{p}{p-1}$.

2. MAIN RESULTS

In this section, we will state our main results and give their proofs.

Lemma 2.1. *Let* $a,b\in\mathbb{T}$, *and* $p>1$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *If two positive functions* $f,g:[a,b]\to\mathbb{R}$ *are rd-continuous and satisfying* $0<m\leq\frac{f^p}{g^q}\leq M<\infty$ *on the set* $[a,b]$, *then we have the following inequality*

$$\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b g^q\Delta x\right)^{\frac{1}{q}}\leq\left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b fg\Delta x. \tag{2.1}$$

Inequality (2.1) is called the reverse Hölder inequality.

Proof. Since $\frac{f^p}{g^q}\leq M,\ g\geq M^{-\frac{1}{q}}f^{\frac{p}{q}}$, therefore

$$fg\geq M^{-\frac{1}{q}}f^{1+\frac{p}{q}}=M^{-\frac{1}{q}}f^p$$

and so,

$$\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}} \le M^{\frac{1}{pq}}\left(\int_a^b fg\Delta x\right)^{\frac{1}{p}}. \tag{2.2}$$

On the other hand, since $m \le \frac{f^p}{g^q},\ \ f \ge m^{\frac{1}{p}} g^{\frac{q}{p}}$, hence

$$\int_a^b fg\Delta x \ge \int_a^b m^{\frac{1}{p}} g^{1+\frac{q}{p}}\Delta x \ge m^{\frac{1}{p}}\int_a^b g^q\Delta x$$

and so,

$$\left(\int_a^b fg\Delta x\right)^{\frac{1}{q}} \ge m^{\frac{1}{pq}}\left(\int_a^b g^q\Delta x\right)^{\frac{1}{q}}.$$

Combining with (2.2), we have the desired inequality

$$\begin{aligned}\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b g^q\Delta x\right)^{\frac{1}{q}} &\le M^{\frac{1}{pq}}\left(\int_a^b fg\Delta x\right)^{\frac{1}{p}} m^{-\frac{1}{pq}}\left(\int_a^b g^q\Delta x\right)^{\frac{1}{q}}\\ &= \left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b fg\Delta x.\end{aligned}$$

□

Corollary 2.2. *In Lemma 2.1, replacing f^p and g^q by f and g, respectively, we obtain the reverse Hölder type inequality,*

$$\left(\int_a^b f\Delta x\right)^{\frac{1}{p}}\left(\int_a^b g\Delta x\right)^{\frac{1}{q}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_a^b f^{\frac{1}{p}} g^{\frac{1}{q}}\Delta x. \tag{2.3}$$

The proof of this corollary can be obtained from (2.1).

Theorem 2.3. *Let $a, b \in \mathbb{T},\ p > 1$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. If $f : [a, b] \to \mathbb{R}$ is rd-continuous and $0 < m^{\frac{1}{p}} \le f \le M^{\frac{1}{p}} < \infty$ on $[a, b]$, then we have the following inequality*

$$\left(\int_a^b f^{\frac{1}{p}}\Delta x\right)^p \ge (b-a)^{\frac{p+1}{q}}\left(\frac{m}{M}\right)^{\frac{p+1}{pq}}\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}. \tag{2.4}$$

Proof. Putting $g \equiv 1$ in Lemma 2.1, we obtain

$$\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}[b-a]^{\frac{1}{q}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_a^b f\Delta x.$$

Therefore, we get

$$\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}}[b-a]^{-\frac{1}{q}}\int_a^b f\Delta x. \tag{2.5}$$

Again, substituting $g \equiv 1$ in Corollary 2.2 leads to

$$\left(\int_a^b f\Delta x\right)^{\frac{1}{p}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}}[b-a]^{-\frac{1}{q}}\int_a^b f^{\frac{1}{p}}\Delta x,$$

and so,

$$\int_a^b f\Delta x \le \left(\frac{m}{M}\right)^{-\frac{1}{q}}[b-a]^{-\frac{p}{q}}\left(\int_a^b f^{\frac{1}{p}}\Delta x\right)^p. \tag{2.6}$$

Combining (2.5) with (2.6), we obtain

$$\left(\int_a^b f^{\frac{1}{p}}\Delta x\right)^p \geq (b-a)^{\frac{p+1}{q}}\left(\frac{m}{M}\right)^{\frac{p+1}{pq}}\left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}.$$

□

Corollary 2.4. *If* $0 < m^{\frac{1}{p}} \leq f \leq M^{\frac{1}{p}} < \infty$ *on* $[a,b]$ *and* $\frac{m}{M} = [b-a]^{-p}$ *for* $p > 1$ *, then*

$$\left(\int_a^b f^{\frac{1}{p}}\Delta x\right)^p \geq \left(\int_a^b f^p\Delta x\right)^{\frac{1}{p}}. \tag{2.7}$$

Remark 2.5. For $\mathbb{T} = \mathbb{R}$, (2.7) is Qi's inequality [9].

Theorem 2.6. *If* $f : [a,b] \to \mathbb{R}$ *is rd-continuous and* $0 < m \leq f(x) \leq M$ *on* $[a,b]$ *, then we have the following inequality*

$$\int_a^b f^{\frac{1}{p}}\Delta x \geq B\left(\int_a^b f\Delta x\right)^{\frac{1}{p}-1}, \tag{2.8}$$

where $B = m(b-a)^{1+\frac{1}{q}}\left(\frac{m}{M}\right)^{\frac{1}{pq}}$ *and* $p > 1$ *,* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1.$

Proof. In Corollary 2.2, putting $g \equiv 1$ yields

$$\left(\int_a^b f\Delta x\right)^{\frac{1}{p}}[b-a]^{\frac{1}{q}} \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_a^b f^{\frac{1}{p}}\Delta x,$$

and so,

$$\int_a^b f^{\frac{1}{p}}\Delta x \geq \left(\frac{m}{M}\right)^{-\frac{1}{pq}}[b-a]^{\frac{1}{q}}\left(\int_a^b f\Delta x\right)^{\frac{1}{p}-1}\left(\int_a^b f\Delta x\right)^{\frac{1}{p}}.$$

Since $0 < m \leq f(x)$, we have

$$\int_a^b f^{\frac{1}{p}}\Delta x \geq \left(\frac{m}{M}\right)^{\frac{1}{pq}} m\,[b-a]^{1+\frac{1}{q}}\left(\int_a^b f\Delta x\right)^{\frac{1}{p}-1}.$$

This proves inequality (2.8). □

Corollary 2.7. *Let* $p > 1$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. If*

$$m\left(\frac{m}{M}\right)^{\frac{1}{pq}} = \frac{1}{[b-a]^{1+\frac{1}{q}}}$$

and $0 < m \leq f(x) \leq M$ *on* $[a,b]$ *, then*

$$\int_a^b f^{\frac{1}{p}}\Delta x \geq \left(\int_a^b f\Delta x\right)^{\frac{1}{p}-1}. \tag{2.9}$$

Remark 2.8. For $\mathbb{T} = \mathbb{R}$, (2.9) is Qi's inequality [9].

Lemma 2.9. *Let* $a, b \in \mathbb{T}$*, and* $f, g : [a,b] \to \mathbb{R}$ *be rd-continuous and nonnegative functions with* $0 < m \leq \frac{f}{g} \leq M < \infty$ *on* $[a,b]$*. Then for* $p > 1$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have the following inequality*

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \leq M^{\frac{1}{p^2}}m^{-\frac{1}{q^2}}\int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\Delta x \tag{2.10}$$

and

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \le M^{\frac{1}{p^2}}m^{-\frac{1}{q^2}}\left(\int_a^b f(x)\Delta x\right)^{\frac{1}{q}}\left(\int_a^b g(x)\Delta x\right)^{\frac{1}{p}}. \tag{2.11}$$

Proof. From Hölder's inequality, we obtain

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \le \left(\int_a^b f(x)\Delta x\right)^{\frac{1}{q}}\left(\int_a^b g(x)\Delta x\right)^{\frac{1}{p}},$$

that is,

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \le \left(\int_a^b [f(x)]^{\frac{1}{p}}[f(x)]^{\frac{1}{q}}\Delta x\right)^{\frac{1}{q}}\left(\int_a^b [g(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x\right)^{\frac{1}{p}}.$$

Since $[f(x)]^{\frac{1}{p}} \le M^{\frac{1}{p}}[g(x)]^{\frac{1}{p}}$ and $[g(x)]^{\frac{1}{q}} \le m^{-\frac{1}{q}}[f(x)]^{\frac{1}{q}}$, from the above inequality it follows that

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x$$
$$\le M^{\frac{1}{p^2}}m^{-\frac{1}{q^2}}\left(\int_a^b [g(x)]^{\frac{1}{p}}[f(x)]^{\frac{1}{q}}\Delta x\right)^{\frac{1}{q}}\left(\int_a^b [g(x)]^{\frac{1}{p}}[f(x)]^{\frac{1}{q}}\Delta x\right)^{\frac{1}{p}},$$

and so,

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \le M^{\frac{1}{p^2}}m^{-\frac{1}{q^2}}\int_a^b [f(x)]^{\frac{1}{q}}[g(x)]^{\frac{1}{p}}\Delta x. \tag{2.12}$$

Hence, the inequality (2.10) is proved.

The inequality (2.11) follows from substituting the following

$$\int_a^b [f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\Delta x \le \left(\int_a^b f(x)\Delta x\right)^{\frac{1}{q}}\left(\int_a^b g(x)\Delta x\right)^{\frac{1}{p}}$$

into (2.12), which can be obtained by Hölder's inequality on time scales. □

Lemma 2.10. *Let $a, b \in \mathbb{T}$. For a given positive integer $p \ge 2$, if $f : [a, b] \to \mathbb{R}$ is rd-continuous and $0 < m \le \frac{f}{g} \le M < \infty$ on $[a, b]$, then*

$$\int_a^b [f(x)]^{\frac{1}{p}}\Delta x \le \left(\int_a^b f(x)\Delta x\right)^{1-\frac{1}{p}}. \tag{2.13}$$

Proof. Putting $g(x) \equiv 1$ in (2.11) yields

$$\int_a^b [f(x)]^{\frac{1}{p}}\Delta x \le K\left(\int_a^b f(x)\Delta x\right)^{1-\frac{1}{p}},$$

where $K = \frac{M^{\frac{1}{p^2}}(b-a)^{\frac{1}{p}}}{m^{(1-\frac{1}{p})^2}}$. From $M \le \frac{m^{(p-1)^2}}{(b-a)^p}$, we conclude that $K \le 1$. Thus the inequality (2.13) is proved. □

In the following we generalize to arbitrary time scales a result in [6].

Theorem 2.11. *Let $a, b \in \mathbb{T}$. If $f, g : [a, b] \to \mathbb{R}$ is rd-continuous and satisfying $0 < m \leq \frac{f}{g} \leq M < \infty$ on $[a, b]$, then we have the following inequality*

$$\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}} \leq c\left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{1-\frac{1}{p}}, \tag{2.14}$$

where $c = (\frac{m}{M})^{\frac{1}{pq}}$.

Proof. It follows from Lemma 2.1 that

$$\begin{aligned}
&\int_a^b (f(x)+g(x))^p \,\Delta x \\
&= \int_a^b (f(x)+g(x))^{p-1} f(x)\Delta x + \int_a^b (f(x)+g(x))^{p-1} g(x)\Delta x \\
&\geq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b (f(x)+g(x))^{q(p-1)} \,\Delta x\right)^{\frac{1}{q}} \\
&\qquad + \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b (f(x)+g(x))^{q(p-1)} \,\Delta x\right)^{\frac{1}{q}} \\
&= \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{\frac{1}{q}} \\
&\qquad \times \left[\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left[\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}}\right] &\leq \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{1-\frac{1}{q}} \\
&= \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{p},
\end{aligned}$$

where $q(p-1) = p$. □

Example 2.1. Let $\mathbb{T} = \mathbb{Z}$. Let $f(x) = 3^x$ and $g(x) = x^2$ on $[3, 4]$ with $M \approx 5.06$ and $m = 3$. Taking $p = 2$, we see that the conditions of Lemma 2.1 are fulfilled. Therefore, for

$$\left(\int_3^4 3^{2x}\Delta x\right)^{\frac{1}{2}} = \left(\frac{1}{8}(3^8 - 3^6)\right)^{\frac{1}{2}} = 3^3,$$

$$\left(\int_3^4 x^4\Delta x\right)^{\frac{1}{2}} = \left(\sum_{x=3}^{4-1} x^4\right)^{\frac{1}{2}} = 3^2$$

and

$$\int_3^4 3^x x^2 \Delta x = \sum_{x=3}^{4-1} 3^x x^2 = 3^5$$

we get

$$\left(\int_3^4 3^{2x}\Delta x\right)^{\frac{1}{2}} \left(\int_3^4 x^4\Delta x\right)^{\frac{1}{2}} = 243 \leq \left(\frac{5.06}{3}\right)^{\frac{1}{4}} \int_3^4 3^x x^2 \Delta x \approx 274.6.$$

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