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DIFFERENTIAL SUBORDINATION RESULTS FOR NEW CLASSES OF THE FAMILY $\mathcal{E}(\Phi, \Psi)$

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ABSTRACT. We define new classes of the family $\mathcal{E}(\Phi, \Psi)$, in a unit disk $U := \{z \in \mathbb{C}, |z| < 1\}$, as follows: for analytic functions $F(z), \Phi(z)$ and $\Psi(z)$ so that $\Re\{\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\} > 0,\ z \in U,\ F(z)*\Psi(z) \neq 0$ where the operator $*$ denotes the convolution or Hadamard product. Moreover, we establish some subordination results for these new classes.

Key words and phrases: Fractional calculus; Subordination; Hadamard product.

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1. INTRODUCTION AND PRELIMINARIES.

Let $\mathcal{B}_\alpha^+$ be the class of all analytic functions $F(z)$ in the open disk $U := \{z \in \mathbb{C}, |z| < 1\}$, of the form

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n z^{n+\alpha-1}, \quad 0 < \alpha \le 1,$$

satisfying $F(0) = 1$. And let $\mathcal{B}_\alpha^-$ be the class of all analytic functions $F(z)$ in the open disk U of the form

$$F(z) = 1 - \sum_{n=1}^{\infty} a_n z^{n+\alpha-1}, \quad 0 < \alpha \le 1, \quad a_n \ge 0; \quad n = 1, 2, 3, \ldots,$$

satisfying $F(0) = 1$. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in U. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in U such that

$$f(z) = g(w(z)), \quad z \in U.$$

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We denote this subordination by $f \prec g$ or $f(z) \prec g(z),\ z \in U$. If the function g is univalent in U the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. Assume that p, ϕ are analytic and univalent in U and p satisfies the differential superordination

$$h(z) \prec \phi(p(z)), zp'(z), z^2p''(z); z), \tag{1.1}$$

then p is called a solution of the differential superordination.

An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent function q such that $p \prec q$ for all subordinants p of (1.1) is said to be the best subordinant.

Let $\mathcal{B}^+$ be the class of analytic functions of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0.$$

Given two functions $f, g \in \mathcal{B}^+$,

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = 1 + \sum_{n=1}^{\infty} a_n b_n z^n, \quad a_n \geq 0,\ b_n \geq 0, \quad z \in U.$$

Juneja et al. [1] define the family $\mathcal{E}(\Phi, \Psi)$, so that

$$\Re\left\{\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)}\right\} > 0, \quad z \in U$$

where

$$\Phi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n \quad \text{and} \quad \Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$$

are analytic in U with the conditions $\varphi_n \geq 0,\ \psi_n \geq 0, \varphi_n \geq \psi_n$ for $n \geq 2$ and $f(z) * \Psi(z) \neq 0$.

Definition 1.1. Let $F(z) \in \mathcal{B}_\alpha^+$, we define the family $\mathcal{E}_\alpha^+(\Phi, \Psi)$ so that

$$\Re\left\{\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)}\right\} > 0, \quad z \in U, \tag{1.2}$$

where

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^{n+\alpha-1} \quad \text{and} \quad \Psi(z) = 1 + \sum_{n=1}^{\infty} \psi_n z^{n+\alpha-1}$$

are analytic in U under the conditions $\varphi_n \geq 0,\ \psi_n \geq 0, \varphi_n \geq \psi_n$ for $n \geq 1$ and $F(z) * \Psi(z) \neq 0$.

Definition 1.2. Letting $F(z) \in \mathcal{B}_\alpha^-$, we define the family $\mathcal{E}_\alpha^-(\Phi, \Psi)$ which satisfies (1.2) where

$$\Phi(z) = 1 - \sum_{n=1}^{\infty} \varphi_n z^{n+\alpha-1} \quad \text{and} \quad \Psi(z) = 1 - \sum_{n=1}^{\infty} \psi_n z^{n+\alpha-1}$$

are analytic in U under the conditions $\varphi_n \geq 0,\ \psi_n \geq 0, \varphi_n \geq \psi_n$ for $n \geq 1$ and $F(z) * \Psi(z) \neq 0$.

In the present paper, we establish some sufficient conditions for functions $F \in \mathcal{B}_\alpha^+$ and $F \in \mathcal{B}_\alpha^-$ to satisfy

$$\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)} \prec q(z), \quad z \in U, \tag{1.3}$$

where $q(z)$ is a given univalent function in U such that $q(0) = 1$. Moreover, we give applications for these results in fractional calculus. We shall need the following known results.

Lemma 1.1 ([2])**.** *Let $q(z)$ be convex in the unit disk U with $q(0) = 1$ and $\Re\{q\} > \frac{1}{2}$, $z \in U$. If $0 \le \mu < 1$, p is an analytic function in U with $p(0) = 1$ and if*

$$(1-\mu)p^2(z) + (2\mu - 1)p(z) - \mu + (1-\mu)zp'(z) \prec (1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.2 ([3])**.** *Let $q(z)$ be univalent in the unit disk U and let $\theta(z)$ be analytic in a domain D containing $q(U)$. If $zq'(z)\theta(q)$ is starlike in U, and*

$$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2. MAIN RESULTS

In this section, we verify some sufficient conditions of subordination for analytic functions in the classes $\mathcal{B}_\alpha^+$ and $\mathcal{B}_\alpha^-$.

Theorem 2.1. *Let the function $q(z)$ be convex in the unit disk U such that $q(0) = 1$ and $\Re\{q\} > \frac{1}{2}$. If $F \in \mathcal{B}_\alpha^+$ and $\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}$ an analytic function in U satisfies the subordination*

$$(1-\mu)\left[\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)}\right]^2 + (2\mu-1)\left[\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)}\right] - \mu + (1-\mu)\left[\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)}\right]\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \prec (1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z),$$

then

$$\frac{F(z) * \Phi(z)}{F(z) * \Psi(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{F(z) * \Phi(z)}{F(z) * \Psi(z)}, \quad z \in U.$$

It is clear that $p(0) = 1$. Then straightforward computation gives us

$$\begin{aligned}
&(1-\mu)p^2(z) + (2\mu-1)p(z) - \mu + (1-\mu)zp'(z) \\
&= (1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]^2 + (2\mu-1)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right] - \mu \\
&\qquad + (1-\mu)z\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]' \\
&= (1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]^2 + (2\mu-1)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right] - \mu \\
&\qquad + (1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \\
&\prec (1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z).
\end{aligned}$$

By the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.1. □

Corollary 2.2. *If* $F \in \mathcal{B}_\alpha^+$ *and* $\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}$ *is an analytic function in* U *satisfying the subordination*

$$\begin{aligned}
&(1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]^2 + (2\mu-1)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right] - \mu \\
&\qquad + (1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \\
&\qquad\qquad \prec (1-\mu)\left(\frac{1+Az}{1+Bz}\right)^2 + (2\mu-1)\left(\frac{1+Az}{1+Bz}\right) \\
&\qquad\qquad\qquad - \mu + (1-\mu)\left(\frac{1+Az}{1+Bz}\right)\frac{(A-B)z}{(1+Az)(1+Bz)},
\end{aligned}$$

then

$$\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)} \prec \left(\frac{1+Az}{1+Bz}\right), \quad -1 \le B < A \le 1$$

and $(\frac{1+Az}{1+Bz})$ *is the best dominant.*

Proof. Let the function $q(z)$ be defined by

$$q(z) := \left(\frac{1+Az}{1+Bz}\right), \quad z \in U.$$

It is clear that $q(0) = 1$ and $\Re\{q\} > \frac{1}{2}$ for arbitrary $A, B,\ z \in U$, then in view of Theorem 2.1 we obtain the result. □

Corollary 2.3. *If* $F \in \mathcal{B}_\alpha^+$ *and* $\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}$ *is an analytic function in* U *satisfying the subordination*

$$(1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]^2+(2\mu-1)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]-\mu$$
$$+(1-\mu)\left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right]\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)}-\frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right]$$
$$\prec(1-\mu)\left(\frac{1+z}{1-z}\right)^2+(2\mu-1)\left(\frac{1+z}{1-z}\right)-\mu$$
$$+(1-\mu)\left(\frac{1+z}{1-z}\right)\left(\frac{2z}{1-z^2}\right),$$

then

$$\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\prec\left(\frac{1+z}{1-z}\right),$$

and $\left(\frac{1+z}{1-z}\right)$ *is the best dominant.*

Define the function $\varphi_\alpha(a,c;z)$ by

$$\varphi_\alpha(a,c;z):=1+\sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}z^{n+\alpha-1},\quad (z\in U;\ a\in\mathbb{R},\ c\in\mathbb{R}\backslash\{0,-1,-2,\dots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n:=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & (n=0);\\ a(a+1)(a+2)\cdots(a+n-1), & (n\in\mathbb{N}).\end{cases}$$

Corresponding to the function $\varphi_\alpha(a,c;z)$, define a linear operator $L_\alpha(a,c)$ by

$$L_\alpha(a,c)F(z):=\varphi_\alpha(a,c;z)*F(z),\quad F(z)\in\mathcal{B}_\alpha^+$$

or equivalently by

$$L_\alpha(a,c)F(z):=1+\sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}a_nz^{n+\alpha-1}.$$

For details see [4]. Hence we have the following result:

Corollary 2.4. *Let the function* $q(z)$ *be convex in the unit disk* U *such that* $q(0)=1$ *and* $\Re\{q\}>\frac{1}{2}$. *If* $\frac{L_\alpha(a,c)\Phi(z)}{L_\alpha(a,c)\Psi(z)}$ *is an analytic function in* U *satisfying the subordination*

$$(1-\mu)\left[\frac{L_\alpha(a,c)\Phi(z)}{L_\alpha(a,c)\Psi(z)}\right]^2+(2\mu-1)\left[\frac{L_\alpha(a,c)\Phi(z)}{L_\alpha(a,c)\Psi(z)}\right]-\mu$$
$$+(1-\mu)\left[\frac{L_\alpha(a,c)\Phi(z)}{L_\alpha(a,c)\Psi(z)}\right]\left[\frac{z(L_\alpha(a,c)\Phi(z))'}{L_\alpha(a,c)\Phi(z)}-\frac{z(L_\alpha(a,c)\Psi(z))'}{L_\alpha(a,c)\Psi(z)}\right]$$
$$\prec(1-\mu)q^2(z)+(2\mu-1)q(z)-\mu+(1-\mu)zq'(z),$$

then

$$\frac{L_\alpha(a,c)\Phi(z)}{L_\alpha(a,c)\Psi(z)}\prec q(z)$$

and $q(z)$ *is the best dominant.*

Theorem 2.5. *Let the function $q(z)$ be univalent in the unit disk U such that $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. If $F \in \mathcal{B}_\alpha^-$ satisfies the subordination*

$$a\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \prec a\frac{zq'(z)}{q(z)},$$

then

$$\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)} \prec q(z), \quad z \in U,$$

and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \left[\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)}\right], \qquad z \in U.$$

By setting

$$\theta(\omega) := \frac{a}{\omega}, \quad a \neq 0,$$

it can easily observed that $\theta(\omega)$ is analytic in $\mathbb{C} - \{0\}$. Then we obtain

$$\begin{aligned} a\frac{zp'(z)}{p(z)} &= a\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \\ &\prec a\frac{zq'(z)}{q(z)}. \end{aligned}$$

By the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.2. □

Corollary 2.6. *If $F \in \mathcal{B}_\alpha^-$ satisfies the subordination*

$$a\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \prec a\frac{(A-B)z}{(1+Az)(1+Bz)}$$

then

$$\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)} \prec \left(\frac{1+Az}{1+Bz}\right), \quad -1 \leq B < A \leq 1$$

and $(\frac{1+Az}{1+Bz})$ is the best dominant.

Corollary 2.7. *If $F \in \mathcal{B}_\alpha^-$ satisfies the subordination*

$$a\left[\frac{z(F(z)*\Phi(z))'}{F(z)*\Phi(z)} - \frac{z(F(z)*\Psi(z))'}{F(z)*\Psi(z)}\right] \prec a\left(\frac{2z}{1-z^2}\right),$$

then

$$\frac{F(z)*\Phi(z)}{F(z)*\Psi(z)} \prec \left(\frac{1+z}{1-z}\right),$$

and $(\frac{1+z}{1-z})$ is the best dominant.

Define the function $\phi_\alpha(a,c;z)$ by

$$\phi_\alpha(a,c;z) := 1 - \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+\alpha-1}, \quad (z \in U;\ a \in \mathbb{R},\ c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}),$$

where $(a)_n$ is the Pochhammer symbol. Corresponding to the function $\phi_\alpha(a,c;z)$, define a linear operator $\mathcal{L}_\alpha(a,c)$ by

$$\mathcal{L}_\alpha(a,c)F(z) := \phi_\alpha(a,c;z) * F(z), \quad F(z) \in \mathcal{B}_\alpha^-$$

or equivalently by

$$\mathcal{L}_\alpha(a,c)F(z) := 1 - \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n+\alpha-1}.$$

Hence we obtain the next result.

Corollary 2.8. *Let the function $q(z)$ be univalent in the unit disk U such that $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U. If $F \in \mathcal{B}_\alpha^-$ satisfies the subordination*

$$a\left[\frac{z(\mathcal{L}_\alpha(a,c)\Phi(z))'}{\mathcal{L}_\alpha(a,c)\Phi(z)} - \frac{z(\mathcal{L}_\alpha(a,c)\Psi(z))'}{\mathcal{L}_\alpha(a,c)\Psi(z)}\right] \prec a\frac{zq'(z)}{q(z)},$$

then

$$\frac{\mathcal{L}_\alpha(a,c)\Phi(z)}{\mathcal{L}_\alpha(a,c)\Psi(z)} \prec q(z), \quad z \in U,$$

and $q(z)$ is the best dominant.

3. APPLICATIONS

In this section, we introduce some applications of Section 2 containing fractional integral operators. Assume that $f(z) = \sum_{n=1}^{\infty} \varphi_n z^n$ and let us begin with the following definitions

Definition 3.1 ([5])**.** The fractional integral of order α for the function $f(z)$ is defined by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad 0 \leq \alpha < 1,$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane $(\mathbb{C})$ containing the origin. The multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when$(z-\zeta) > 0$. Note that, $I_z^\alpha f(z) = \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right] f(z)$, for $z > 0$ and 0 for $z \leq 0$ (see [6]).

From Definition 3.1, we have

$$I_z^\alpha f(z) = \left[\frac{z^{\alpha-1}}{\Gamma(\alpha)}\right] f(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \varphi_n z^n = \sum_{n=1}^{\infty} a_n z^{n+\alpha-1}$$

where $a_n := \frac{\varphi_n}{\Gamma(\alpha)}$ for all $n = 1, 2, 3, \ldots$, thus $1 + I_z^\alpha f(z) \in \mathcal{B}_\alpha^+$ and $1 - I_z^\alpha f(z) \in \mathcal{B}_\alpha^-$ $(\varphi_n \geq 0)$. Then we have the following results.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold. Then*

$$\left[\frac{(1 + I_z^\alpha f(z)) * \Phi(z)}{(1 + I_z^\alpha f(z)) * \Psi(z)}\right] \prec q(z), \quad z \neq 0, \ z \in U$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := 1 + I_z^\alpha f(z), \quad z \in U.$$

□

Theorem 3.2. *Let the assumptions of Theorem 2.5 hold. Then*

$$\left[\frac{(1 - I_z^\alpha f(z)) * \Phi(z)}{(1 - I_z^\alpha f(z)) * \Psi(z)}\right] \prec q(z), \quad z \in U$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := 1 - I_z^{\alpha} f(z), \quad z \in U.$$

□

Let $F(a, b; c; z)$ be the Gauss hypergeometric function (see [7]) defined, for $z \in U$, by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n.$$

We need the following definitions of fractional operators in Saigo type fractional calculus (see [8, 9]).

Definition 3.2. For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\zeta}{z}\right) f(\zeta) d\zeta,$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin, with order

$$f(z) = O(|z|^{\epsilon})(z \to 0), \quad \epsilon > \max\{0, \beta - \eta\} - 1$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

From Definition 3.2, with $\beta < 0$, we have

$$\begin{aligned}
I_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_n (1)_n} \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} \left(1 - \frac{\zeta}{z}\right)^n f(\zeta) d\zeta \\
&= \sum_{n=0}^{\infty} B_n \frac{z^{-\alpha-\beta-n}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{n+\alpha-1} f(\zeta) d\zeta \\
&= \sum_{n=0}^{\infty} B_n \frac{z^{-\beta-1}}{\Gamma(\alpha)} f(\zeta) \\
&= \frac{\overline{B}}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \varphi_n z^{n-\beta-1},
\end{aligned}$$

where

$$B_n := \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_n (1)_n} \quad \text{and} \quad \overline{B} := \sum_{n=0}^{\infty} B_n.$$

Denote $a_n := \frac{\overline{B}\varphi_n}{\Gamma(\alpha)}$ for all $n = 1, 2, 3, \ldots$, and let $\alpha = -\beta$. Thus,

$$1 + I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{B}_{\alpha}^{+} \quad \text{and} \quad 1 - I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{B}_{\alpha}^{-} \ (\varphi_n \geq 0),$$

andn we have the following results

Theorem 3.3. *Let the assumptions of Theorem 2.1 hold. Then*

$$\left[\frac{(1 + I_{0,z}^{\alpha,\beta,\eta} f(z)) * \Phi(z)}{(1 + I_{0,z}^{\alpha,\beta,\eta} f(z)) * \Psi(z)}\right] \prec q(z), \quad z \in U$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := 1 + I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U.$$

□

Theorem 3.4. *Let the assumptions of Theorem 2.5 hold. Then*

$$\left[\frac{(1 - I_{0,z}^{\alpha,\beta,\eta} f(z)) * \Phi(z)}{(1 - I_{0,z}^{\alpha,\beta,\eta} f(z)) * \Psi(z)}\right] \prec q(z), \quad z \in U$$

and $q(z)$ is the best dominant.

Proof. Let the function $F(z)$ be defined by

$$F(z) := 1 - I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U.$$

□

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