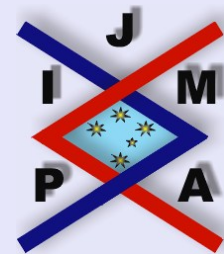


DIRECT APPROXIMATION THEOREMS FOR DISCRETE TYPE OPERATORS

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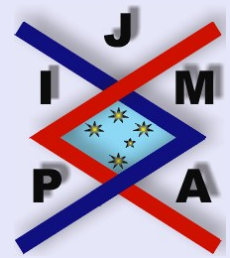


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Abstract

In the present paper we prove direct approximation theorems for discrete type operators

$$(L_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x) \lambda_{n,k}(f),$$

$f \in C[0, \infty)$, $x \in [0, \infty)$ using a modified K -functional. As applications we give direct theorems for Baskakov type operators, Szász-Mirakjan type operators and Lupaş operator.

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Key words: Direct approximation theorem, K -functional, Ditzian-Totik modulus of smoothness.

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1. Introduction

We introduce the following discrete type operators L_n , $n \in \{1, 2, 3, \dots\}$, defined by

$$(1.1) \quad (L_n f)(x) \equiv L_n(f, x) = \sum_{k=0}^{\infty} u_{n,k}(x) \lambda_{n,k}(f),$$

where $f \in C[0, \infty)$, $x \geq 0$, $u_{n,k} \in C[0, \infty)$ with $u_{n,k} \geq 0$ on $[0, \infty)$ and $\lambda_{n,k} : C[0, \infty) \rightarrow \mathbb{R}$ are linear positive functionals, $k \in \{0, 1, 2, \dots\}$.

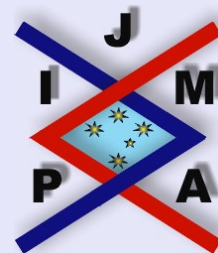
The purpose of this paper is to establish sufficient conditions with the aim of obtaining direct local and global approximation theorems for (1.1). In [3] Ditzian gave the following interesting estimate:

$$(1.2) \quad |B_n(f, x) - f(x)| \leq C \omega_{\varphi, \lambda}^2 \left(f, \frac{1}{\sqrt{n}} \varphi^{1-\lambda}(x) \right),$$

where

$$B_n(f, x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0, 1], \quad x \in [0, 1]$$

is the Bernstein-polynomial, $C > 0$ is an absolute constant and $\varphi(x) = \sqrt{x(1-x)}$. This estimate unifies the classical estimate for $\lambda = 0$ and the norm estimate for $\lambda = 1$. Guo et al. in [7] proved a similar estimate to (1.2) for the Baskakov operator. For the more general operator (1.1) we shall give a result similar to the estimate (1.2) and to the result established in [7].



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To formulate the main results we need some notations: let $C_B[0, \infty)$ be the space of all bounded continuous functions on $[0, \infty)$ with the norm $\|f\| = \sup_{x \geq 0} |f(x)|$. Furthermore, let

$$\omega_{\varphi}^{\lambda}(f, t) = \sup_{0 < h \leq t} \sup_{x \pm h\varphi^{\lambda}(x) \in [0, \infty)} |f(x + h\varphi^{\lambda}(x)) - 2f(x) + f(x - h\varphi^{\lambda}(x))|$$

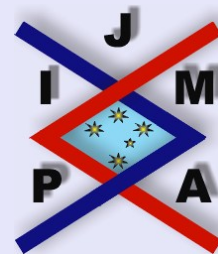
be the second order modulus of smoothness of Ditzian-Totik and let

$$\overline{K}_{\varphi^{\lambda}}(f, t) = \inf \{ \|f - g\| + t \|\varphi^{2\lambda} g''\| + t^{2/(2-\lambda)} \|g''\| : g'', \varphi^{2\lambda} g'' \in C_B[0, \infty) \}$$

be the corresponding modified weighted K -functional, where $\lambda \in [0, 1]$ and $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is an admissible weight function (cf. [4, Section 1.2]) such that $\varphi^2(x) \sim x^{\lambda}$ as $x \rightarrow 0+$ and $\varphi^2(x) \sim x^{\lambda}$ as $x \rightarrow \infty$, respectively. Then, in view of [4, p.24, Theorem 3.1.2] we have

$$(1.3) \quad \overline{K}_{\varphi^{\lambda}}(f, t^2) \sim \omega_{\varphi^{\lambda}}^2(f, t)$$

($x \sim y$ means that there exists an absolute constant $C > 0$ such that $C^{-1}y \leq x \leq Cy$). Throughout this paper C_1, C_2, \dots, C_6 denote positive constants and $C > 0$ is an absolute constant which can be different at each occurrence.



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2. Main Results

Our first theorem is the following:

Theorem 2.1. Let $(L_n)_{n \geq 1}$ be defined as in (1.1) satisfying

- (i) $L_n(1, x) = 1, \quad x \geq 0;$
- (ii) $L_n(t, x) = x, \quad x \geq 0;$
- (iii) $L_n(t^2, x) \leq x^2 + C_1 n^{-1} \varphi^2(x), \quad x \geq 0;$
- (iv) $\|L_n f\| \leq C_2 \|f\|, \quad f \in C_B[0, \infty);$
- (v) $L_n \left(\left| \int_x^t |t-u| \frac{du}{\varphi^{2\lambda}(u)} \right|, x \right) \leq C_3 n^{-1} \varphi^{2(1-\lambda)}(x), \quad x \in [1/n, \infty) \quad \text{and}$
- (vi) $n^{-1} \varphi^2(x) \leq C_4 \left(n^{-1} \varphi^{2(1-\lambda)}(x) \right)^{2/(2-\lambda)}, \quad x \in [0, 1/n).$

Then for every $f \in C_B[0, \infty)$, $n \in \{1, 2, 3, \dots\}$ and $x \geq 0$ one has

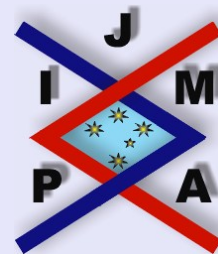
$$|(L_n f)(x) - f(x)| \leq \max\{1 + C_2, C_3, C_1 C_4\} \cdot \bar{K}_{\varphi^\lambda}(f, n^{-1} \varphi^{2(1-\lambda)}(x)).$$

Proof. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \geq 0$$

and the assumptions (i), (ii), (iii), (v) and (vi) we obtain

$$(2.1) \quad |(L_n g)(x) - g(x)| \leq \left| L_n \left(\int_x^t (t-u)g''(u)du, x \right) \right|$$



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$$\begin{aligned}
&\leq L_n \left(\left| \int_x^t |t-u| \cdot |g''(u)| du \right|, x \right) \\
&\leq L_n \left(\left| \int_x^t |t-u| \cdot \frac{du}{\varphi^{2\lambda}(u)} \right|, x \right) \cdot \|\varphi^{2\lambda} g''\| \\
&\leq \frac{C_3}{n} \cdot \varphi^{2(1-\lambda)}(x) \cdot \|\varphi^{2\lambda} g''\|,
\end{aligned}$$

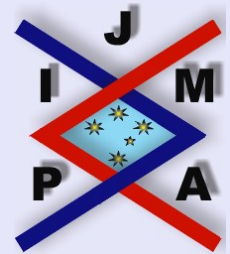
where $x \in [1/n, \infty)$, and

$$\begin{aligned}
(2.2) \quad |(L_n g)(x) - g(x)| &\leq L_n \left(\left| \int_x^t |t-u| \cdot |g''(u)| du \right|, x \right) \\
&\leq L_n \left((t-x)^2, x \right) \cdot \|g''\| \\
&\leq C_1 \frac{\varphi^2(x)}{n} \cdot \|g''\| \\
&\leq C_1 C_4 \left(\frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \right)^{2/(2-\lambda)} \cdot \|g''\|,
\end{aligned}$$

where $x \in [0, 1/n)$.

In conclusion, by (2.1) and (2.2),

$$\begin{aligned}
(2.3) \quad |(L_n g)(x) - g(x)| &\leq \max\{C_3, C_1 C_4\} \cdot \left\{ \frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \cdot \|\varphi^{2\lambda} g''\| \right. \\
&\quad \left. + \left(\frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \right)^{2/(2-\lambda)} \cdot \|g''\| \right\}
\end{aligned}$$



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for $x \geq 0$. Using (iv) and (2.3) we get

$$\begin{aligned}
 & |(L_n f)(x) - f(x)| \\
 & \leq |L_n(f - g, x) - (f - g)(x)| + |(L_n g)(x) - g(x)| \\
 & \leq (C_2 + 1)\|f - g\| + \max\{C_3, C_1 C_4\} \\
 & \quad \cdot \left\{ \frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \cdot \|\varphi^{2\lambda} g''\| + \left(\frac{1}{n} \cdot \varphi^{2(1-\lambda)}(x) \right)^{2/(2-\lambda)} \cdot \|g''\| \right\} \\
 & \leq \max\{1 + C_2, C_3, C_1 C_4\} \cdot \{ \|f - g\| \\
 & \quad + (n^{-1/2} \cdot \varphi^{1-\lambda}(x))^2 \cdot \|\varphi^{2\lambda} g''\| + (n^{-1/2} \cdot \varphi^{1-\lambda}(x))^{4/(2-\lambda)} \cdot \|g''\| \}.
 \end{aligned}$$

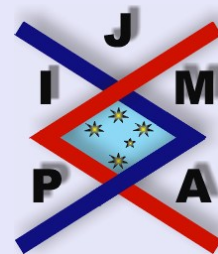
Now taking the infimum on the right-hand side over g and using the definition of $\overline{K}_{\varphi^\lambda}(f, n^{-1}\varphi^{2(1-\lambda)}(x))$ we get the assertion of the theorem. \square

Corollary 2.2. *Under the assumptions of Theorem 2.1 and for arbitrary $f \in C_B[0, \infty)$, $n \in \{1, 2, 3, \dots\}$ and $x \geq 0$ we have the estimate*

$$|(L_n f)(x) - f(x)| \leq C\omega_{\varphi^\lambda}^2(f, n^{-1/2}\varphi^{1-\lambda}(x)).$$

Proof. It is an immediate consequence of Theorem 2.1 and (1.3). \square

Remark 1. *In Corollary 2.2 the case $\lambda = 0$ gives the local estimate and for $\lambda = 1$ we obtain a global estimate.*



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3. Applications

The applications are in connection with Baskakov type operators, Szász - Mirakjan type operators and the Lupaş operator. To be more precise, we shall study the following operators:

$$(L_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) \lambda_{n,k}(f), \quad v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)};$$

$$(L_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \lambda_{n,k}(f), \quad s_{n,k}(x) = e^{-nx} \cdot \frac{(nx)^k}{k!},$$

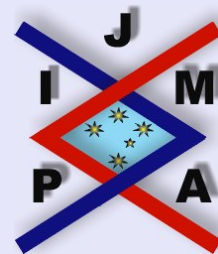
and their generalizations:

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x) \lambda_{n,k}(f),$$

$$v_{n,k}^{(\alpha)}(x) = \binom{n+k-1}{k} \frac{\prod_{i=0}^{k-1} (x+i\alpha) \prod_{j=1}^n (1+j\alpha)}{\prod_{r=1}^{n+k} (x+1+r\alpha)}, \quad \alpha \geq 0;$$

$$(L_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \lambda_{n,k}(f),$$

$$s_{n,k}^{(\alpha)}(x) = (1+n\alpha)^{-x/\alpha} \frac{nx(nx+n\alpha) \cdots (nx+n(k-1)\alpha)}{k!(1+n\alpha)^k}, \quad \alpha \geq 0$$



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(the parameter α may depend only on the natural number n), and the Lupas operator [8] defined by

$$(\tilde{L}_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{nx(nx+1) \cdots (nx+k-1)}{2^k k!} f\left(\frac{k}{n}\right).$$

For different values of $\lambda_{n,k}$ we obtain the following explicit forms of the above operators:

1) *the Baskakov operator* [2]

$$(V_n f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right);$$

2) *the generalized Baskakov operator* [5]

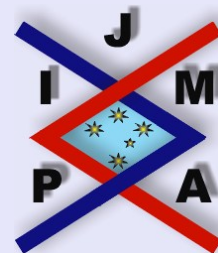
$$(V_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right);$$

3) *the modified Agrawal and Thamer operator* [1]

$$(L_{1,n} f)(x) = v_{n,0}(x) f(0) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t) dt;$$

4) *the generalized Agrawal and Thamer type operator*

$$(L_{1,n}^{(\alpha)} f)(x) = v_{n,0}^{(\alpha)}(x) f(0) + \sum_{k=1}^{\infty} v_{n,k}^{(\alpha)}(x) \frac{1}{B(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t) dt;$$



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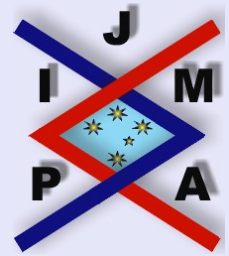


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5) Szász - Mirakjan operator [12]

$$(S_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right);$$

6) Mastroianni operator [9]

$$(S_n^{(\alpha)} f)(x) = \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right);$$

7) Phillips operator [10], [11]

$$(L_{2,n} f)(x) = s_{n,0}(x) f(0) + n \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt;$$

8) the generalized Phillips operator

$$(L_{2,n}^{(\alpha)} f)(x) = s_{n,0}^{(\alpha)}(x) f(0) + n \sum_{k=1}^{\infty} s_{n,k}^{(\alpha)}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt;$$

9) a new generalized Phillips type operator [6] defined as follows:

let $I = \{k_i : 0 = k_0 \leq k_1 \leq k_2 \leq \dots\} \subseteq \{0, 1, 2, \dots\}$. Then we can introduce the operators

$$(L_{3,n} f)(x) = \sum_{\substack{k=0 \\ k \in I}}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right) + \sum_{\substack{k=0 \\ k \notin I}}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt$$

and its generalization

$$(L_{3,n}^{(\alpha)} f)(x) = \sum_{\substack{k=0 \\ k \in I}}^{\infty} s_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) + \sum_{\substack{k=0 \\ k \notin I}}^{\infty} s_{n,k}^{(\alpha)}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt.$$

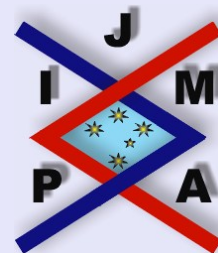
For the above enumerated operators we have the following theorem:

Theorem 3.1. *If $f \in C_B[0, \infty)$, $x \geq 0$, $\varphi(x) = \sqrt{x(1+x)}$, $\lambda \in [0, 1]$ then*

- a) $|(V_n f)(x) - f(x)| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 1;$
- b) $|(V_n^\alpha f)(x) - f(x)| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 1,$
 $\alpha = \alpha(n) \leq C_5/(4n), C_5 < 1;$
- c) $|(L_{1,n} f)(x) - f(x)| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 9;$
- d) $|(L_{1,n}^{(\alpha)} f)(x) - f(x)| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 9,$
 $\alpha = \alpha(n) \leq C_6/(4n), C_6 < 1.$

For $f \in C_B[0, \infty)$, $x \geq 0$, $\varphi(x) = \sqrt{x}$, $\lambda \in [0, 1]$ and $L_n \in \{S_n, L_{2,n}, L_{3,n}, \tilde{L}_n\}$ resp. $L_n^{(\alpha)} \in \{S_n^{(\alpha)}, L_{2,n}^{(\alpha)}, L_{3,n}^{(\alpha)}\}$ we have

- e) $|(L_n f)(x) - f(x)| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 1;$
- f) $\left| \left(L_n^{(\alpha)} f \right) (x) - f(x) \right| \leq C \omega_{\varphi\lambda}^2(f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 1,$
 $\alpha = \alpha(n) \leq 1/n;$



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$$g) \left| \left(\tilde{L}_n f \right) (x) - f(x) \right| \leq C \omega_{\varphi^\lambda}^2 (f, n^{-1/2} \varphi^{1-\lambda}(x)), \quad n \geq 1.$$

Proof. First of all let us observe that we have the integral representation

$$(3.1) \quad (L_n^{(\alpha)} f) (x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} (L_n f)(\theta) d\theta,$$

where $0 < \alpha < 1$ and $(L_n, L_n^{(\alpha)}) \in \left\{ (V_n, V_n^{(\alpha)}), (L_{1,n}, L_{1,n}^{(\alpha)}) \right\}$.

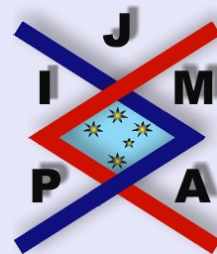
Analogously

$$(3.2) \quad (L_n^{(\alpha)} f) (x) = \frac{\left(\frac{1}{\alpha}\right)^{\frac{x}{\alpha}}}{\Gamma\left(\frac{x}{\alpha}\right)} \int_0^\infty e^{-\frac{\theta}{\alpha}} \theta^{\frac{x}{\alpha}-1} (L_n f)(\theta) d\theta,$$

where $\alpha > 0$, and $(L_n, L_n^{(\alpha)}) \in \left\{ (S_n, S_n^{(\alpha)}), (L_{2,n}, L_{2,n}^{(\alpha)}), (L_{3,n}, L_{3,n}^{(\alpha)}) \right\}$.

The relations (3.1) and (3.2) can be proved with the same idea. For example, if $L_n^{(\alpha)} = V_n^{(\alpha)}$ and $L_n = V_n$ then

$$\begin{aligned} & \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} V_n(f, \theta) d\theta \\ &= \sum_{k=0}^\infty \binom{n+k-1}{k} \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} \frac{\theta^k}{(1+\theta)^{n+k}} d\theta f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^\infty \binom{n+k-1}{k} \frac{B\left(\frac{x}{\alpha} + k, \frac{1}{\alpha} + n + 1\right)}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^\infty v_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) = V_n^{(\alpha)}(f, x). \end{aligned}$$



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The statements of our theorem follow from Corollary 2.2 if we verify the conditions (i) – (vi). It is easy to show that each operator preserves the linear functions and

$$(3.3) \quad V_n((t-x)^2, x) = \frac{1}{n}x(1+x),$$

$$V_n^{(\alpha)}((t-x)^2, x) = \frac{x(1+x)}{(1-\alpha)n} + \frac{\alpha x(1+x)}{1-\alpha} \leq \frac{5}{3n}x(1+x),$$

$$L_{1,n}((t-x)^2, x) = \frac{2x(1+x)}{n-1} \leq \frac{4}{n}x(1+x),$$

$$L_{1,n}^{(\alpha)}((t-x)^2, x) = \frac{2x(1+x)}{(1-\alpha)(n-1)} + \frac{\alpha x(1+x)}{1-\alpha} \leq \frac{17}{3n}x(1+x),$$

$$S_n((t-x)^2, x) = \frac{1}{n}x,$$

$$S_n^{(\alpha)}((t-x)^2, x) = \left(\alpha + \frac{1}{n}\right)x + \alpha x \leq \frac{3}{n}x,$$

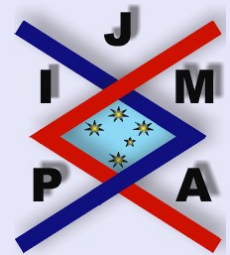
$$L_{2,n}((t-x)^2, x) = \frac{2}{n}x,$$

$$L_{2,n}^{(\alpha)}((t-x)^2, x) = \frac{2}{n}x + \alpha x \leq \frac{3}{n}x,$$

$$L_{3,n}((t-x)^2, x) \leq \frac{2}{n}x,$$

$$L_{3,n}^{(\alpha)}((t-x)^2, x) \leq \frac{2}{n}x + \alpha x \leq \frac{3}{n}x \quad (\text{see [6, p. 179]}),$$

$$\tilde{L}_n((t-x)^2, x) = \frac{2}{n}x,$$



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which imply (i), (ii) and (iii). The condition (iv) can be obtained from the integral representations (3.1) – (3.2) and the definition of \tilde{L}_n .

For (v) we have in view of [4, p.140, Lemma 9.6.1] that

$$(3.4) \quad \left| \int_x^t |t-u| \frac{du}{\varphi^{2\lambda}(u)} \right| = \left| \int_x^t |t-u| \frac{du}{u^\lambda(1+u)^\lambda} \right| \\ \leq \frac{(t-x)^2}{x^\lambda} \cdot \left(\frac{1}{(1+x)^\lambda} + \frac{1}{(1+t)^\lambda} \right)$$

or

$$(3.5) \quad \left| \int_x^t |t-u| \frac{du}{\varphi^{2\lambda}(u)} \right| = \left| \int_x^t |t-u| \frac{du}{u^\lambda} \right| \leq \frac{(t-x)^2}{x^\lambda}.$$

Because L_n is a linear positive operator, therefore either (3.4) and (3.3) or (3.5) and (3.3) imply

$$L_n \left(\left| \int_x^t |t-u| \frac{du}{\varphi^{2\lambda}(u)} \right|, x \right) \leq \frac{17}{3n} \cdot \frac{x(1+x)}{x^\lambda(1+x)^\lambda} + \frac{1}{x^\lambda} L_n \left((t-x)^2(1+t)^{-\lambda}, x \right),$$

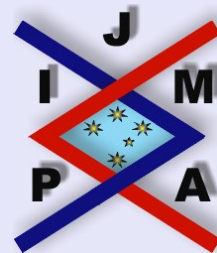
and

$$L_n \left(\left| \int_x^t |t-u| \frac{du}{\varphi^{2\lambda}(u)} \right|, x \right) \leq \frac{3}{n} \cdot \frac{x}{x^\lambda} = \frac{3}{n} x^{1-\lambda},$$

respectively. Thus we have to prove the estimation

$$(3.6) \quad L_n \left((t-x)^2(1+t)^{-\lambda}, x \right) \leq \frac{C}{n} \cdot \frac{x(1+x)}{(1+x)^\lambda}, \quad x \in [1/n, \infty)$$

for each Baskakov type operator defined in this section.



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1. By Hölder's inequality and [4, p.128, Lemma 9.4.3 and p.141, Lemma 9.6.2] we have

$$\begin{aligned} V_n((t-x)^2(1+t)^{-\lambda}, x) &\leq \{V_n((t-x)^4, x)\}^{\frac{1}{2}} \cdot \{V_n((1+t)^{-4}, x)\}^{\frac{\lambda}{4}} \\ &\leq C(n^{-2}x^2(1+x)^2)^{\frac{1}{2}} \cdot ((1+x)^{-4})^{\frac{\lambda}{4}} \\ &= \frac{C}{n} \cdot \frac{x(1+x)}{(1+x)^\lambda}, \end{aligned}$$

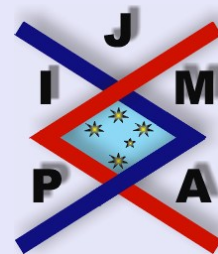
where $x \in [1/n, \infty)$;

2. Using

$$V_n((t-x)^4, x) = \frac{3}{n^2} \left(1 + \frac{2}{n}\right) \cdot x^2(1+x)^2 + \frac{1}{n^3} \cdot x(1+x),$$

(3.1) and [4, p.141, Lemma 9.6.2] we obtain

$$\begin{aligned} (3.7) \quad &V_n^{(\alpha)}((t-x)^4, x) \\ &= \frac{3}{n^2} \left(1 + \frac{2}{n}\right) \cdot \frac{x(x+\alpha)(x+1)(x+1-\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)} \\ &\quad + \frac{1}{n^3} \cdot \frac{x(x+1)}{1-\alpha} \\ &\leq \frac{3}{n^2} \left(1 + \frac{2}{n}\right) \cdot \frac{5}{4} \cdot \frac{1}{(1-C_5)^4} \cdot x^2(1+x)^2 \\ &\quad + \frac{1}{n^2} \cdot \frac{1}{(1-C_5)^2} \cdot x^2(1+x)^2 \\ &\leq C(n^{-1}x(1+x))^2, \end{aligned}$$



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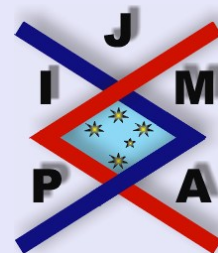


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where $x \in [1/n, \infty)$, and

$$\begin{aligned}
 (3.8) \quad & V_n^{(\alpha)}((1+t)^{-4}, x) \\
 & \leq \frac{C}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} \cdot \frac{d\theta}{(1+\theta)^4} \\
 & = C \frac{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}{(1+x+\alpha)(1+x+2\alpha)(1+x+3\alpha)(1+x+4\alpha)} \\
 & \leq C(1+x)^{-4},
 \end{aligned}$$

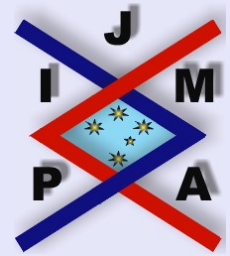
where $x \in [0, \infty)$, $\alpha = \alpha(n) \leq C_5/(4n)$, $n \geq 1$, $C_5 < 1$. Therefore the Hölder inequality, (3.7) and (3.8) imply (3.6) for $L_n = V_n^{(\alpha)}$;

3. We have

$$\begin{aligned}
 (3.9) \quad & L_{1,n}((t-x)^2(1+t)^{-\lambda}, x) \\
 & \leq \{L_{1,n}((t-x)^4, x)\}^{\frac{1}{2}} \cdot \{L_{1,n}((1+t)^{-4}, x)\}^{\frac{\lambda}{4}}.
 \end{aligned}$$

By direct computation we get

$$\begin{aligned}
 (3.10) \quad & L_{1,n}((t-x)^4, x) \\
 & = v_{n,0}(x)x^4 \\
 & \quad + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k+1}} (t-x)^4 dt \\
 & = v_{n,0}(x)x^4 + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k, n+1)} \{B(k+4, n-3) \\
 & \quad - 4xB(k+3, n-2) + 6x^2B(k+2, n-1)
 \end{aligned}$$



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$$\begin{aligned}
 & - 4x^3 B(k+1, n) + x^4 B(k, n+1) \} \\
 = & v_{n,0}(x)x^4 + \sum_{k=1}^{\infty} v_{n,k}(x) \left\{ \frac{k(k+1)(k+2)(k+3)}{n(n-1)(n-2)(n-3)} \right. \\
 & \left. - 4x \cdot \frac{k(k+1)(k+2)}{n(n-1)(n-2)} + 6x^2 \cdot \frac{k(k+1)}{n(n-1)} - 4x^3 \cdot \frac{k}{n} + x^4 \right\} \\
 = & \frac{(12n+84)x^4 + (24n+168)x^3 + (12n+108)x^2 + 11x}{(n-1)(n-2)(n-3)}.
 \end{aligned}$$

Hence, for $x \in [1/n, \infty)$ and $n \geq 9$ one has

$$\begin{aligned}
 (3.11) \quad & L_{1,n}((t-x)^4, x) \\
 & \leq \frac{(12n+84)x^4 + (24n+168)x^3 + (12n+108)x^2 + 11nx^2}{(n-1)(n-2)(n-3)} \\
 & \leq \frac{C}{n^2} x^2 (1+x)^2.
 \end{aligned}$$

Further,

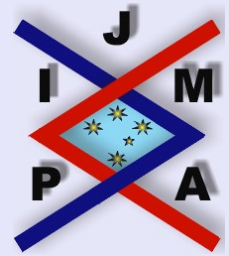
$$\begin{aligned}
 (3.12) \quad & L_{1,n}((1+t)^{-4}, x) \\
 & = v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{1}{B(k, n+1)} \\
 & \quad \times \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} \cdot \frac{dt}{(1+t)^4} \\
 & = v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{B(k, n+5)}{B(k, n+1)}
 \end{aligned}$$

$$\begin{aligned}
&= v_{n,0}(x) + \sum_{k=1}^{\infty} v_{n,k}(x) \frac{(n+1)(n+2)(n+3)(n+4)}{(n+k+1)(n+k+2)(n+k+3)(n+k+4)} \\
&= v_{n-4,0}(x) \cdot \frac{1}{(1+x)^4} + \sum_{k=1}^{\infty} \frac{v_{n-4,k}(x)}{(1+x)^4} \\
&\quad \cdot \frac{(n+k-4)(n+k-3)(n+k-2)(n+k-1)}{(n+k+1)(n+k+2)(n+k+3)(n+k+4)} \\
&\quad \cdot \frac{(n+1)(n+2)(n+3)(n+4)}{(n-4)(n-3)(n-2)(n-1)} \\
&\leq 16(1+x)^{-4},
\end{aligned}$$

where $n \geq 9$. Now (3.9), (3.11) and (3.12) imply (3.6) for $L_n = L_{1,n}$;

4. Using (3.1), Hölder's inequality, (3.10) and (3.12) we have

$$\begin{aligned}
&L_{1,n}^{(\alpha)}((t-x)^2(1+t)^{-\lambda}, x) \\
&= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha} + \frac{1}{\alpha} + 1}} L_{1,n}((t-x)^2(1+t)^{-\lambda}, \theta) d\theta \\
&\leq \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha} + \frac{1}{\alpha} + 1}} L_{1,n}((t-x)^4, \theta) d\theta \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^{\infty} \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha} + \frac{1}{\alpha} + 1}} L_{1,n}((1+t)^{-4}, \theta) d\theta \right)^{\frac{\lambda}{4}}
\end{aligned}$$



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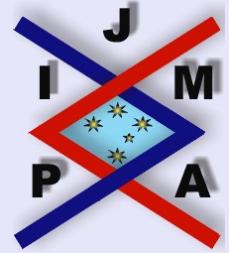


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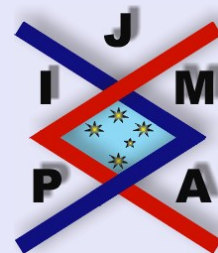
$$\begin{aligned}
 &\leq C \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} [L_{1,n}((t-\theta)^4, \theta) + (\theta-x)^4] d\theta \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\frac{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 5\right)}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \right)^{\frac{\lambda}{4}} \\
 &\leq C \left(\frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \cdot \frac{1}{n^2} \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} (\theta^4 + \theta^3 + \theta^2 + n^{-1}\theta) d\theta \right. \\
 &\quad \left. + \int_0^\infty \frac{\theta^{\frac{x}{\alpha}-1}}{(1+\theta)^{\frac{x}{\alpha}+\frac{1}{\alpha}+1}} (\theta-x)^4 d\theta \right)^{\frac{1}{2}} \\
 &\quad \cdot \left(\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}{(1+x+\alpha)(1+x+2\alpha)(1+x+3\alpha)(1+x+4\alpha)} \right)^{\frac{\lambda}{4}} \\
 &\leq C [n^{-2}(x^4 + x^3 + x^2) + n^{-2}(6\alpha^3 + 2\alpha^2 + \alpha + n^{-1})x + (18\alpha^3 + 3\alpha^2)x^4 \\
 &\quad + (36\alpha^3 + 6\alpha^2)x^3 + (24\alpha^3 + 3\alpha^2)x^2 + 6\alpha^3x]^{\frac{1}{2}} \cdot (1+x)^{-\lambda} \\
 &\leq \frac{C}{n} \cdot \frac{x(1+x)}{(1+x)^\lambda}
 \end{aligned}$$

for $x \in [1/n, \infty)$, $n \geq 9$, $\alpha = \alpha(n) \leq C_6/(4n)$, $C_6 < 1$.

Condition (vi) follows by direct computation if $\varphi^2(x) = x(1+cx)$, $c \in \{0, 1\}$ and $x \in [0, 1/n)$. Thus the theorem is proved. \square

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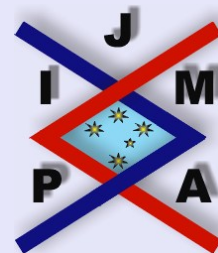
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