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## NOTE ON CERTAIN INEQUALITIES FOR MEANS IN TWO VARIABLES

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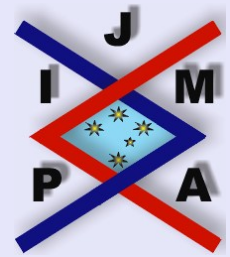


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## Abstract

Given the positive real numbers  $x$  and  $y$ , let  $A(x, y)$ ,  $G(x, y)$ , and  $I(x, y)$  denote their arithmetic mean, geometric mean, and identric mean, respectively. It is proved that for  $p \geq 2$ , the double inequality

$$\alpha A^p(x, y) + (1 - \alpha)G^p(x, y) < I^p(x, y) < \beta A^p(x, y) + (1 - \beta)G^p(x, y)$$

holds true for all positive real numbers  $x \neq y$  and only if  $\alpha \leq (\frac{2}{e})^p$  and  $\beta \geq \frac{2}{3}$ . This result complements a similar one established by H. Alzer and S.-L. Qiu [Inequalities for means in two variables, *Arch. Math. (Basel)* **80** (2003), 201–215].

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*Key words:* Arithmetic mean, Geometric mean, Identric mean.

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# 1. Introduction and Main Result

The means in two variables are special and they have found a number of applications (see, for instance, [1, 5] and the references therein). In this note we focus on certain inequalities involving the arithmetic mean, the geometric mean, and the identric mean of two positive real numbers  $x$  and  $y$ . Recall that these means are defined by  $A(x, y) = \frac{x+y}{2}$ ,  $G(x, y) = \sqrt{xy}$ , and

$$I(x, y) = \frac{1}{e} \left( \frac{x^x}{y^y} \right)^{\frac{1}{x-y}} \quad \text{if } x \neq y,$$
$$I(x, x) = x,$$

respectively. It is well-known that

$$(1.1) \quad G(x, y) < I(x, y) < A(x, y)$$

for all positive real numbers  $x \neq y$ . On the other hand, J. Sándor [6] proved that

$$(1.2) \quad \frac{2}{3}A(x, y) + \frac{1}{3}G(x, y) < I(x, y)$$

for all positive real numbers  $x \neq y$ . Note that inequality (1.2) is a refinement of the first inequality in (1.1). Also, (1.2) is sharp in the sense that  $\frac{2}{3}$  cannot be replaced by any greater constant. An interesting counterpart of (1.2) has been recently obtained by H. Alzer and S.-L. Qiu [1, Theorem 1]. Their result reads as follows:



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**Theorem 1.1.** *The double inequality*

$$\alpha A(x, y) + (1 - \alpha)G(x, y) < I(x, y) < \beta A(x, y) + (1 - \beta)G(x, y)$$

holds true for all positive real numbers  $x \neq y$ , if and only if  $\alpha \leq \frac{2}{3}$  and  $\beta \geq \frac{2}{e}$ .

Another counterpart of (1.2) has been obtained by J. Sándor and T. Trif [8, Theorem 2.5]. More precisely, they proved that

$$(1.3) \quad I^2(x, y) < \frac{2}{3}A^2(x, y) + \frac{1}{3}G^2(x, y)$$

for all positive real numbers  $x \neq y$ . We note that (1.3) is a refinement of the second inequality in (1.1). Moreover, (1.3) is the best possible inequality of the type

$$(1.4) \quad I^2(x, y) < \beta A^2(x, y) + (1 - \beta)G^2(x, y)$$

in the sense that (1.4) holds true for all positive real numbers  $x \neq y$  if and only if  $\beta \geq \frac{2}{3}$ .

It should be mentioned that (1.3) was derived in [8] as a consequence of certain power series expansions discovered by J. Sándor [7]. We present here an alternative proof of (1.3), based on the Gauss quadrature formula with two knots (see [2, pp. 343–344] or [3, p. 36])

$$\int_0^1 f(t)dt = \frac{1}{2}f\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right) + \frac{1}{2}f\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + \frac{1}{4320}f^{(4)}(\xi), \quad 0 < \xi < 1.$$



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Choosing  $f(t) = \log(tx + (1 - t)y)$  and taking into account that

$$\int_0^1 f(t)dt = \log I(x, y),$$

we get

$$\log I(x, y) = \frac{1}{2} \log \left( \frac{2}{3}A^2(x, y) + \frac{1}{3}G^2(x, y) \right) - \frac{(x - y)^4}{720(\xi x + (1 - \xi)y)^4}.$$

Consequently, it holds that

$$\begin{aligned} \exp \left( \frac{1}{360} \left( \frac{x - y}{\max(x, y)} \right)^4 \right) &< \frac{\frac{2}{3}A^2(x, y) + \frac{1}{3}G^2(x, y)}{I^2(x, y)} \\ &< \exp \left( \frac{1}{360} \left( \frac{x - y}{\min(x, y)} \right)^4 \right). \end{aligned}$$

This inequality yields (1.3) and estimates the sharpness of (1.3). However, we note that the double inequality (2.33) in [8] provides better bounds for the ratio

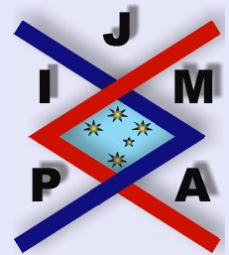
$$\left( \frac{2}{3}A^2(x, y) + \frac{1}{3}G^2(x, y) \right) / I^2(x, y).$$

The next theorem is the main result of this note and it is motivated by (1.3) and Theorem 1.1.

**Theorem 1.2.** *Given the real number  $p \geq 2$ , the double inequality*

$$(1.5) \quad \alpha A^p(x, y) + (1 - \alpha)G^p(x, y) < I^p(x, y) < \beta A^p(x, y) + (1 - \beta)G^p(x, y)$$

*holds true for all positive real numbers  $x \neq y$  if and only if  $\alpha \leq \left(\frac{2}{e}\right)^p$  and  $\beta \geq \frac{2}{3}$ .*



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## 2. Proof of Theorem 1.2

*Proof.* In order to prove that the first inequality in (1.5) holds true for  $\alpha = \left(\frac{2}{e}\right)^p$ , we will use the ingenious method of E.B. Leach and M.C. Sholander [4] (see also [1]). More precisely, we show that

$$(2.1) \quad \left(\frac{2}{e}\right)^p A^p(e^t, e^{-t}) + \left(1 - \left(\frac{2}{e}\right)^p\right) G^p(e^t, e^{-t}) < I^p(e^t, e^{-t}), \quad \text{for all } t > 0.$$

It is easily seen that (2.1) is equivalent to  $f_p(t) < 0$  for all  $t > 0$ , where  $f_p : (0, \infty) \rightarrow \mathbb{R}$  is the function defined by

$$f_p(t) = (2 \cosh t)^p + e^p - 2^p - \exp(pt \coth t).$$

We have

$$f'_p(t) = \frac{4p \sinh^3 t (2 \cosh t)^{p-1} - p(\sinh(2t) - 2t) \exp(pt \coth t)}{2 \sinh^2 t}.$$

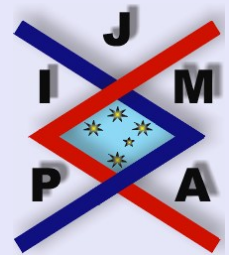
By means of the logarithmic mean of two variables,

$$L(x, y) = \frac{x - y}{\log x - \log y} \quad \text{if } x \neq y,$$

$$L(x, x) = x,$$

the derivative  $f'_p$  may be expressed as

$$(2.2) \quad f'_p(t) = p \frac{L(u(t), v(t))}{2 \sinh^2 t} g(t),$$



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where

$$u(t) = 4 \sinh^3 t (2 \cosh t)^{p-1},$$

$$v(t) = (\sinh(2t) - 2t) \exp(pt \coth t),$$

$$g(t) = \log u(t) - \log v(t)$$

$$= (p + 1) \log 2 + 3 \log(\sinh t) + (p - 1) \log(\cosh t) \\ - \log(\sinh(2t) - 2t) - pt \coth t.$$

We have

$$g'(t) = \frac{3 \cosh t}{\sinh t} + \frac{(p - 1) \sinh t}{\cosh t} - \frac{2 \cosh(2t) - 2}{\sinh(2t) - 2t} - \frac{p \cosh t}{\sinh t} + \frac{pt}{\sinh^2 t} \\ = \frac{3 \cosh^2 t - \sinh^2 t - p(\cosh^2 t - \sinh^2 t)}{\sinh t \cosh t} + \frac{pt}{\sinh^2 t} - \frac{2 \cosh(2t) - 2}{\sinh(2t) - 2t} \\ = \frac{\cosh(2t) + 2 - p}{\sinh t \cosh t} + \frac{pt}{\sinh^2 t} - \frac{2 \cosh(2t) - 2}{\sinh(2t) - 2t},$$

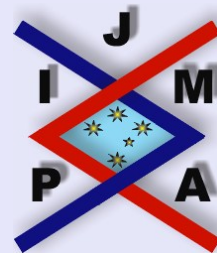
hence

$$(2.3) \quad g'(t) = g_1(t) + g_2(t),$$

where

$$g_1(t) = \frac{\cosh(2t)}{\sinh t \cosh t} + \frac{2t}{\sinh^2 t} - \frac{2 \cosh(2t) - 2}{\sinh(2t) - 2t},$$

$$g_2(t) = \frac{(p - 2)t}{\sinh^2 t} - \frac{p - 2}{\sinh t \cosh t}.$$



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But

$$\begin{aligned} g_2(t) &= \frac{p-2}{\sinh^2 t \cosh t} (t \cosh t - \sinh t) \\ &= \frac{p-2}{\sinh^2 t \cosh t} \sum_{k=1}^{\infty} \left( \frac{1}{(2k)!} - \frac{1}{(2k+1)!} \right) t^{2k+1}. \end{aligned}$$

Taking into account that  $p \geq 2$ , we deduce that

$$(2.4) \quad g_2(t) \geq 0 \quad \text{for all } t > 0.$$

Further, let  $h : (0, \infty) \rightarrow \mathbb{R}$  be the function defined by

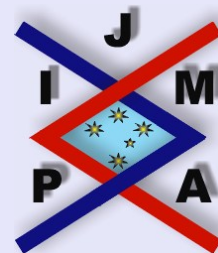
$$h(t) = \sinh^2 t \cosh t (\sinh(2t) - 2t) g_1(t).$$

Then we have

$$\begin{aligned} h(t) &= 2t \sinh t + \sinh t \sinh 2t - 4t^2 \\ &= 2t \sinh t + \frac{1}{2} \cosh(3t) - \frac{1}{2} \cosh t - 4t^2 \\ &= \sum_{k=2}^{\infty} \left( \frac{2}{(2k-1)!} + \frac{3^{2k}-1}{2(2k)!} \right) t^{2k}. \end{aligned}$$

Therefore  $h(t) > 0$  for  $t > 0$ , hence

$$(2.5) \quad g_1(t) > 0 \quad \text{for all } t > 0.$$



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By (2.3), (2.4), and (2.5) we conclude that  $g'(t) > 0$  for  $t > 0$ , hence  $g$  is increasing on  $(0, \infty)$ . Taking into account that

$$\begin{aligned} \lim_{t \rightarrow \infty} g(t) &= (p+1) \log 2 + \log \left( \lim_{t \rightarrow \infty} \frac{\sinh^3 t}{\cosh t (\sinh(2t) - 2t)} \right) \\ &\quad + p \lim_{t \rightarrow \infty} (\log(\cosh t) - t) + p \lim_{t \rightarrow \infty} t(1 - \coth t) \\ &= (p+1) \log 2 + \log \frac{1}{2} + p \log \frac{1}{2} \\ &= 0, \end{aligned}$$

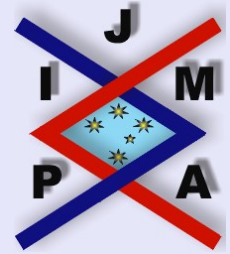
it follows that  $g(t) < 0$  for all  $t > 0$ . By virtue of (2.2), we deduce that  $f'_p(t) < 0$  for all  $t > 0$ , hence  $f_p$  is decreasing on  $(0, \infty)$ . Since  $\lim_{t \searrow 0} f_p(t) = 0$ , we conclude that  $f_p(t) < 0$  for all  $t > 0$ . This proves the validity of (2.1).

Now let  $x \neq y$  be two arbitrary positive real numbers. Letting  $t = \log \sqrt{\frac{x}{y}}$  in (2.1) and multiplying the obtained inequality by  $(\sqrt{xy})^p$ , we obtain

$$\left(\frac{2}{e}\right)^p A^p(x, y) + \left(1 - \left(\frac{2}{e}\right)^p\right) G^p(x, y) < I^p(x, y).$$

Consequently, the first inequality in (1.5) holds true for  $\alpha = \left(\frac{2}{e}\right)^p$ .

Let us prove now that the second inequality in (1.5) holds true for  $\beta = \frac{2}{3}$ . Indeed, taking into account (1.3) as well as the convexity of the function  $t \in$



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$(0, \infty) \mapsto t^{\frac{p}{2}} \in (0, \infty)$  (recall that  $p \geq 2$ ), we get

$$\begin{aligned} I^p(x, y) &= [I^2(x, y)]^{\frac{p}{2}} \\ &< \left[ \frac{2}{3}A^2(x, y) + \frac{1}{3}G^2(x, y) \right]^{\frac{p}{2}} \\ &\leq \frac{2}{3} [A^2(x, y)]^{\frac{p}{2}} + \frac{1}{3} [G^2(x, y)]^{\frac{p}{2}} \\ &= \frac{2}{3} A^p(x, y) + \frac{1}{3} G^p(x, y). \end{aligned}$$

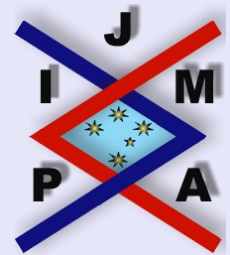
Conversely, suppose that (1.5) holds true for all positive real numbers  $x \neq y$ . Then we have

$$\alpha < \frac{I^p(x, y) - G^p(x, y)}{A^p(x, y) - G^p(x, y)} < \beta.$$

The limits

$$\lim_{x \rightarrow 0} \frac{I^p(x, 1) - G^p(x, 1)}{A^p(x, 1) - G^p(x, 1)} = \left(\frac{2}{e}\right)^p \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{I^p(x, 1) - G^p(x, 1)}{A^p(x, 1) - G^p(x, 1)} = \frac{2}{3}$$

yield  $\alpha \leq \left(\frac{2}{e}\right)^p$  and  $\beta \geq \frac{2}{3}$ . □



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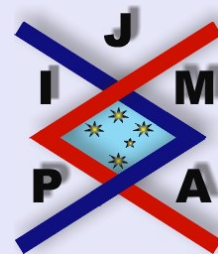
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