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GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING GAMMA FUNCTION AND APPLICATIONS TO INEQUALITY THEORY

XIAO-MING ZHANG, TIE-QUAN XU, AND LING-BO SITU

HAINING TV UNIVERSITY,
HAINING CITY, ZHEJIANG PROVINCE,
314400, CHINA
zjzxm79@sohu.com

QINGDAO VOCATIONAL AND TECHNICAL COLLEGE,
QINGDAO CITY, 266071, CHINA

CANGJIANG MIDDLE SCHOOL,
KAIPING CITY, GUANGDONG PROVINCE,
529300, CHINA

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ABSTRACT. In this paper, the geometric convexity of a function involving gamma function is studied, as applications to inequality theory, some important inequalities which improve some known inequalities, including Wallis' inequality, are obtained.

Key words and phrases: Gamma function, Geometrically Convex function, Wallis' inequality, Application, Inequality.

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1. INTRODUCTION AND MAIN RESULTS

The geometrically convex functions are as defined below.

Definition 1.1 ([10, 11, 12]). Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be a continuous function. Then f is called a geometrically convex function on I if there exists an integer $n \geq 2$ such that one of the following two inequalities holds:

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1)f(x_2)}\,, \tag{1.1}$$

$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leq \prod_{i=1}^{n} [f(x_i)]^{\lambda_i}, \tag{1.2}$$

where $x_1, x_2, \dots, x_n \in I$ and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. If inequalities (1.1) and (1.2) are reversed, then f is called a geometrically concave function on I.

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For more literature on geometrically convex functions and their properties, see [12, 29, 30, 31, 32] and the references therein.

It is well known that Euler's gamma function $\Gamma(x)$ and the psi function $\psi(x)$ are defined for $x > 0$ respectively by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}\,t$ and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. For $x > 0$, let

$$f(x) = \frac{e^x\Gamma(x)}{x^x}. \tag{1.3}$$

This function has been studied extensively by many mathematicians, for example, see [6] and the references therein.

In this article, we would like to discuss the geometric convexity of the function f defined by (1.3) and apply this property to obtain, from a new viewpoint, some new inequalities related to the gamma function.

Our main results are as follows.

Theorem 1.1. *The function f defined by* (1.3) *is geometrically convex.*

Theorem 1.2. *For $x > 0$ and $y > 0$, the double inequality*

$$\frac{x^x}{y^y}\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]} e^{y-x} \le \frac{\Gamma(x)}{\Gamma(y)} \le \frac{x^x}{y^y}\left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]} e^{y-x} \tag{1.4}$$

holds.

As consequences of above theorems, the following corollaries can be deduced.

Corollary 1.3. *The function f is logarithmically convex.*

Remark 1.4. More generally, the function f is logarithmically completely monotonic in $(0,\infty)$. See [6].

Corollary 1.5 ([7, 13])**.** *For $0 < y < x$ and $0 < s < 1$, inequalities*

$$e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)} \tag{1.5}$$

and

$$\frac{x^{x-1}}{y^{y-1}}e^{y-x} < \frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}e^{y-x} \tag{1.6}$$

are valid.

Remark 1.6. Note that inequality (1.4) is better than (1.5) and (1.6). The lower and upper bounds for $\frac{\Gamma(x)}{\Gamma(y)}$ have been established in many papers such as [14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26].

Corollary 1.7. *For $x > 0$ and $n \in \mathbb{N}$, the following double inequalities hold:*

$$\sqrt{ex}\left(1+\frac{1}{2x}\right)^{-x} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{ex}\left(1+\frac{1}{2x}\right)^{\frac{1}{12x}-x} \tag{1.7}$$

and

$$\begin{aligned}\sqrt{e(x+n)}\left(1+\frac{1}{2x+2n}\right)^{-x-n}\prod_{k=1}^{n}\left(1-\frac{1}{2x+2k}\right)\\ < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{e(x+n)}\left(1+\frac{1}{2x+2n}\right)^{\frac{1}{12x+12n}-x-n}\prod_{k=1}^{n}\left(1-\frac{1}{2x+2k}\right).\end{aligned} \tag{1.8}$$

Corollary 1.8. *For $n \in \mathbb{N}$, the double inequality*

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}} \tag{1.9}$$

is valid.

Remark 1.9. Inequality (1.9) is related to the well known Wallis inequality. If $n \geq 2$, inequality (1.9) is better than

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \leq \frac{(2n-1)!!}{(2n)!!} \leq \frac{1}{\sqrt{\pi(n+1/4)}} \tag{1.10}$$

in [3]. For more details, please refer to [2, 8, 33, 34, 35] and the references therein.

Corollary 1.10 ([28]). *Let $S_n = \sum_{k=1}^{n}\frac{1}{k}$ for $n \in \mathbb{N}$. Then*

$$\frac{2^{n+1}n!}{(2n+1)!!}\left(\frac{2n+3}{2n+2}\right)^{3/2+n} e^{(S_n-1-\gamma)/2} < \sqrt{\pi}\,. \tag{1.11}$$

2. LEMMAS

In order to prove our main results, the following lemmas are necessary.

Lemma 2.1 ([1, 5, 22]). *For $x > 0$,*

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \tag{2.1}$$

$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \qquad \psi'(x) > \frac{1}{x} + \frac{1}{2x^2}.$$

Lemma 2.2. *For $x > 0$,*

$$2\psi'(x) + x\psi''(x) < \frac{1}{x}. \tag{2.2}$$

Remark 2.3. The complete monotonicity of the function $2\psi'(x)+x\psi''(x)$ was obtained in [27].

Proof. It is a well known fact that

$$\psi'(x) = \sum_{k=1}^{\infty}\frac{1}{(k-1+x)^2} \quad \text{and} \quad \psi''(x) = -\sum_{k=1}^{\infty}\frac{2}{(k-1+x)^3}. \tag{2.3}$$

From this, it follows that

$$\begin{aligned}
2\psi'(x) + x\psi''(x) - \frac{1}{x} &= 2\sum_{k=1}^{\infty}\frac{k}{(k+x)^3} - \frac{1}{x} \\
&< 2\sum_{k=1}^{\infty}\frac{k}{(k-1+x)(k+x)(k+1+x)} - \frac{1}{x} \\
&= \sum_{k=1}^{\infty}\left[\frac{k}{(k-1+x)(k+x)} - \frac{k}{(k+x)(k+1+x)}\right] - \frac{1}{x} \\
&= \sum_{k=1}^{\infty}\frac{1}{(k-1+x)(k+x)} - \frac{1}{x} \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{k-1+x} - \frac{1}{k+x}\right) - \frac{1}{x} = 0.
\end{aligned}$$

Thus the proof of Lemma 2.2 is completed. □

Lemma 2.4 ([12]). *Let $(a,b)\subset(0,\infty)$ and $f:(a,b)\to(0,\infty)$ be a differentiable function. Then f is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.*

Lemma 2.5 ([12]). *Let $(a,b)\subset(0,\infty)$ and $f:(a,b)\to(0,\infty)$ be a differentiable function. Then f is a geometrically convex function if and only if $\frac{f(x)}{f(y)}\geq\left(\frac{x}{y}\right)^{yf'(y)/f(y)}$ holds for any $x,y\in(a,b)$.*

Lemma 2.6 ([4, 9]). *Let $S_n=\sum_{k=1}^{n}\frac{1}{k}$ and $C_n=S_n-\ln\left(n+\frac{1}{2}\right)-\gamma$ for $n\in\mathbb{N}$, where $\gamma=0.5772156\ldots$ is Euler-Mascheroni's constant. Then*

$$\frac{1}{24(n+1)^2}<C_n<\frac{1}{24n^2}. \tag{2.4}$$

3. PROOFS OF THEOREMS AND COROLLARIES

Now we are in a position to prove our main results.

Proof of Theorem 1.1. Easy calculation yields

$$\ln f(x)=\ln\Gamma(x)-x\ln x+x \quad\text{and}\quad \frac{f'(x)}{f(x)}=\psi(x)-\ln x. \tag{3.1}$$

Let $F(x)=\left[\frac{xf'(x)}{f(x)}\right]'$. Then

$$F(x)=\psi(x)+x\psi'(x)-\ln x-1, \quad\text{and}\quad F'(x)=2\psi'(x)+x\psi''(x)-\frac{1}{x}.$$

By virtue of Lemma 2.2, it follows that $F'(x)<0$, thus F is decreasing in $x>0$. By Lemma 2.1, we deduce that

$$F(x)=\psi(x)+x\psi'(x)-\ln x-1>\ln x-\frac{1}{x}+x\left(\frac{1}{x}+\frac{1}{2x^2}\right)-\ln x-1=-\frac{1}{2x}.$$

Hence $\lim_{x\to\infty}F(x)\geq 0$. This implies that $F(x)>0$ and, by Lemma 2.4, the function f is geometrically convex. The proof is completed. □

Proof of Theorem 1.2. Combining Theorem 1.1, Lemma 2.5 and (3.1) leads to

$$\frac{e^x\Gamma(x)}{x^x}\geq\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]}\frac{e^y\Gamma(y)}{y^y} \quad\text{and}\quad \frac{e^y\Gamma(y)}{y^y}\geq\left(\frac{y}{x}\right)^{x[\psi(x)-\ln x]}\frac{e^x\Gamma(x)}{x^x}.$$

Inequality (1.4) is established. □

Proof of Corollary 1.3. A combination of (3.1) with Lemma 2.1 reveals the decreasing monotonicity of f in $(0,\infty)$. Considering the geometric convexity and the decreasing monotonicity of f and the arithmetic-geometric mean inequality, we have

$$f\left(\frac{x_1+x_2}{2}\right)\leq f(\sqrt{x_1x_2})\leq\sqrt{f(x_1)f(x_2)}\leq\frac{f(x_1)+f(x_2)}{2}.$$

Hence, f is convex and logarithmic convex in $(0,\infty)$. □

Proof of Corollary 1.5. A property of mean values [9] and direct argument gives

$$\frac{1}{x}<\frac{\ln x-\ln y}{x-y}<\frac{1}{y}, \qquad \ln x-\ln y>1-\frac{y}{x}, \tag{3.2}$$

$$-1+\ln x+\frac{y}{x}>\psi(y)+y[\ln y-\psi(y)]\frac{1}{y}.$$

Hence,

$$-1+\ln x+y\frac{\ln x-\ln y}{x-y}>\psi(y)+y[\ln y-\psi(y)]\frac{\ln x-\ln y}{x-y}, \tag{3.3}$$
$$(y-x)+(x-y)\ln x+y(\ln x-\ln y)>(x-y)\psi(y)+y[\ln y-\psi(y)](\ln x-\ln y),$$
$$(y-x)+x\ln x-y\ln y+y[\psi(y)-\ln y](\ln x-\ln y)>(x-y)\psi(y),$$
$$\left(\frac{x}{y}\right)^{y[\psi(y)-\ln y]}\frac{e^y x^x}{e^x y^y}>e^{(x-y)\psi(y)}.$$

Similarly,

$$-1+\ln x+y\frac{1}{y}=x[\ln x-\psi(x)]\frac{1}{x}+\psi(x), \tag{3.4}$$
$$-1+\ln x+y\frac{\ln x-\ln y}{x-y}<x[\ln x-\psi(x)]\frac{\ln x-\ln y}{x-y}+\psi(x),$$
$$(y-x)+(x-y)\ln x+y(\ln x-\ln y)<x[\ln x-\psi(x)](\ln x-\ln y)+(x-y)\psi(x),$$
$$(y-x)+x\ln x-y\ln y+x[\psi(x)-\ln x](\ln x-\ln y)<(x-y)\psi(x),$$
$$\left(\frac{x}{y}\right)^{x[\psi(x)-\ln x]}\frac{e^y x^x}{e^x y^y}<e^{(x-y)\psi(x)}.$$

Combination of (3.3) and (3.4) leads to (1.5).

By (2.1), it is easy to see that

$$1<\left(\frac{x}{y}\right)^{y[\ln y-\psi(y)]}\frac{x}{y},\qquad \frac{x^{x-1}}{y^{y-1}}e^{y-x}<\left(\frac{x}{y}\right)^{y[\ln y-\psi(y)]}\frac{e^y x^x}{e^x y^y}.$$

Similarly,

$$\frac{e^y x^x}{e^x y^y}\left(\frac{x}{y}\right)^{x[\ln x-\psi(x)]}<\frac{x^{x-\frac{1}{2}}}{y^{y-\frac{1}{2}}}e^{y-x}.$$

By virtue of (1.4), inequality (1.6) follows. □

Proof of Corollary 1.7. Let $y=x+\frac{1}{2}$ in inequality (1.4). Then

$$\begin{aligned}\frac{e^{\frac{1}{2}}x^x}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(\frac{x}{x+\frac{1}{2}}\right)^{\left(x+\frac{1}{2}\right)\left[\psi\left(x+\frac{1}{2}\right)-\ln\left(x+\frac{1}{2}\right)\right]}&\le\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)}\\&\le\frac{e^{\frac{1}{2}}x^x}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(\frac{x}{x+\frac{1}{2}}\right)^{x[\psi(x)-\ln x]},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{e^{\frac{1}{2}}x^{x+1}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(\frac{x+\frac{1}{2}}{x}\right)^{\left(x+\frac{1}{2}\right)\left[\ln\left(x+\frac{1}{2}\right)-\psi\left(x+\frac{1}{2}\right)\right]}&\le\frac{x\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)}\\&\le\frac{e^{\frac{1}{2}}x^{x+1}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(\frac{x+\frac{1}{2}}{x}\right)^{x[\ln x-\psi(x)]}.\end{aligned}$$

From inequality (2.2), we obtain

$$\frac{\sqrt{ex}\,x^{x+\frac{1}{2}}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(1+\frac{1}{2x}\right)^{\frac{1}{2}}<\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}<\frac{\sqrt{ex}\,x^{x+\frac{1}{2}}}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}}\left(1+\frac{1}{2x}\right)^{\frac{1}{2}+\frac{1}{12x}},$$
$$\sqrt{ex}\left(1+\frac{1}{2x}\right)^{-x}<\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{ex}\left(1+\frac{1}{2x}\right)^{\frac{1}{12x}-x}.$$

The proof of inequality (1.7) is completed.

Substituting

$$\frac{\Gamma\left(x+n+1\right)}{\Gamma\left(x+n+\frac{1}{2}\right)}=\frac{\left(x+n\right)\Gamma\left(x+n\right)}{\left(x+n-\frac{1}{2}\right)\Gamma\left(x+n-\frac{1}{2}\right)}=\cdots=\frac{\Gamma\left(x+1\right)\prod_{k=1}^{n}\left(x+k\right)}{\Gamma\left(x+\frac{1}{2}\right)\prod_{k=1}^{n}\left(x+k-\frac{1}{2}\right)}$$

into (1.7) shows that inequality (1.8) is valid. □

Proof of Corollary 1.8. For $n=1,2$, inequality (1.9) can be verified readily.

For $n\geq 3$, in view of formulas $\Gamma(n+1)=n!$, $\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$ and inequality (1.7), we have

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}<\sqrt{en}\left(1+\frac{1}{2n}\right)^{\frac{1}{12n}-n},\qquad \frac{2^n n!}{(2n-1)!!}<\sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\frac{1}{12n}-n},$$

and

$$\frac{(2n-1)!!}{(2n)!!}>\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}}. \tag{3.6}$$

Further, taking $x=n$ in inequality (3.5) reveals

$$\frac{e^{\frac{1}{2}}n^{n+1}}{\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}}}\left(\frac{n+\frac{1}{2}}{n}\right)^{\left(n+\frac{1}{2}\right)\left(\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)\right)}\leq\frac{n\Gamma\left(n\right)}{\Gamma\left(n+\frac{1}{2}\right)},$$
$$\frac{2^n n!}{(2n-1)!!}\geq\sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)-1\right]},$$
$$\frac{2^n n!}{(2n-1)!!}\geq\sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n+\frac{1}{2}\right)-1\right]}.$$

Employing formulas

$$\psi(x+1)=\psi(x)+\frac{1}{x},\qquad \psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2,\qquad C_n=S_n-\ln\left(n+\frac{1}{2}\right)-\gamma \tag{3.7}$$

yields

$$\begin{aligned}
\frac{2^n n!}{(2n-1)!!} &\ge \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(n-\frac{1}{2}\right)-\frac{1}{n-\frac{1}{2}}-1\right]}\\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)-\frac{1}{n-\frac{1}{2}}-\cdots-\frac{1}{\frac{1}{2}}-1\right]}\\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)+2\ln 2+\gamma-2\sum_{k=1}^{n}\frac{1}{2k-1}-1\right]}\\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\ln\left(n+\frac{1}{2}\right)+2\ln 2+\gamma-2\sum_{k=1}^{2n}\frac{1}{k}+\sum_{k=1}^{n}\frac{1}{k}-1\right]}\\
&= \sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[2\ln(2n+1)-2C_{2n}-2\ln\left(2n+\frac{1}{2}\right)+C_n-1\right]}.
\end{aligned}\tag{3.8}$$

Letting $x=\frac{1}{1+4n}$ in $\ln(1+x)>\frac{x}{1+\frac{x}{2}}$ for $x>0$, we obtain

$$\ln\left(1+\frac{1}{1+4n}\right)>\frac{2}{8n+3}.\tag{3.9}$$

In view of Lemma 2.6 and inequalities (3.8) and (3.9), we have

$$\frac{2^n n!}{(2n-1)!!}>\sqrt{e\pi n}\left(1+\frac{1}{2n}\right)^{\left(n+\frac{1}{2}\right)\left[\frac{4}{8n+3}-\frac{1}{48n^2}+\frac{1}{24(n+1)^2}-1\right]}.\tag{3.10}$$

It is easy to verify that

$$\left(n+\frac{1}{2}\right)\left[\frac{4}{8n+3}-\frac{1}{48n^2}+\frac{1}{24(n+1)^2}-1\right]>-n+\frac{1}{12n+16}\tag{3.11}$$

with $n\ge 3$. By virtue of (3.6), (3.10) and (3.11), Corollary 1.8 is proved. □

Proof of Corollary 1.10. Letting $x=n+\frac{3}{2}$ and $y=n+1$ in inequality (1.4) yields

$$\frac{1}{\sqrt{e\pi(n+1)}}\left(1+\frac{1}{2n+2}\right)^{(n+1)[\psi(n+1)-\ln(n+1)+1]+\frac{1}{2}}\le\frac{(2n+1)!!}{(2n+2)!!}.\tag{3.12}$$

By using inequality (2.1), $\psi(n+1)=\sum_{k=1}^{n}\frac{1}{k}-\gamma$ and $\frac{1}{\sqrt{e}}\left(\frac{2n+3}{2n+2}\right)^{n+1}<1$ for $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\frac{(2n+1)!!}{(2n)!!}\left(\frac{2n+2}{2n+3}\right)^{\frac{3}{2}+n}e^{-\frac{1}{2}(S_n-1-\gamma)}\\
&=(2n+2)\frac{(2n+1)!!}{(2n+2)!!}\left(\frac{2n+2}{2n+3}\right)^{\frac{3}{2}+n}e^{-\frac{1}{2}[\psi(n+1)-1]}\\
&>\frac{2\sqrt{n+1}}{\sqrt{\pi}}\left(\frac{2n+3}{2n+2}\right)^{-(n+1)\ln(n+1)}\left[\frac{1}{\sqrt{e}}\left(\frac{2n+3}{2n+2}\right)^{n+1}\right]^{\ln(n+1)-\frac{1}{2(n+1)}}\\
&=\frac{2\sqrt{n+1}}{\sqrt{\pi}}\sqrt{\frac{2n+2}{2n+3}}\,e^{-\frac{1}{2}\ln(n+1)+\frac{1}{4(n+1)}}=\frac{2}{\sqrt{\pi}}\sqrt{\frac{2n+2}{2n+3}}\,e^{\frac{1}{4(n+1)}}>\frac{2}{\sqrt{\pi}}.
\end{aligned}$$

The proof of Corollary 1.10 is completed. □

REFERENCES

[1] G.D. ANDERSON AND S.L. QIU, A monotoneity property of the gamma function, *Proc. Amer. Math. Soc.*, **125**(11) (1997), 3355–3362.

[2] J. CAO, D.-W. NIU AND F. QI, A Wallis type inequality and a double inequality for probability integral, *Austral. J. Math. Anal. Appl.*, **4**(1) (2007), Art. 3. [ONLINE: `http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/V4I1P3.tex`].

[3] C.P. CHEN AND F. QI, The best bounds in Wallis' inequality, *Proc. Amer. Math., Soc.*, **133**(2) (2005), 397–401.

[4] D.W. DE TEMPLE, A quicker convergence to Euler's constant, *Amer. Math. Monthly*, **100**(5) (1993), 468–470.

[5] Á. ELBERT AND A. LAFORGIA, On some properties of the gamma function, *Proc. Amer. Math. Soc.*, **128**(9) (2000), 2667–2673.

[6] S. GUO, Monotonicity and concavity properties of some functions involving the gamma function with applications, *J. Inequal. Pure Appl. Math.*, **7**(2) (2006), Art. 45. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=662`].

[7] J. D. KEČLIĆ AND P. M. VASIĆ, Some inequalities for the gamma function, *Publ. Inst. Math. (Beograd) (N.S.)*, **11** (1971), 107–114.

[8] S. KOUMANDOS, Remarks on a paper by Chao-Ping Chen and Feng Qi, *Proc. Amer. Math. Soc.*, **134** (2006), 1365–1367.

[9] J.-Ch. KUANG, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè, Jinan City, Shandong Province, China, 2004. (Chinese)

[10] J. MATKOWSKI, L^p-like paranorms, *Selected topics in functional equations and iteration theory (Graz, 1991)*, 103–138, Grazer Math. Ber., 316, *Karl-Franzens-Univ. Graz, Graz*, 1992.

[11] P. MONTEL, Sur les functions convexes et les fonctions sousharmoniques, *J. de Math.*, **9**(7) (1928), 29–60.

[12] C.P. NICULESCU, Convexity according to the geometric mean, *Math. Inequal. Appl.*, **2**(2) (2000), 155–167.

[13] J. PEČARIĆ, G. ALLASIA AND C. GIORDANO, Convexity and the gamma function, *Indian J. Math.*, **41**(1) (1999), 79–93.

[14] F. QI, A class of logarithmically completely monotonic functions and application to the best bounds in the second Gautschi-Kershaw's inequality, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 11. [ONLINE: `http://rgmia.vu.edu.au/v9n4.html`].

[15] F. QI, A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality, *J. Comput. Appl. Math.*, (2007), in press. [ONLINE: `http://dx.doi.org/10.1016/j.cam.2006.09.005`]. *RGMIA Res. Rep. Coll.*, **9**(2) (2006), Art. 16. [ONLINE: `http://rgmia.vu.edu.au/v9n2.html`].

[16] F. QI, A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 5. [ONLINE: `http://rgmia.vu.edu.au/v9n4.html`].

[17] F. QI, A completely monotonic function involving divided differences of psi and polygamma functions and an application, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 8. [ONLINE: `http://rgmia.vu.edu.au/v9n4.html`].

[18] F. QI, A new lower bound in the second Kershaw's double inequality, *RGMIA Res. Rep. Coll.*, **10**(1) (2007), Art. 9. [ONLINE: `http://rgmia.vu.edu.au/v10n1.html`].

[19] F. QI, Monotonicity results and inequalities for the gamma and incomplete gamma functions, *Math. Inequal. Appl.*, **5** (1) (2002), 61–67. *RGMIA Res. Rep. Coll.*, **2**(7) (1999), Art. 7, 1027–1034. [ONLINE: `http://rgmia.vu.edu.au/v2n7.html`].

[20] F. QI, Refinements, extensions and generalizations of the second Kershaw's double inequality, *RGMIA Res. Rep. Coll.*, **10** (2) (2007), Art. 8. [ONLINE: `http://rgmia.vu.edu.au/v10n2.html`].

[21] F. QI, The best bounds in Kershaw's inequality and two completely monotonic functions, *RGMIA Res. Rep. Coll.*, **9**(4) (2006), Art. 2. [ONLINE: `http://rgmia.vu.edu.au/v9n4.html`].

[22] F. QI, R.-Q. CUI AND Ch.-P. CHEN, AND B.-N. GUO, Some completely monotonic functions involving polygamma functions and an application, *J. Math. Anal. Appl.*, **310**(1) (2005), 303–308.

[23] F. QI AND B.-N. GUO, A class of logarithmically completely monotonic functions and the best bounds in the second Kershaw's double inequality, *J. Comput. Appl. Math.*, (2007), in press. [ONLINE: `http://dx.doi.org/10.1016/j.cam.2006.12.022`].

[24] F. QI AND B.-N. GUO, Wendel-Gautschi-Kershaw's inequalities and sufficient and necessary conditions that a class of functions involving ratio of gamma functions are logarithmically completely monotonic, *RGMIA Res. Rep. Coll.*, **10**(1) (2007), Art 2. [ONLINE: `http://rgmia.vu.edu.au/v10n1.html`].

[25] F. QI, B.-N. GUO AND Ch.-P. CHEN, *The best bounds in Gautschi-Kershaw inequalities*, *Math. Inequal. Appl.*, **9** (3) (2006), 427–436. *RGMIA Res. Rep. Coll.*, **8**(2) (2005), Art. 17. . [ONLINE: `http://rgmia.vu.edu.au/v8n2.html`].

[26] F. QI AND S. GUO, New upper bounds in the second Kershaw's double inequality and its generalizations, *RGMIA Res. Rep. Coll.*, **10**(2) (2007), Art. 1. [ONLINE: `http://rgmia.vu.edu.au/v10n2.html`].

[27] F. QI, S. GUO AND B.-N. GUO, Note on a class of completely monotonic functions involving the polygamma functions, *RGMIA Res. Rep. Coll.*, **10**(1) (2006), Art. 5. [ONLINE: `http://rgmia.vu.edu.au/v10n1.html`].

[28] Z. STARC, Power product inequalities for the Gamma function, *Kragujevac J. Math.*, **24** (2002), 81–84.

[29] L. YANG, Some inequalities on geometric convex function, *Héběi Dàxué Xuébào (Zìrán Kēxué Bǎn) (J. Hebei Univ. (Nat. Sci. Ed.))*, **22**(4) (2002), 325–328. (Chinese)

[30] X.-M. ZHANG, Some theorem on geometric convex function and its applications, *Shǒudū Shīfàn Dàxué Xuébào (Zìrán Kēxué Bǎn) (J. Capital Norm. Univ. (Nat. Sci. Ed.))*, **25**(2) (2004), 11–13. (Chinese)

[31] X.-M. ZHANG AND Y.-D. WU, Geometrically convex functions and solution of a question, *RGMIA Res. Rep. Coll.*, **7**(4) (2004), Art. 11. [ONLINE: `http://rgmia.vu.edu.au/v7n4.html`].

[32] N.-G. ZHENG AND X.-M. ZHANG, An important property and application of geometrically concave functions, *Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory)*, **35**(8) (2005), 200–205. (Chinese)

[33] D.-J. ZHAO, On a two-sided inequality involving Wallis' formula, *Shùxué de Shíjiàn yǔ Rènshí (Mathematics in Practice and Theory)*, **34**(7) (2004), 166–168. (Chinese)

[34] Y.-Q. ZHAO AND Q.-B. WU, An improvement of the Wallis inequality, *Zhējiāng Dàxué Xuébào (Lǐxué Bǎn) (Journal of Zhejiang University (Science Edition))*, **33**(2) (2006), 372–375. (Chinese)

[35] Y.-Q. ZHAO AND Q.-B. WU, Wallis inequality with a parameter, *J. Inequal. Pure Appl. Math.*, **7**(2) (2006), Art. 56. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=673`].