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## NEW SUBCLASSES OF MEROMORPHIC $p$ -VALENT FUNCTIONS

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Abstract

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## Abstract

In this paper, we introduce two subclasses  $\Omega_p^*(\alpha)$  and  $\Lambda_p^*(\alpha)$  of meromorphic  $p$ -valent functions in the punctured disk  $\mathcal{D} = \{z : 0 < |z| < 1\}$ . Coefficient inequalities, distortion theorems, the radii of starlikeness and convexity, closure theorems and Hadamard product ( or convolution) of functions belonging to these classes are obtained.

*2000 Mathematics Subject Classification:* 30C45, 30C50.

*Key words:* Meromorphic  $p$ -valent functions, Meromorphically starlike and convex functions.

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# 1. Introduction and Definitions

Let  $\Sigma_p$  denote the class of functions of the form:

$$(1.1) \quad f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \quad (p \in \mathbb{N}),$$

which are analytic and  $p$ -valent in the punctured unit disk  $\mathcal{D} = \{z : 0 < |z| < 1\}$ . A function  $f \in \Sigma_p$  is said to be in the class  $\Omega_p(\alpha)$  of meromorphic  $p$ -valently starlike functions of order  $\alpha$  in  $\mathcal{D}$  if and only if

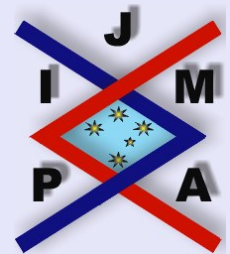
$$(1.2) \quad \operatorname{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{D}; 0 \leq \alpha < p; p \in \mathbb{N}).$$

Furthermore, a function  $f \in \Sigma_p$  is said to be in the class  $\Lambda_p(\alpha)$  of meromorphic  $p$ -valently convex functions of order  $\alpha$  in  $\mathcal{D}$  if and only if

$$(1.3) \quad \operatorname{Re} \left\{ -1 - \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathcal{D}; 0 \leq \alpha < p; p \in \mathbb{N}).$$

The classes  $\Omega_p(\alpha)$ ,  $\Lambda_p(\alpha)$  and various other subclasses of  $\Sigma_p$  have been studied rather extensively by Aouf *et al.* [1] – [3], Joshi and Srivastava [4], Kulkarni *et al.* [5], Mogra [6], Owa *et al.* [7], Srivastava and Owa [8], Uralegaddi and Somantha [9], and Yang [10].

In the next section we derive sufficient conditions for  $f(z)$  to be in the classes  $\Omega_p(\alpha)$  and  $\Lambda_p(\alpha)$ , which are obtained by using coefficient inequalities.



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## 2. Coefficient Inequalities

**Theorem 2.1.** Let  $\sigma_n(p, k, \alpha) = (p + n + k - 1) + |p + n + 2\alpha - k - 1|$ . If  $f(z) \in \Sigma_p$  satisfies

$$(2.1) \quad \sum_{n=1}^{\infty} \sigma_n(p, k, \alpha) |a_{p+n-1}| < 2(p - \alpha)$$

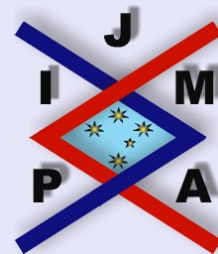
for some  $\alpha$  ( $0 \leq \alpha < p$ ) and some  $k$  ( $k \geq p$ ), then  $f(z) \in \Omega_p(\alpha)$ .

*Proof.* Suppose that (2.1) holds true for  $\alpha$  ( $0 \leq \alpha < p$ ) and  $k$  ( $k \geq p$ ). For  $f(z) \in \Sigma_p$ , it suffices to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} + k}{\frac{zf'(z)}{f(z)} + (2\alpha - k)} \right| < 1 \quad (z \in \mathcal{D}).$$

We note that

$$\begin{aligned} & \left| \frac{\frac{zf'(z)}{f(z)} + k}{\frac{zf'(z)}{f(z)} + (2\alpha - k)} \right| \\ &= \left| \frac{k - p + \sum_{n=1}^{\infty} (p + n + k - 1) a_{p+n-1} z^{2p+n-1}}{2\alpha - k - p + \sum_{n=1}^{\infty} (p + n + 2\alpha - k - 1) a_{p+n-1} z^{2p+n-1}} \right| \\ &\leq \frac{k - p + \sum_{n=1}^{\infty} (p + n + k - 1) |a_{p+n-1}| |z|^{2p+n-1}}{p + k - 2\alpha - \sum_{n=1}^{\infty} |p + n + 2\alpha - k - 1| |a_{p+n-1}| |z|^{2p+n-1}} \end{aligned}$$



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$$< \frac{k - p + \sum_{n=1}^{\infty} (p + n + k - 1) |a_{p+n-1}|}{p + k - 2\alpha - \sum_{n=1}^{\infty} |p + n + 2\alpha - k - 1| |a_{p+n-1}|}.$$

The last expression is bounded above by 1 if

$$k - p + \sum_{n=1}^{\infty} (p + n + k - 1) |a_{p+n-1}| < p + k - 2\alpha - \sum_{n=1}^{\infty} |p + n + 2\alpha - k - 1| |a_{p+n-1}|$$

which is equivalent to our condition (2.1) of the theorem.  $\square$

**Example 2.1.** The function  $f(z)$  given by

$$(2.2) \quad f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{4(p - \alpha)}{n(n + 1)\sigma_n(p, k, \alpha)} z^{p+n-1} \quad (p \in \mathbb{N})$$

belongs to the class  $\Omega_p(\alpha)$ .

Since  $f(z) \in \Omega_p(\alpha)$  if and only if  $zf'(z) \in \Lambda_p(\alpha)$ , we can prove:

**Theorem 2.2.** If  $f(z) \in \Sigma_p$  satisfies

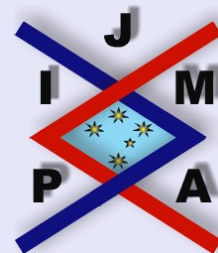
$$(2.3) \quad \sum_{n=1}^{\infty} (p + n - 1)\sigma_n(p, k, \alpha) |a_{p+n-1}| < 2(p - \alpha)$$

for some  $\alpha(0 \leq \alpha < p)$  and some  $k(k \geq p)$ , then  $f(z) \in \Lambda_p(\alpha)$ .

**Example 2.2.** The function  $f(z)$  given by

$$(2.4) \quad f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{4(p - \alpha)}{n(n + 1)(p + n - 1)\sigma_n(p, k, \alpha)} z^{p+n-1}$$

belongs to the class  $\Lambda_p(\alpha)$ .



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In view of Theorem 2.1 and Theorem 2.2, we now define the subclasses:

$$\Omega_p^*(\alpha) \subset \Omega_p(\alpha) \text{ and } \Lambda_p^*(\alpha) \subset \Lambda_p(\alpha),$$

which consist of functions  $f(z) \in \Sigma_p$  satisfying the conditions (2.1) and (2.3), respectively.

Letting  $p = 1$ ,  $1 \leq k \leq n + 2\alpha$ , where  $0 \leq \alpha < 1$  in Theorem 2.1 and Theorem 2.2, we have the following corollaries:

**Corollary 2.3.** *If  $f(z) \in \Sigma_1$  satisfies*

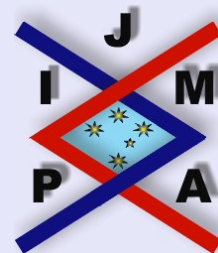
$$\sum_{n=1}^{\infty} (n + \alpha) |a_n| < 1 - \alpha$$

*then  $f(z) \in \Omega_1(\alpha) = \Sigma^*(\alpha)$  the class of meromorphically starlike functions of order  $\alpha$  in  $\mathcal{D}$ .*

**Corollary 2.4.** *If  $f(z) \in \Sigma_1$  satisfies*

$$\sum_{n=1}^{\infty} n(n + \alpha) |a_n| < 1 - \alpha$$

*then  $f(z) \in \Lambda_1(\alpha) = \Sigma_{\mathcal{K}}^*(\alpha)$  the class of meromorphically convex functions of order  $\alpha$  in  $\mathcal{D}$ .*



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### 3. Distortion Theorems

A distortion property for functions in the class  $\Omega_p^*(\alpha)$  is contained in

**Theorem 3.1.** *If the function  $f(z)$  defined by (1.1) is in the class  $\Omega_p^*(\alpha)$ , then for  $0 < |z| = r < 1$ , we have*

$$(3.1) \quad \frac{1}{r^p} - \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p \leq |f(z)| \\ \leq \frac{1}{r^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p,$$

and

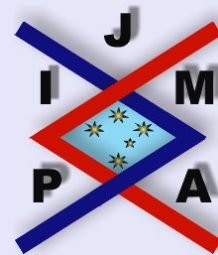
$$(3.2) \quad \frac{p}{r^{p+1}} - \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1} \\ \leq |f'(z)| \\ \leq \frac{p}{r^{p+1}} + \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}.$$

The bounds in (3.1) and (3.2) are attained for the functions  $f(z)$  given by

$$(3.3) \quad f(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} z^p \quad (p \in \mathbb{N}; z \in \mathcal{D}).$$

*Proof.* Since  $f \in \Omega_p^*(\alpha)$ , from the inequality (2.1), we have

$$(3.4) \quad \sum_{n=1}^{\infty} |a_{p+n-1}| \leq \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|}.$$



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Thus, for  $0 < |z| = r < 1$ , and making use of (3.4) we have

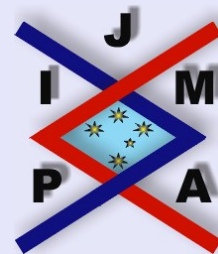
$$\begin{aligned}
 (3.5) \quad |f(z)| &\leq \left| \frac{1}{z^p} \right| + \sum_{n=1}^{\infty} |a_{p+n-1}| |z|^{p+n-1} \\
 &\leq \frac{1}{r^p} + r^p \sum_{n=1}^{\infty} |a_{p+n-1}| \\
 &\leq \frac{1}{r^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p
 \end{aligned}$$

and

$$\begin{aligned}
 (3.6) \quad |f(z)| &\geq \left| \frac{1}{z^p} \right| - \sum_{n=1}^{\infty} |a_{p+n-1}| |z|^{p+n-1} \\
 &\geq \frac{1}{r^p} - r^p \sum_{n=1}^{\infty} |a_{p+n-1}| \\
 &\geq \frac{1}{r^p} - \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p.
 \end{aligned}$$

We also observe that

$$(3.7) \quad \frac{p+k+|p+2\alpha-k|}{p} \sum_{n=1}^{\infty} (p+n-1) |a_{p+n-1}| \leq 2(p-\alpha)$$




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which readily yields the following distortion inequalities:

$$\begin{aligned}
 (3.8) \quad |f'(z)| &\leq \frac{p}{|z|^{p+1}} + \sum_{n=1}^{\infty} (p+n-1) |a_{p+n-1}| |z|^{p+n-2} \\
 &\leq \frac{p}{r^{p+1}} + r^{p-1} \sum_{n=1}^{\infty} (p+n-1) |a_{p+n-1}| \\
 &\leq \frac{p}{r^{p+1}} + \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}
 \end{aligned}$$

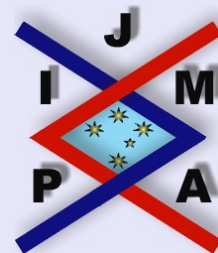
and

$$\begin{aligned}
 (3.9) \quad |f'(z)| &\geq \frac{p}{|z|^{p+1}} - \sum_{n=1}^{\infty} (p+n-1) |a_{p+n-1}| |z|^{p+n-2} \\
 &\geq \frac{p}{r^{p+1}} - r^{p-1} \sum_{n=1}^{\infty} (p+n-1) |a_{p+n-1}| \\
 &\geq \frac{p}{r^{p+1}} - \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}.
 \end{aligned}$$

This completes the proof of Theorem 3.1. □

Similarly, for function  $f(z) \in \Lambda_p^*(\alpha)$ , and making use of (2.3), we can prove

**Theorem 3.2.** *If the function  $f(z)$  defined by (1.1) is in the class  $\Lambda_p^*(\alpha)$ , then*




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for  $0 < |z| = r < 1$ , we have

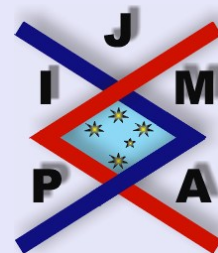
$$(3.10) \quad \frac{1}{r^p} - \frac{2(p-\alpha)}{p[p+k+|p+2\alpha-k|]} r^p \leq |f(z)| \\ \leq \frac{1}{r^p} + \frac{2(p-\alpha)}{p[p+k+|p+2\alpha-k|]} r^p,$$

and

$$(3.11) \quad \frac{p}{r^{p+1}} - \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1} \\ \leq |f'(z)| \\ \leq \frac{p}{r^{p+1}} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}.$$

The bounds in (3.10) and (3.11) are attained for the functions  $f(z)$  given by

$$(3.12) \quad g(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{p[p+k-1+|p+2\alpha-k|]} z^p \quad (p \in \mathbb{N}; z \in \mathcal{D}).$$



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## 4. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the classes  $\Omega_p^*(\alpha)$  is given by

**Theorem 4.1.** *If the function  $f(z)$  be defined by (1.1) is in the class  $\Omega_p^*(\alpha)$ , then  $f(z)$  is meromorphically  $p$ -valently starlike of order  $\delta$  ( $0 \leq \delta < p$ ) in  $|z| < r_1$ , where*

$$(4.1) \quad r_1 = \inf_{n \geq 1} \left\{ \frac{(p - \delta)\sigma_n(p, k, \alpha)}{2(3p + n + 1 - \delta)(p - \alpha)} \right\}^{\frac{1}{2p+n-1}} \quad (p \in \mathbb{N}).$$

Furthermore,  $f(z)$  is meromorphically  $p$ -valently convex of order  $\delta$  ( $0 \leq \delta < p$ ) in  $|z| < r_2$ , where

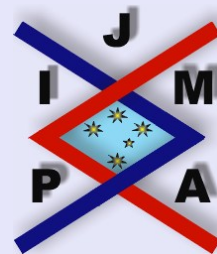
$$(4.2) \quad r_2 = \inf_{n \geq 1} \left\{ \frac{p(p - \delta)\sigma_n(p, k, \alpha)}{2[(p + n - 1)[3p + n - 1 - \delta](p - \alpha)} \right\}^{\frac{1}{2p+n-1}} \quad (p \in \mathbb{N}).$$

The results (4.1) and (4.2) are sharp for the function  $f(z)$  given by

$$(4.3) \quad f(z) = \frac{1}{z^p} + \frac{2(p - \alpha)}{\sigma_n(p, k, \alpha)} z^{p+n-1} \quad (p \in \mathbb{N}; z \in \mathcal{D}).$$

*Proof.* It suffices to prove that

$$(4.4) \quad \left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \delta,$$



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for  $|z| \leq r_1$ . We have

$$(4.5) \quad \left| \frac{zf'(z)}{f(z)} + p \right| = \left| \frac{\sum_{n=1}^{\infty} (2p+n-1)a_{p+n-1}z^{p+n-1}}{\frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1}z^{p+n-1}} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} (2p+n-1)|a_{p+n-1}||z|^{2p+n-1}}{1 - \sum_{n=1}^{\infty} |a_{p+n-1}||z|^{2p+n-1}}.$$

Hence (4.5) holds true if

$$(4.6) \quad \sum_{n=1}^{\infty} (2p+n-1)|a_{p+n-1}||z|^{2p+n-1}$$

$$\leq (p-\delta) \left( 1 - \sum_{n=1}^{\infty} |a_{p+n-1}||z|^{2p+n-1} \right),$$

or

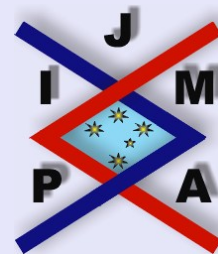
$$(4.7) \quad \sum_{n=1}^{\infty} \frac{3p+n-1-\delta}{(p-\delta)} |a_{p+n-1}||z|^{2p+n-1} \leq 1,$$

with the aid of (2.1), (4.7) is true if

$$(4.8) \quad \frac{3p+n-1-\delta}{(p-\delta)} |z|^{2p+n-1} \leq \frac{\sigma_n(p, k, \alpha)}{2(p-\alpha)} \quad (n \geq 1).$$

Solving (4.8) for  $|z|$ , we obtain

$$(4.9) \quad |z| < \left\{ \frac{(p-\delta)\sigma_n(p, k, \alpha)}{2(3p+n+1-\delta)(p-\alpha)} \right\}^{\frac{1}{2p+n-1}} \quad (n \geq 1).$$



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In precisely the same manner, we can find the radius of convexity asserted by (4.2), by requiring that

$$(4.10) \quad \left| \frac{zf''(z)}{f'(z)} + p + 1 \right| \leq p - \delta,$$

in view of (2.1). This completes the proof of Theorem 4.1.  $\square$

Similarly, we can get the radii of starlikeness and convexity for functions in the class  $\Lambda_p^*(\alpha)$ .

**Theorem 4.2.** *If the function  $f(z)$  be defined by (1.1) is in the class  $\Lambda_p^*(\alpha)$ , then  $f(z)$  is meromorphically  $p$ -valently starlike of order  $\delta$  ( $0 \leq \delta < p$ ) in  $|z| < r_3$ , where*

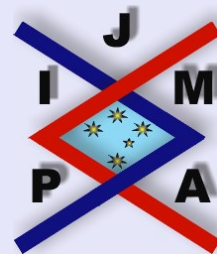
$$(4.11) \quad r_3 = \inf_{n \geq 1} \left\{ \frac{(p - \delta)(p + n - 1)\sigma_n(p, k, \alpha)}{2(3p + n + 1 - \delta)(p - \alpha)} \right\}^{\frac{1}{2p+n-1}} \quad (p \in \mathbb{N}).$$

Furthermore,  $f(z)$  is meromorphically  $p$ -valently convex of order  $\delta$  ( $0 \leq \delta < p$ ) in  $|z| < r_4$ , where

$$(4.12) \quad r_4 = \inf_{n \geq 1} \left\{ \frac{p(p - \delta)(p + n - 1)\sigma_n(p, k, \alpha)}{2[(p + n - 1)[3p + n - 1 - \delta](p - \alpha)} \right\}^{\frac{1}{2p+n-1}} \quad (p \in \mathbb{N}).$$

The results (4.11) and (4.12) are sharp for the function  $g(z)$  given by

$$(4.13) \quad g(z) = \frac{1}{z^p} + \frac{2(p - \alpha)}{(p + n - 1)\sigma_n(p, k, \alpha)} z^{p+n-1} \quad (p \in \mathbb{N}; z \in \mathcal{D}).$$



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## 5. Closure Theorems

Let the functions  $f_j(z)$  be defined, for  $j \in \{1, 2, \dots, m\}$ , by

$$(5.1) \quad f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j} z^{p+n-1}, \quad (z \in \mathcal{D}).$$

Now, we shall prove the following results for the closure of functions in the classes  $\Omega_p^*(\alpha)$  and  $\Lambda_p^*(\alpha)$ .

**Theorem 5.1.** *Let the functions  $f_j(z)$ ,  $j \in \{1, 2, \dots, m\}$ , defined by (5.1) be in the class  $\Omega_p^*(\alpha)$ . Then the function  $h(z) \in \Omega_p^*(\alpha)$  where*

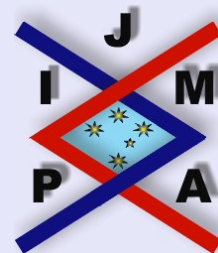
$$(5.2) \quad h(z) = \sum_{j=1}^m b_j f_j(z), \quad b_j \geq 0 \quad \text{and} \quad \sum_{j=1}^m b_j = 1).$$

*Proof.* From (5.2), we can write  $h(z)$  as

$$(5.3) \quad h(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} c_{p+n-1} z^{p+n-1},$$

where

$$(5.4) \quad c_{p+n-1} = \sum_{j=1}^m b_j a_{p+n-1,j}, \quad j \in \{1, 2, \dots, m\}.$$



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Since  $f_j(z) \in \Omega_p^*(\alpha)$ , ( $j \in \{1, 2, \dots, m\}$ ), from (2.1), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left[ \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} \right] & \left( \sum_{j=1}^m b_j |a_{p+n-1, j}| \right) \\ & = \sum_{j=1}^m b_j \left( \sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} |a_{p+n-1, j}| \right) \\ & \leq \sum_{j=1}^m b_j = 1, \end{aligned}$$

which shows that  $h(z) \in \Omega_p^*(\alpha)$ . This completes the proof of Theorem 5.1.  $\square$

Using the same technique as in the proof of Theorem 5.1, we have

**Theorem 5.2.** *Let the functions  $f_j(z)$ ,  $j \in \{1, 2, \dots, m\}$ , defined by (5.1) be in the class  $\Lambda_p^*(\alpha)$ . Then the function  $h(z) \in \Lambda_p^*(\alpha)$ , where  $h(z)$  defined by (5.2).*

**Theorem 5.3.** *Let*

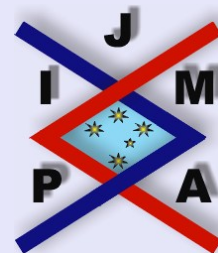
$$(5.5) \quad f_{p-1}(z) = \frac{1}{z^p} \quad (z \in \mathcal{D})$$

and

$$(5.6) \quad f_{p+n-1}(z) = \frac{1}{z^p} + \frac{2(p - \alpha)}{\sigma_n(p, k, \alpha)} z^{p+n-1},$$

where  $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ;  $z \in \mathcal{D}$ . Then  $f(z) \in \Omega_p^*(\alpha)$  if and only if it can be expressed in the form

$$(5.7) \quad f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z)$$



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where  $\lambda_{p+n-1} \geq 0$ , ( $n \in \mathbb{N}_0$ ) and  $\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1$ .

*Proof.* From (5.5), (5.6) and (5.7), it is easily seen that

$$(5.8) \quad \begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{n+p-1}(z) \\ &= \frac{1}{z^p} + \frac{2(p-\alpha)}{\sigma_n(p, k, \alpha)} \lambda_{p+n-1} z^{p+n-1}. \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \alpha)}{2(p-\alpha)} \cdot \frac{2(p-\alpha)}{\sigma_n(p, k, \alpha)} \lambda_{p+n-1} = \sum_{n=1}^{\infty} \lambda_{p+n-1} = 1 - \lambda_{p-1} \leq 1,$$

it follows from Theorem 2.1 that the function  $f(z)$  given by (5.6) is in the class  $\Omega_p^*(\alpha)$ .

Conversely, let us suppose that  $f(z) \in \Omega_p^*(\alpha)$ . Since

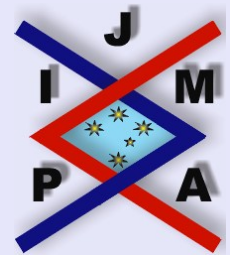
$$|a_{p+n-1}| \leq \frac{2(p-\alpha)}{\sigma_n(p, k, \alpha)} \quad (n \geq 1),$$

setting

$$\lambda_{p+n-1} = \frac{\sigma_n(p, k, \alpha)}{2(p-\alpha)} |a_{p+n-1}|, \quad (n \geq 1)$$

and

$$\lambda_{p-1} = 1 - \sum_{n=1}^{\infty} \lambda_{p+n-1},$$



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it follows that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z).$$

This completes the proof of the theorem. □

Similarly, we can prove the same result for the class  $\Lambda_p^*(\alpha)$ .

**Theorem 5.4.** *Let*

$$(5.9) \quad g_{p-1}(z) = \frac{1}{z^p} \quad (z \in \mathcal{D})$$

and

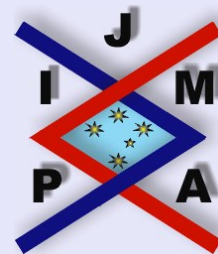
$$(5.10) \quad g_{p+n-1}(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{(p+n-1)\sigma_n(p, k, \alpha)} z^{p+n-1}$$

where  $n \in \mathbb{N}_0$  and  $z \in \mathcal{D}$ . Then  $g(z) \in \Lambda_p^*(\alpha)$  if and only if it can be expressed in the form

$$(5.11) \quad g(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} g_{p+n-1}(z)$$

where  $\lambda_{p+n-1} \geq 0$ , ( $n \in \mathbb{N}_0$ ) and  $\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1$ .

Next, we state a theorem which exhibits the fact that the classes  $\Omega^*(\alpha)$  and  $\Lambda_p^*(\alpha)$  are closed under convex linear combinations. The proof is fairly straightforward so we omit it.



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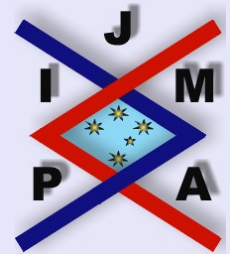
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**Theorem 5.5.** *Suppose that  $f(z)$  and  $g(z)$  are in the class  $\Omega^*(\alpha)$  (or in  $\Lambda_p^*(\alpha)$ ). Then the function  $h(z)$  defined by*

$$(5.12) \quad h(z) = tf(z) + (1 - t)g(z), \quad (0 \leq t \leq 1)$$

*is also in the class  $\Omega^*(\alpha)$  (or in  $\Lambda_p^*(\alpha)$ ).*



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## 6. Convolution Properties

For functions

$$(6.1) \quad f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j} z^{p+n-1}, \quad (j = 1, 2)$$

belonging to the class  $\Sigma_p$ , we denote by  $(f_1 * f_2)(z)$  the Hadamard product (or convolution) of the functions  $f_1(z)$  and  $f_2(z)$ , that is,

$$(6.2) \quad (f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,1} a_{p+n-1,2} z^{p+n-1}.$$

Finally, we prove the following.

**Theorem 6.1.** *Let each of the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (6.1) be in the class  $\Omega^*(\alpha)$ . Then  $(f_1 * f_2)(z) \in \Omega^*(\eta)$ , where*

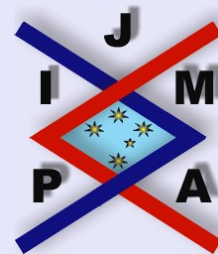
$$(6.3) \quad \frac{1}{2}(k+1-p-n) \leq \eta = \frac{p([p+k+|p+2\alpha-k|]^2 - 4(p-\alpha)^2)}{4(p-\alpha)^2 + [p+k+|p+2\alpha-k|]^2},$$

$$(k \geq p; p, n \in \mathbb{N}).$$

*The result is sharp.*

*Proof.* For  $f_j(z) \in \Omega^*(\alpha)$  ( $j = 1, 2$ ), we need to find the largest  $\eta$  such that

$$(6.4) \quad \sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \eta)}{2(p-\eta)} |a_{p+n-1,1}| |a_{p+n-1,2}| \leq 1.$$



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From (2.1), we have

$$(6.5) \quad \sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} |a_{p+n-1,1}| \leq 1$$

and

$$(6.6) \quad \sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} |a_{p+n-1,2}| \leq 1.$$

Therefore, by the Cauchy-Schwarz inequality, we have

$$(6.7) \quad \sum_{n=1}^{\infty} \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} \sqrt{|a_{p+n-1,1}| |a_{p+n-1,2}|} \leq 1.$$

Thus it is sufficient to show that

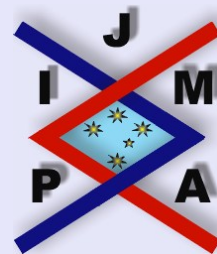
$$(6.8) \quad \frac{\sigma_n(p, k, \eta)}{2(p - \eta)} |a_{p+n-1,1}| |a_{p+n-1,2}| \leq \frac{\sigma_n(p, k, \alpha)}{2(p - \alpha)} \sqrt{|a_{p+n-1,1}| |a_{p+n-1,2}|}, \quad (n \geq 1)$$

that is, that

$$(6.9) \quad \sqrt{|a_{p+n-1,1}| |a_{p+n-1,2}|} \leq \frac{(p - \eta)\sigma_n(p, k, \alpha)}{(p - \alpha)\sigma_n(p, k, \eta)}, \quad (n \geq 1).$$

From (6.7), we have

$$\sqrt{|a_{p+n-1,1}| |a_{p+n-1,2}|} \leq \frac{2(p - \alpha)}{\sigma_n(p, k, \alpha)}.$$



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Consequently, we need only to prove that

$$(6.10) \quad \frac{2(p - \alpha)}{\sigma_n(p, k, \alpha)} \leq \frac{(p - \eta)\sigma_n(p, k, \alpha)}{(p - \alpha)\sigma_n(p, k, \eta)}, \quad (n \geq 1).$$

Let  $\eta \geq \frac{1}{2}(k + 1 - p - n)$ , where  $k \geq p$  and  $p, n \in \mathbb{N}$ . It follows from (6.10) that

$$(6.11) \quad \eta \leq \frac{p[\sigma_n(p, k, \alpha)]^2 - 4(p - \alpha)^2(p + n - 1)}{4(p - \alpha)^2 + [\sigma_n(p, k, \alpha)]^2} = \Psi(n).$$

Since  $\Psi(k)$  is an increasing function of  $n$  ( $n \geq 1$ ), letting  $n = 1$  in (6.11), we obtain

$$(6.12) \quad \eta \leq \Psi(1) = \frac{p([p + k + |p + 2\alpha - k|]^2 - 4(p - \alpha)^2)}{4(p - \alpha)^2 + [p + k + |p + 2\alpha - k|]^2},$$

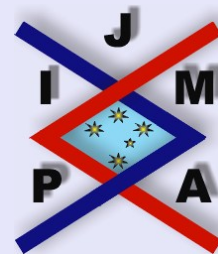
which proves the main assertion of Theorem 6.1.

Finally, by taking the functions

$$(6.13) \quad f_j(z) = \frac{1}{z^p} + \frac{2(p - \alpha)}{\sigma_n(p, k, \alpha)} z^{p+n-1}, \quad (j = 1, 2)$$

we can see the result is sharp. □

Similarly, and as the above proof, we can prove the following.




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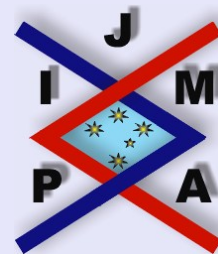
**Theorem 6.2.** Let each of the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (6.1) be in the class  $\Lambda_p^*(\alpha)$ . Then  $(f_1 * f_2)(z) \in \Lambda_p^*(\xi)$ , where

$$(6.14) \quad \frac{1}{2}(k + 1 - p - n) \leq \xi = \frac{p(p[p + k + |p + 2\alpha - k|]^2 - 4(p - \alpha)^2)}{4(p - \alpha)^2 + p[p + k + |p + 2\alpha - k|]^2},$$

$$(k \geq p; p, n \in \mathbb{N}).$$

The result is sharp for the functions

$$(6.15) \quad f_j(z) = \frac{1}{z^p} + \frac{2(p - \alpha)}{(p + n - 1)\sigma_n(p, k, \alpha)} z^{p+n-1}, \quad (j = 1, 2).$$




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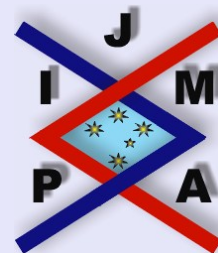
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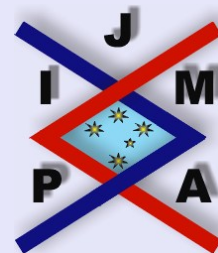
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