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INEQUALITIES INVOLVING LOGARITHMIC, POWER AND SYMMETRIC MEANS

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ABSTRACT. Inequalities involving the logarithmic mean, power means, symmetric means, and the Heronian mean are derived. They provide generalizations of some known inequalities for the logarithmic mean and the power means.

Key words and phrases: Logarithmic mean, Power means, Symmetric means, Inequalities.

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1. INTRODUCTION AND NOTATION

Let $x > 0$ and $y > 0$. In order to avoid trivialities we will always assume that $x \neq y$. The logarithmic mean of x and y is defined as

$$L(x,y) = \frac{x-y}{\ln x - \ln y}. \tag{1.1}$$

Other means used in this paper include the extended logarithmic mean E_p, where

$$E_p(x,y) = \begin{cases} \left[\dfrac{x^p - y^p}{p(x-y)}\right]^{\frac{1}{p-1}}, & p \neq 0,1; \\ L(x,y), & p = 0; \\ \exp\left(-1 + \dfrac{x\ln x - y\ln y}{x-y}\right), & p = 1, \end{cases} \tag{1.2}$$

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the power mean A_p, where

$$A_p(x,y) = \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{\frac{1}{p}}, & p \neq 0; \\ G(x,y), & p = 0 \end{cases} \tag{1.3}$$

with $G(x,y) = \sqrt{xy}$ being the geometric mean of x and y, and the symmetric mean s_k, where

$$s_k(x,y;\alpha) = \frac{1}{k}\sum_{i=1}^{k} x^{1-\alpha_i} y^{\alpha_i}, \tag{1.4}$$

($\alpha = (\alpha_1, \ldots, \alpha_k)$, $0 \le \alpha_1 < \cdots < \alpha_k \le 1$). A special case of (1.4) is the Heronian mean H, where

$$H(x,y) = \frac{x + (xy)^{1/2} + y}{3}. \tag{1.5}$$

Substituting $k = 3$ and $\alpha = \left(0, \frac{1}{2}, 1\right)$ in (1.4) we obtain $s_3(x,y;\alpha) = H(x,y)$.

The following result is known

$$\frac{1}{2}\left(x^{1/4}y^{3/4} + x^{3/4}y^{1/4}\right) < L(x,y) < A_{1/3}(x,y). \tag{1.6}$$

The first inequality in (1.6) has been established by B. Carlson [1], while the second one is proven in [3]. It is worth mentioning that the left inequality in (1.6) has been sharpened by A. Pittenger [6] who proved that

$$\frac{1}{2}(x^{\alpha_1}y^{\alpha_2} + x^{\alpha_2}y^{\alpha_1}) < L(x,y), \tag{1.7}$$

where $\alpha_1 = \left(1 - \frac{1}{\sqrt{3}}\right)\Big/ 2$, $\alpha_2 = 1 - \alpha_1$. Let us note that the numbers α_1 and α_2 are the roots of the second-degree Legendre polynomial $P_2(t) = t^2 - t + \frac{1}{6}$ on $[0,1]$. The inequality (1.7) can be derived easily by applying the two-point Gauss-Legendre quadrature formula to the integral formula for the logarithmic mean

$$L(x,y) = \int_0^1 x^t y^{1-t} dt$$

(see [4, (2.1)]).

The goal of this note is to obtain new inequalities which involve the logarithmic mean, power means, and the symmetric means. These results are given in the next section and they provide improvements and generalizations of known results.

2. Main Results

In what follows we will always assume that $\alpha_i = (2i-1)/2k$ ($i = 1, 2, \ldots, k$) and we will write $s_k(x,y)$ instead of $s_k(x,y;\alpha)$.

We are in a position to prove the following.

Theorem 2.1. *Let x and y denote positive numbers and let $t \in \mathbb{R}$. Then the following inequalities*

$$\left[A_{2t}(x,y)G^2(x,y)\right]^{2t/3} \le L(x,y)E_{2t}^{2t-1}(x,y) \le \left(A_{2t/3}(x,y)\right)^{2t} \tag{2.1}$$

are valid. They become equalities if $t = 0$.

Proof. Let $\lambda = t\ln(x/y)$ ($t \neq 0$). We substitute $x := e^{\lambda}$ and $y := e^{-\lambda}$ into (1.1) and next multiply the numerator and denominator by $(xy)^t(x-y)$ to obtain

$$L(e^{\lambda}, e^{-\lambda}) = \frac{L(x,y)E_{2t}^{2t-1}(x,y)}{G^{2t}(x,y)}. \tag{2.2}$$

Let $A(x,y) = A_1(x,y)$. Letting $x := e^{\lambda}$, $y := e^{-\lambda}$ and multiplying and dividing by $(xy)^t$ we obtain easily

$$A(e^{\lambda}, e^{-\lambda}) = \left(\frac{A_{2t}(x,y)}{G(x,y)}\right)^{2t}. \tag{2.3}$$

Also, we need the following formula

$$A_{1/3}(e^{\lambda}, e^{-\lambda}) = \left(\frac{A_{2t/3}(x,y)}{G(x,y)}\right)^{2t}. \tag{2.4}$$

We have

$$\begin{aligned} A_{1/3}^{1/3}(e^{\lambda}, e^{-\lambda}) &= \frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{t}{3}} + \left(\frac{y}{x}\right)^{\frac{t}{3}}\right] \\ &= \frac{(xy)^{\frac{t}{3}}\left[\left(\frac{x}{y}\right)^{\frac{t}{3}} + \left(\frac{y}{x}\right)^{\frac{t}{3}}\right]}{2} \cdot \frac{1}{(xy)^{\frac{t}{3}}} \\ &= \frac{x^{\frac{2t}{3}} + y^{\frac{2t}{3}}}{2} \cdot \frac{1}{(xy)^{\frac{t}{3}}} \\ &= \left(\frac{A_{2t/3}(x,y)}{G(x,y)}\right)^{\frac{2t}{3}}. \end{aligned}$$

Hence (2.4) follows. In order to establish the inequalities (2.1) we employ the following ones

$$\left[A(u,v)G^2(u,v)\right]^{\frac{1}{3}} \le L(u,v) \le A_{1/3}(u,v) \tag{2.5}$$

($u, v > 0$). The first inequality in (2.5) has been proven by E. Leach and M. Sholander in [2] (see also [5, (3.10)] for its generalization) while the second one is due to T. Lin [3]. To complete the proof of inequalities (2.1) we substitute $u = e^{\lambda}$ and $v = e^{-\lambda}$ into (2.5) and next utilize (2.3) and (2.4) to obtain the desired result. When $t = 0$, then the inequalities (2.1) become equalities because $E_0^{-1}(x,y) = 1/L(x,y)$. The proof is complete. □

Corollary 2.2. *Let $x > 0$, $y > 0$ and let $k = 1, 2, \ldots$. Then*

$$\left[\frac{A_{1/k}(x,y)}{G(x,y)}\right]^{\frac{1}{3k}} \le \frac{L(x,y)}{s_k(x,y)} \le \left[\frac{A_{1/3k}(x,y)}{G(x,y)}\right]^{\frac{1}{k}} \tag{2.6}$$

and

$$\lim_{k\to\infty} s_k(x,y) = L(x,y). \tag{2.7}$$

Proof. In order to establish inequalities (2.6) we use (2.1) with $t = 1/2k$ to obtain

$$\left[A_{1/k}(x,y)G^2(x,y)\right]^{\frac{1}{3k}} \le L(x,y)E_{1/k}^{1/k-1}(x,y) \le \left[A_{1/3k}(x,y)\right]^{\frac{1}{k}}. \tag{2.8}$$

It follows from (1.2) that

$$E_{1/k}^{1/k-1}(x,y) = \frac{x^{1/k} - y^{1/k}}{\frac{1}{k}(x-y)} = \left[\frac{1}{k}\sum_{i=1}^{k} x^{(k-i)/k} y^{(i-1)/k}\right]^{-1}.$$

Substituting this into (2.8) and next multiplying all terms of the resulting inequality by $1/(xy)^{1/2k}$ gives the desired result (2.6). For the proof of (2.7) we use

$$\lim_{k\to\infty} A_{1/k}(x,y) = \lim_{k\to\infty} A_{1/3k}(x,y) = G(x,y)$$

together with (2.6). The proof is complete. □

The first inequality in (2.6), with $k = 2$, provides a refinement of the first inequality in (1.6). We have

$$1 < \left[\frac{A_{1/2}(x,y)}{G(x,y)}\right]^{\frac{1}{6}} \le \frac{L(x,y)}{s_2(x,y)},$$

where the first inequality is an obvious consequence of $G(x,y) < A_{1/2}(x,y)$.

Corollary 2.3. *The following inequalities*

(2.9) $$\left[A_{1/2}(x,y)A_{3/2}(x,y)G^2(x,y)\right]^{\frac{1}{4}} \le \left[L(x,y)H(x,y)\right]^{\frac{1}{2}} \le A_{1/2}(x,y),$$

(2.10) $$\frac{1}{L(x,y)} + \frac{1}{H(x,y)} \ge \frac{2}{A_{1/2}(x,y)}$$

and

(2.11) $$\frac{1}{A_{1/2}(x,y)} + \frac{1}{A_{3/2}(x,y)} + \frac{2}{G(x,y)} \ge \frac{4}{\sqrt{L(x,y)H(x,y)}}$$

hold true.

Proof. Inequalities (2.9) follow from (2.1) by letting $t = 3/4$ and from

$$E_{3/2}^{1/2}(x,y) = \frac{H(x,y)}{A_{1/2}^{1/2}(x,y)},$$

where the last result is a special case of (1.2) when $p = 3/2$. For the proof of (2.10) we use the second inequality in (2.9) together with the inequality of arithmetic and geometric means. We have

$$\frac{1}{A_{1/2}(x,y)} \le \left[\frac{1}{L(x,y)}\cdot\frac{1}{H(x,y)}\right]^{\frac{1}{2}} \le \frac{1}{2}\left[\frac{1}{L(x,y)} + \frac{1}{H(x,y)}\right].$$

Inequality (2.11) is a consequence of the first inequality in (2.9) and the inequality for the weighted arithmetic and geometric means. Proceeding as in the proof of (2.10) we obtain

$$\begin{aligned}\frac{1}{\sqrt{L(x,y)H(x,y)}} &\le \left[\frac{1}{A_{1/2}(x,y)}\right]^{\frac{1}{4}}\left[\frac{1}{A_{3/2}(x,y)}\right]^{\frac{1}{4}}\left[\frac{1}{G(x,y)}\right]^{\frac{1}{2}} \\ &\le \frac{1}{4A_{1/2}(x,y)} + \frac{1}{4A_{3/2}(x,y)} + \frac{1}{2G(x,y)}.\end{aligned}$$

This completes the proof. □

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