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A NOTE ON A PAPER OF H. ALZER AND S. KOUMANDOS

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Abstract

In the paper " Sharp inequalities for trigonometric sums in two variables," (*Illinois Journal of Mathematics*, Vol. 48, No.3, (2004), 887–907) Alzer and Koumandos investigated some special trigonometric sums. One of them is the sum

$$A_n^*(x,y) := \sum_{k=1}^{n} \frac{\cos(k-\frac{1}{2})x \sin(k-\frac{1}{2})y}{k-\frac{1}{2}}.$$

In the present note we show that the results of [1] can be easily obtained by a very simple elementary argument. And the results we obtained are more exact.

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In a recent long paper [1], Alzer and Koumandos investigated the trigonometric sums:

$$A_n(x,y) = \sum_{k=1}^{n} \frac{\cos kx \sin ky}{k}, \qquad A_n^*(x,y) = \sum_{k=1}^{n} \frac{\cos\left(k-\frac{1}{2}\right) x \sin\left(k-\frac{1}{2}\right) y}{k-\frac{1}{2}},$$

$$B_n(x,y) = \sum_{k=1}^{n} \frac{\sin kx \sin ky}{k}.$$

Their results can be restated as follows:

(A) $|A_n(x,y)| < \sup\{A_n(x,y) :\ x,y \in [0,\pi], n \in \mathbb{N}\} = \int_0^\pi \frac{\sin t}{t}\,dt$;

(B) $\min\{B_n(x,y) :\ x,y \in [0,\pi], n \in \mathbb{N}\} = -\frac{1}{8}$,

$$B_n(x,y) = -\frac{1}{8} \Longleftrightarrow n = 2 \text{ and } (x,y) = \left(\frac{5\pi}{6}, \frac{\pi}{6}\right) \text{ or } (x,y) = \left(\frac{\pi}{6}, \frac{5\pi}{6}\right);$$

(C) $-\frac{2}{3}(\sqrt{2}-1) \le A_n^*(x,y) \le 2$,

$$A_n^*(x,y) = -\frac{2}{3}(\sqrt{2}-1) \Longleftrightarrow n = 2, (x,y) = \left(\frac{3\pi}{4}, \frac{\pi}{4}\right),$$

$$A_n^*(x,y) = 2 \Longleftrightarrow n = 1, (x,y) = (0,\pi).$$

The purpose of the present note is to give more exact results by very much simpler proof.

For a continuous function f on $D := [0,\pi] \times [0,\pi]$ we define

$$\min(f) = \min\{f(x,y) : (x,y) \in D\}, \quad \max(f) = \max\{f(x,y) : (x,y) \in D\}.$$

Our results are

(A′)

$$\max(A_n) = A_n\left(0, \frac{\pi}{n+1}\right) = \int_0^{\frac{\pi}{2(n+1)}} \frac{2\cos(n+1)t\sin nt}{\sin t}\,dt,$$

$$\min(A_n) = -\max(A_n) = A_n\left(\pi, \pi - \frac{\pi}{n+1}\right),$$

$$\max(A_n) < \lim_{n\to\infty} \max(A_n) = \int_0^\pi \frac{\sin t}{t}\,dt.$$

(B′) For $n \geq 2$

$$\min(B_n) = B_n\left(\frac{(2n+1)\pi}{n(n+1)}, \frac{\pi}{n(n+1)}\right)$$
$$= \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\sin(n+1)t \sin nt}{\sin t}\, dt < \min(B_{n+1}),$$

$$\lim_{n\to\infty} \min(B_n) = 0.$$

(C′) For all n

$$\max(A_n^*) = A_n^*\left(0, \frac{\pi}{n}\right)$$
$$= \int_0^{\frac{\pi}{2n}} \frac{\sin 2nt}{\sin t}\, dt$$
$$> \max(A_{n+1}^*) \to \int_0^{\pi} \frac{\sin t}{t}\, dt, \ (n \to \infty),$$

and for $n \geq 2$

$$\min(A_n^*) = A_n^*\left(\frac{3\pi}{2n}, \frac{\pi}{2n}\right) = \int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} \frac{\sin 2nt}{2\sin t}\, dt < \min(A_{n+1}^*) \to 0 \ \ (n \to \infty).$$

In particular, $\min(A_2^*) = \frac{2}{3}(1 - \sqrt{2})$.

The results (A), (B), (C) are easy consequences of (A′), (B′) and (C′) respectively.

Proof of (A'). We have

$$\begin{aligned} A_n(x,y) &= \sum_{k=1}^{n} \frac{\sin k(x+y) - \sin k(x-y)}{2k} \\ &= \sum_{k=1}^{n} \frac{1}{2k} \int_{k(x-y)}^{k(x+y)} \cos t\, dt \\ &= \sum_{k=1}^{n} \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \cos 2kt\, dt \\ &= \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \sum_{k=1}^{n} \frac{2\cos 2kt \sin t}{2\sin t} \\ &= \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\sin(2n+1)t - \sin t}{2\sin t}\, dt = \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\cos(n+1)t \sin nt}{\sin t}\, dt. \end{aligned}$$

Then we get

$$\begin{aligned} \max(A_n) &= A_n\left(0, \frac{\pi}{n+1}\right) \\ &= \int_{\frac{-\pi}{2(n+1)}}^{\frac{\pi}{2(n+1)}} \frac{\cos(n+1)t \sin nt}{\sin t}\, dt \\ &= \int_{0}^{\frac{\pi}{2(n+1)}} \frac{2\cos(n+1)t \sin nt}{\sin t}\, dt \\ &< \int_{0}^{\frac{\pi}{2(n+1)}} \frac{2\cos(n+1)t \sin(n+1)t}{t}\, dt = \int_0^{\pi} \frac{\sin t}{t}\, dt; \end{aligned}$$

$$\lim_{n\to\infty} \max(A_n) = \int_0^\pi \frac{\sin t}{t} dt;$$

$$\begin{aligned}\min(A_n) &= \min\{A_n(\pi - x, \pi - y) : (x, y) \in D\} \\ &= -\max(A_n) = A_n\left(\pi, \pi - \frac{\pi}{n+1}\right).\end{aligned}$$

□

Proof of (B′) and (C′). We have

$$\begin{aligned}B_n(x, y) &= \sum_{k=1}^{n} \frac{\cos k|x - y| - \cos k(x + y)}{2k} \\ &= \sum_{k=1}^{n} \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \sin 2kt\, dt \\ &= \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \frac{\cos t - \cos(2n + 1)t}{2 \sin t} dt \\ &= \int_{\frac{|x-y|}{2}}^{\frac{x+y}{2}} \frac{\sin(n + 1)t \sin nt}{\sin t} dt,\end{aligned}$$

$$A_n^*(x, y) = \frac{1}{2} \sum_{k=1}^{n} \int_{x-y}^{x+y} \cos\left(k - \frac{1}{2}\right) t\, dt = \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \frac{\sin 2nt}{2 \sin t} dt.$$

Then we get (B′) and (C′). □

References

[1] H. ALZER AND S. KOUMANDOS, Sharp inequalities for trigonometric sums in two variables, *Illinois J. Math.*, **48**(3) (2004), 887–907.