journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 3, Article 84, 8 pp.

THE SIMULTANEOUS NONLINEAR INEQUALITIES PROBLEM AND APPLICATIONS

ZORAN D. MITROVIĆ

FACULTY OF ELECTRICAL ENGINEERING
UNIVERSITY OF BANJA LUKA
78000 BANJA LUKA, PATRE 5
BOSNIA AND HERZEGOVINA
zmitrovic@etfbl.net

Received 22 June, 2007; accepted 02 August, 2007
Communicated by S.P. Singh

ABSTRACT. In this paper, we prove the existence of a solution to the simultaneous nonlinear inequality problem. As applications, we derive the results on the simultaneous approximations, variational inequalities and saddle points. The results of this paper generalize some known results in the literature.

Key words and phrases: KKM map, Best approximations, Fixed points and coincidences, Variational inequalities, Saddle points.

2000 *Mathematics Subject Classification.* 47J20, 41A50, 54H25.

1. INTRODUCTION AND PRELIMINARIES

In this paper, using the methods of KKM-theory, see for example, Singh, Watson and Srivastava [17] and Yuan [20], we prove some results on simultaneous nonlinear inequalities. As corollaries, some results on the simultaneous approximations, variational inequalities and saddle points are obtained.

Let X be a set. We shall denote by 2^X the family of all non-empty subsets of X. If A is a subset of a vector space X, then coA denotes the convex hull of A in X. Let K be a subset of a topological vector space X. Then a multivalued map $G : K \to 2^X$ is called a KKM-map if

$$co\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i)$$

for each finite subset $\{x_1, \ldots, x_n\}$ of K.

Let K be a nonempty convex subset of a vector space X. For a map $f : K \to \mathbb{R}$, the set

$$Ep(f) = \{(x, r) \in K \times \mathbb{R} : f(x) \leq r\}$$

is called the epigraph of f. Note that a map f is convex if and only if the set $Ep(f)$ is convex.

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Let K be a nonempty set, $n \in \mathbb{N}$ and $f_i : K \times K \to \mathbb{R}$ maps for all $i \in [n]$, where $[n] = \{1, \dots, n\}$. A simultaneous nonlinear inequalities problem is to find $x_0 \in K$ such that it satisfies the following inequality

$$\sum_{i=1}^{n} f_i(x_0, y) \geq 0 \quad \text{for all } y \in K. \tag{1.1}$$

When $n = 1$ and $f(x, x) = 0$ for all $x \in K$, (1.1) reduces to the scalar equilibrium problem considered by Blum and Oettli [5], that is, to find $x_0 \in K$ such that

$$f(x_0, y) \geq 0 \quad \text{for all } y \in K.$$

This problem has been generalized and applied in various directions, see for example [1], [2], [3], [9], [10], [14].

The following result of Ky Fan [8] will be used to prove the main result of this paper.

Theorem 1.1 ([8])**.** *Let X be a topological vector space, K a nonempty subset of X and $G : K \to 2^X$ be a KKM-map with closed values. If $G(x)$ is compact for at least one $x \in K$, then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

2. Main Result

Now we will apply Theorem 1.1 to show the existence of a solution for our simultaneous nonlinear inequalities problem.

Theorem 2.1. *Let K be a nonempty compact convex subset of a topological vector space X and $f_i : K \times K \to \mathbb{R}, i \in [n]$, continuous maps. If there exists $\lambda \geq 0$, such that*

$$coEp(f_i(x, \cdot)) \subseteq Ep(f_i(x, \cdot) - \lambda) \tag{2.1}$$

for all $x \in K, i \in [n]$, then there exists $x_0 \in K$ such that

$$\lambda n + \sum_{i=1}^{n} f_i(x_0, y) \geq \sum_{i=1}^{n} f_i(x_0, x_0) \quad \textit{for all } y \in K.$$

Proof. Let us define the map $G : K \to 2^K$ by

$$G(y) = \left\{ x \in K : \lambda n + \sum_{i=1}^{n} f_i(x, y) \geq \sum_{i=1}^{n} f_i(x, x) \right\}, \quad \text{for all } y \in K.$$

We have that $G(y)$ is nonempty for all $y \in K$, because $y \in G(y)$ for all $y \in K$.

The $f_i, i \in [n]$ are continuous maps and we obtain that $G(y)$ is closed for each $y \in K$. Since K is a compact set, we have that $G(y)$ is compact for each $y \in K$.

Now, we prove that G is a KKM-map. If G is not a KKM-map, then there exists a subset $\{y_1, \dots, y_m\}$ of K and there exists $\mu_j \geq 0,\ j \in [m]$ with $\sum_{j=1}^{m} \mu_j = 1$, such that

$$y_\mu = \sum_{j=1}^{m} \mu_j y_j \notin \bigcup_{j=1}^{m} G(y_j).$$

So, we have

$$\lambda n + \sum_{i=1}^{n} f_i(y_\mu, y_j) < \sum_{i=1}^{n} f_i(y_\mu, y_\mu), \quad \text{for all } j \in [n].$$

On the other hand, since,

$$(y_j, f_i(y_\mu, y_j)) \in Ep(f_i(y_\mu, \cdot)), \quad \text{for all } i \in [n], j \in [m],$$

from condition (2.1) we obtain

$$\left(y_\mu, \sum_{j=1}^{m} \mu_j f_i(y_\mu, y_j)\right) \in Ep(f_i(y_\mu, \cdot) - \lambda) \quad \text{for all } i \in [n].$$

Therefore, it follows that

$$f_i(y_\mu, y_\mu) - \lambda \leq \sum_{j=1}^{m} \mu_j f_i(y_\mu, y_j) \quad \text{for all } i \in [n].$$

This implies that

$$\sum_{i=1}^{n} f_i(y_\mu, y_\mu) \leq \lambda n + \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_j f_i(y_\mu, y_j).$$

Further, since

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \mu_j f_i(y_\mu, y_j) = \sum_{j=1}^{m} \mu_j \sum_{i=1}^{n} f_i(y_\mu, y_j) \leq \max_{1\leq j\leq m} \sum_{i=1}^{n} f_i(y_\mu, y_j),$$

and

$$\lambda n + \sum_{i=1}^{n} f_i(y_\mu, y_j) < \sum_{i=1}^{n} f_i(y_\mu, y_\mu), \quad \text{for all } j \in [m],$$

we obtain

$$\sum_{i=1}^{n} f_i(y_\mu, y_\mu) < \sum_{i=1}^{n} f_i(y_\mu, y_\mu).$$

This is a contradiction. Thus, G is a KKM-map.

By Theorem 1.1, there exists $x_0 \in K$ such that $x_0 \in G(y)$ for all $y \in K$, that is,

$$\lambda n + \sum_{i=1}^{n} f_i(x_0, y) \geq \sum_{i=1}^{n} f_i(x_0, x_0) \quad \text{for all } y \in K.$$

□

Corollary 2.2. *Let K be a nonempty compact convex subset of a topological vector space X and $f_i : K \times K \to \mathbb{R}$, $i \in [n]$, continuous maps. If $y \mapsto f_i(x, y)$ are convex for all $x \in K$, $i \in [n]$, then there exists $x_0 \in K$ such that*

$$\sum_{i=1}^{n} f_i(x_0, y) \geq \sum_{i=1}^{n} f_i(x_0, x_0) \quad \textit{for all } y \in K.$$

Note that, if in Theorem 2.1 the maps $x \mapsto f_i(x, y)$ are upper semicontinuous for all $y \in K$ and $f_i(x, x) \geq 0$ for all $x \in K, i \in [n]$, we obtain the following result.

Corollary 2.3. *Let K be a nonempty compact convex subset of a topological vector space X and $f_i : K \times K \to \mathbb{R}, i \in [n]$, maps such that*

(i) $f_i(x, x) \geq 0$ for all $x \in K$,
(ii) $x \mapsto f_i(x, y)$ are upper semicontinuous for all $y \in K$,
(iii) $y \mapsto f_i(x, y)$ are convex for all $x \in K$,

for all $i \in [n]$. Then there exists $x_0 \in K$ such that

$$\sum_{i=1}^{n} f_i(x_0, y) \geq 0 \quad \textit{for all } y \in K.$$

From Theorem 2.1, we have the theorem on the existence of zeros of bifunctions.

Theorem 2.4. *Let K be a nonempty compact convex subset of a topological vector space X, $f_i : K \times K \to \mathbb{R}, i \in [n]$ continuous maps and there exists $\lambda \geq 0$ such that the condition (2.1) is satisfied for all $x \in K$ and $i \in [n]$. If for every $x \in K$, with $f_i(x,x) \neq 0$ for all $i \in [n]$,*

$$\bigcap_{i=1}^{n} \{y \in K : f_i(x,x) - f_i(x,y) > \lambda\} \neq \emptyset$$

then the set

$$S = \left\{ x \in K : \lambda n + \sum_{i=1}^{n} f_i(x,y) \geq \sum_{i=1}^{n} f_i(x,x) \quad \text{for all } y \in K \right\}$$

is nonempty and for each $x \in S$ there exists $i \in [n]$ such that $f_i(x,x) = 0$.

Proof. By Theorem 2.1, there exists $x_0 \in S$. We claim that such x_0 is a zero of f_i for any $i \in [n]$. Suppose not, i.e., $f_i(x_0, x_0) \neq 0$ for all $i \in [n]$. Then we have the existence of $y_0 \in K$, such that

$$f_i(x_0, x_0) - f_i(x_0, y_0) > \lambda \quad \text{for all } i \in [n].$$

Consequently,

$$\lambda n + \sum_{i=1}^{n} f_i(x_0, y_0) < \sum_{i=1}^{n} f_i(x_0, x_0),$$

so, $x_0 \notin S$ and that is a contradiction. Therefore, $f_i(x_0, x_0) = 0$ for any $i \in [n]$. □

3. Applications

From Theorem 2.1, we have the following simultaneous approximations theorem for metric spaces.

Theorem 3.1. *Let K be a nonempty compact convex subset of a topological vector space X with metric d and $f_i, g_i : K \to X, i \in [n]$ continuous maps. Suppose there exists $\lambda \geq 0$, such that f_i, g_i satisfy the condition*

$$co\{(y,r) : d(g_i(y), f_i(x)) \leq r\} \subseteq \{(y,r) : d(g_i(y), f_i(x)) \leq r + \lambda\} \tag{3.1}$$

for all $x \in K$, $i \in [n]$. Then there exists $x_0 \in K$ such that

$$\lambda n + \sum_{i=1}^{n} d(g_i(y), f_i(x_0)) \geq \sum_{i=1}^{n} d(g_i(x_0), f_i(x_0)) \quad \text{for all } y \in K.$$

Proof. Define

$$f_i(x,y) = d(g_i(y), f_i(x)), \quad \text{for } x, y \in K, i \in [n].$$

Now, the result follows by Theorem 2.1. □

Remark 3.2. Let X be a normed space and let $g_i : K \to X$ be almost affine maps, see, for example [6], [13], [15], [16], [17], [18], i. e.

$$||g_i(\alpha x_1 + (1-\alpha)x_2) - y|| \leq \alpha ||g_i(x_1) - y|| + (1-\alpha)||g_i(x_2) - y||$$

for all $x_1, x_2 \in K, y \in X, \alpha \in [0,1], i \in [n]$. Then for $\lambda = 0$, assumption (3.1) is satisfied.

Corollary 3.3. *Let K be a nonempty compact convex subset of a normed space X, $f_i, g_i : K \to X$ continuous maps and g_i almost affine maps for all $i \in [n]$. Then there exists $x_0 \in K$ such that*

$$\sum_{i=1}^{n} ||g_i(y) - f_i(x_0)|| \geq \sum_{i=1}^{n} ||g_i(x_0) - f_i(x_0)|| \quad \text{for all } y \in K.$$

Corollary 3.4. *Let K be a nonempty compact convex subset of a normed space X and $f_i : K \to X, i \in [n]$, continuous maps. Then there exists $x_0 \in K$ such that*

$$\sum_{i=1}^{n} ||y - f_i(x_0)|| \geq \sum_{i=1}^{n} ||x_0 - f_i(x_0)|| \quad \textit{for all } y \in K.$$

Remark 3.5.

- (i) If $n = 1$ then Corollary 3.4 reduces to the well-known best approximations theorem of Ky Fan [8] and Corollary 3.3 reduces to the result of J.B. Prolla [15].
- (ii) Note that, if X is a Hilbert space and $n = 2$, from Corollary 3.4 we obtain the result of D. Delbosco [7].

As application of Theorem 2.4, we have the following coincidence point theorem for metric spaces.

Theorem 3.6. *Let K be a nonempty compact convex subset of a topological vector space X with metric d and $f_i, g_i : K \to X, i \in [n]$, continuous maps. Suppose there exists $\lambda \geq 0$, such that f_i, g_i satisfy the condition (3.1) for all $x \in K$, $i \in [n]$. If for every $x \in K$, with $f_i(x) \neq g_i(x)$ for all $i \in [n]$,*

$$\bigcap_{i=1}^{n} \{y \in K : d(g_i(x), f_i(x)) > d(g_i(y), f_i(x)) + \lambda\} \neq \emptyset,$$

then the set

$$S = \left\{ x \in K : \lambda n + \sum_{i=1}^{n} d(g_i(y), f_i(x)) \geq \sum_{i=1}^{n} d(g_i(x), f_i(x)) \quad \textit{for all } y \in K \right\}$$

is nonempty and for each $x \in S$ there exists $i \in [n]$ such that $f_i(x) = g_i(x)$.

Proof. Put

$$f_i(x, y) = d(g_i(y), f_i(x)), \quad \text{for } x, y \in K, i \in [n].$$

Then f_i, g_i, $i \in [n]$ satisfy all of the requirements of Theorem 2.4. □

Corollary 3.7. *Let K be a nonempty compact convex subset of a metric space X, $f, g : K \to X$ continuous maps and*

$$d(g(\lambda x_1 + (1 - \lambda)x_2), f(y)) \leq \lambda d(g(x_1), f(y)) + (1 - \lambda)d(g(x_2), f(y)),$$

for all $x_1, x_2 \in K, y \in X, \lambda \in [0, 1]$. If for every $x \in K$, with $f(x) \neq g(x)$ there exists a $y \in K$ such that

$$d(g(x), f(x)) > d(g(y), f(x)),$$

then the set

$$S = \{x \in K : d(g(y), f(x)) \geq d(g(x), f(x)) \quad \textit{for all } y \in K\}$$

is nonempty and $f(x) = g(x)$ for each $x \in S$.

Corollary 3.8. *Let K be a nonempty compact convex subset of a metric space X and $f : K \to X$ a continuous map such that*

$$x \mapsto d(x, f(y)) \textit{ is a convex map for all } y \in X.$$

If for every $x \in K$, with $f(x) \neq x$ there exists a $y \in K$ such that

$$d(x, f(x)) > d(y, f(x)),$$

then the set

$$S = \{x \in K : d(y, f(x)) \geq d(x, f(x)) \quad \textit{for all } y \in K\}$$

is nonempty and $f(x) = x$ *for each* $x \in S$.

We note that if $f : K \to K$, then, from Corollary 3.8, we obtain the famous Schauder fixed point theorem.

Now, we establish an existence result for our simultaneous variational inequality problem by using Corollary 2.3.

Theorem 3.9. *Let* X *be a reflexive Banach space with its dual* $X^\star$ *and* K *a compact convex subset of* X. *Let* $T_i : K \to X^\star, i \in [n]$, *be maps. If* $x \mapsto \langle T_i(x), y-x\rangle$ *are upper semicontinuous for all* $y \in K, i \in [n]$, *then there exists* $x_0 \in K$ *such that*

$$\sum_{i=1}^{n} \langle T_i(x_0), y - x_0\rangle \geq 0 \quad \textit{for all } y \in K.$$

Proof. Let $f_i(x, y) = \langle T_i(x), y - x\rangle$, for all $x, y \in K, i \in [n]$. By our assumptions, the maps f_i satisfy all the hypotheses of Corollary 2.3, and it follows that there exists $x_0 \in K$ such that

$$\sum_{i=1}^{n} \langle T_i(x_0), y - x_0\rangle \geq 0 \quad \text{for all } y \in K.$$

□

Remark 3.10.

(i) If $n = 1$ then Theorem 3.9 reduces to the classical result of F. E. Browder and W. Takahashi, see for example [17, Theorem 4. 33].

(ii) Given two maps $T : K \to X^\star$ and $\mu : K \times K \to X$, the variational-like inequality problem, see for example [12], is to find $x_0 \in K$ such that

$$\langle T(x_0), \mu(y, x_0)\rangle \geq 0 \quad \text{for all } y \in K.$$

If in Corollary 2.3 a map

$$f_1(x, y) = \langle T(x), \mu(y, x)\rangle \quad \text{for all } x, y \in K$$

and $n = 1$, we obtain the result of X.Q. Yang and G.Y. Chen [19, Theorem 8], and the result of A. Behera and L. Nayak [4, Theorem 2.1]. Also, if in Corollary 2.3 a map

$$f_1(x, y) = \langle T(x), \mu(y, x)\rangle - \langle A(x), \mu(y, x)\rangle \quad \text{for all } x, y \in K,$$

where $A : K \to X^\star$, we obtain the result of G. K. Panda and N. Dash, [11, Theorem 2.1].

Finally, we give the following application to the existence for saddle points.

Theorem 3.11. *Let* K *be a nonempty compact convex subset of a topological vector space* X *and* $f_i : K \times K \to \mathbb{R}$ *continuous maps and* $f_i(x, x) = 0$ *for all* $x \in K$, $i \in [n]$. *If there exists* $\lambda \geq 0$, *such that*

$$coEp(f_i(x, \cdot)) \subseteq Ep(f_i(x, \cdot) - \lambda) \quad \textit{for all } x \in K$$

and

$$coEp(-f_i(\cdot, y)) \subseteq Ep(-f_i(\cdot, y) - \lambda) \quad \textit{for all } y \in K,$$

for all $i \in [n]$, *then*

$$0 \leq \min_{x\in K} \max_{y\in K} \sum_{i=1}^{n} f_i(x, y) - \max_{y\in K} \min_{x\in K} \sum_{i=1}^{n} f_i(x, y) \leq 2\lambda n.$$

Proof. Note that

$$0 \leq \min_{x\in K}\max_{y\in K}\sum_{i=1}^{n} f_i(x,y) - \max_{y\in K}\min_{x\in K}\sum_{i=1}^{n} f_i(x,y)$$

holds in general. By our assumptions, $f_i, i \in [n]$ satisfy all the hypotheses of Theorem 2.1, and it follows that there exists $x_0 \in K$ such that

$$\min_{y\in K}\sum_{i=1}^{n} f_i(x_0,y) \geq -\lambda n.$$

So, we obtain,

$$\max_{x\in K}\min_{y\in K}\sum_{i=1}^{n} f_i(x,y) \geq -\lambda n. \tag{3.2}$$

Let $g_i(x,y) = -f_i(y,x)$ for all $(x,y) \in K \times K, i \in [n]$. By Theorem 2.1, it follows that there exists $y_0 \in K$ such that

$$\min_{y\in K}\sum_{i=1}^{n} g_i(y_0,y) \geq -\lambda n,$$

so

$$\max_{x\in K}\sum_{i=1}^{n} f_i(x,y_0) \leq \lambda n.$$

Therefore, we obtain,

$$\min_{y\in K}\max_{x\in K}\sum_{i=1}^{n} f_i(x,y) \leq \lambda n. \tag{3.3}$$

By combining (3.2) and (3.3), it follows that

$$\min_{x\in K}\max_{y\in K}\sum_{i=1}^{n} f_i(x,y) - \max_{y\in K}\min_{x\in K}\sum_{i=1}^{n} f_i(x,y) \leq 2\lambda n.$$

□

Corollary 3.12. *Let K be a nonempty compact convex subset of a topological vector space X. Suppose $f_i : K \times K \to \mathbb{R}, i \in [n]$ are continuous maps such that*

(1) $f_i(x,x) = 0$ *for all* $x \in K$,
(2) $y \mapsto f_i(x,y)$ *is convex for all* $x \in K$,
(3) $x \mapsto f_i(x,y)$ *is concave for all* $y \in K$,

for all $i \in [n]$. Then we have

$$\max_{y\in K}\min_{x\in K}\sum_{i=1}^{n} f_i(x,y) = \min_{x\in K}\max_{y\in K}\sum_{i=1}^{n} f_i(x,y).$$

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