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ITERATION SEMIGROUPS WITH GENERALIZED CONVEX, CONCAVE AND AFFINE ELEMENTS

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ABSTRACT. Given continuous functions M and N of two variables, it is shown that if in a continuous iteration semigroup with only (M, N)-convex or (M, N)-concave elements there are two (M, N)-affine elements, then $M = N$ and every element of the semigroup is M-affine. Moreover, all functions in the semigroup either are M-convex or M-concave.

Key words and phrases: Iteration semigroup, Convex (concave, affine)-functions with respect to some functions.

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1. INTRODUCTION

In this paper we use the definition of (M, N)-convex, (M, N)-concave and (M, N)-affine functions, introduced earlier by G. Aumann [1]. For a given M in $(0, \infty) \times (0, \infty)$ J. Matkowski [5] considered a continuous multiplicative iteration group of homeomorphisms $f^t : (0, \infty) \to (0, \infty)$, consisting of M-convex or M-concave elements. In the present paper we generalize some results of Matkowski considering the problem proposed in [5]. Let M and N be arbitrary continuous functions. We prove that, if in a continuous iteration semigroup with only (M, N)-convex or (M, N)-concave elements there are two (M, N)-affine functions, then every element of the semigroup is M-affine. Moreover, we show that if in a semigroup there exist f^{t_0}, which is (M, N)-affine, and two iterates with indices greater than t_0, one (M, N)-convex and the second (M, N)-concave, then the thesis is the same (all elements in a semigroup are M-affine). We end the paper with theorems describing the regularity of semigroups containing generalized convex and concave elements.

2. PRELIMINARIES

Let $I, J \subset \mathbb{R}$ be open intervals and let $M : I^2 \to I$, $N : J^2 \to J$ be arbitrary functions.
A function $f : I \to J$ is said to be

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(M,N)-*convex*, if

$$f(M(x,y)) \le N(f(x),f(y)), \qquad x,y \in I;$$

(M,N)-*concave*, if

$$f(M(x,y)) \ge N(f(x),f(y)), \qquad x,y \in I;$$

(M,N)-*affine*, if it is both (M,N)-convex and (M,N)-concave.

In the case when $M = N$, the respective functions are called M-convex, M-concave, and M-affine, respectively.

We start with three remarks which can easily be verified.

Remark 1. If a function f is increasing and (M,N)-convex, then for all M_1 and N_1 satisfying $M_1 \le M$ and $N_1 \ge N$ it is (M_1,N_1)-convex. Analogously, if a function f is decreasing and (M,N)-concave, then for all M_1 and N_1 satisfying $M_1 \le M$ and $N_1 \le N$ it is (M_1,N_1)-concave.

Remark 2. Let $f : I \to J$ be strictly increasing and onto J. If f is (M,N)-convex then its inverse function f^{-1} is (N,M)-concave.

If $f : I \to J$ is strictly decreasing, onto and (M,N)-convex, then its inverse function is (N,M)-convex.

If $f : I \to J$ is (M,N)-affine, then its inverse function is (N,M)-affine.

Remark 3. Let $I, J, K \subset \mathbb{R}$ be open intervals and $M : I^2 \to I$, $N : J^2 \to J$, $P : K^2 \to K$ be arbitrary functions.

If $g : I \to K$ is (M,P)-affine and $f : K \to J$ is (P,N)-affine, then $f \circ g$ is (M,N)-affine.

Under some additional conditions on f and g, the converse implication also holds true. Namely, we have the following:

Lemma 2.1. *Suppose that $g : I \to K$ is onto and (M,P)-convex and $f : K \to J$ is strictly increasing and (P,N)-convex. If $f \circ g$ is (M,N)-affine, then g is (M,P)-affine and f is (P,N)-affine.*

Proof. Let $f \circ g$ be (M,N)-affine. Assume, to the contrary, that f is not (P,N)-affine. Then $u_0, v_0 \in K$ would exist such that

$$f(P(u_0,v_0)) < N(f(u_0),f(v_0)).$$

Since g is onto K, there are $x_0, y_0 \in I$ such that $g(x_0) = u_0$ and $g(y_0) = v_0$. Hence, by the monotonicity of f and the (M,P)-convexity of g,

$$\begin{aligned} f \circ g(M(x_0,y_0)) &\le f(P(g(x_0),g(y_0))) \\ &= f(P(u_0,v_0)) \\ &< N(f(u_0),f(v_0)) \\ &= N(f \circ g(x_0), f \circ g(y_0)), \end{aligned}$$

which contradicts the assumption that $f \circ g$ is (M,N)-affine.

Similarly, if g were not (M,P)-affine then we would have

$$g(M(x_0,y_0)) < P(g(x_0),g(y_0))$$

for some $x_0, y_0 \in I$. By the monotonicity and the (P,N)-convexity of f we would obtain

$$f(g(M(x_0,y_0))) < f(P(g(x_0),g(y_0))) \le N(f(g(x_0)),f(g(y_0))),$$

which contradicts the (M,N)-affinity of $f \circ g$. This contradiction completes the proof. □

In a similar way one can show the following:

Remark 4. Suppose that $g : I \to K$ is onto and (M, P)-concave and $f : K \to J$ is strictly increasing and (P, N)-concave. If $f \circ g$ is (M, N)-affine, then g is (M, P)-affine and f is (P, N)-affine.

Remark 5. Observe that, without any loss of generality, considering the (M, N)-affinity, the (M, N)-convexity or the (M, N)-concavity of a function f we can assume that $I = J = (0, \infty)$.

Indeed, let $\varphi : (0, \infty) \to I$ and $\psi : J \to (0, \infty)$ be one-to-one and onto. Put $M_\varphi(s, t) := \varphi^{-1}(M(\varphi(s), \varphi(t)))$ and $N_\psi(u, v) := \psi(N(\psi^{-1}(u), \psi^{-1}(v)))$. A function $f : I \to J$ satisfies the equation

$$f(M(x, y)) = N(f(x), f(y)), \qquad x, y \in I,$$

if and only if the function $f^* := \psi \circ f \circ \varphi : (0, \infty) \to (0, \infty)$ satisfies

$$f^*(M_\varphi(s, t)) = N_\psi(f^*(s), f^*(t)), \qquad s, t \in (0, \infty).$$

Moreover, if ψ is strictly increasing, then f is (M, N)-convex ((M, N)-concave) if and only if f^* is (M_φ, N_ψ)-convex ((M_φ, N_ψ)-concave); if ψ is strictly decreasing, then f is (M, N)-convex ((M, N)-concave) if and only if f^* is (M_φ, N_ψ)-concave ((M_φ, N_ψ)-convex).

In what follows, we assume that $I = J$.

In the proof of the main result we need the following

Lemma 2.2. *Suppose that a non-decreasing function $f : I \to I$ is M-convex (or M-concave) and one-to-one or onto. If, for a positive integer m, the m-th iterate of f is M-affine, then f is M-affine.*

Proof. Assume that f is M-convex. Using, in turn, the convexity, the monotonicity, and again the convexity of f, we get, for $x, y \in I$,

$$f^2(M(x, y)) = f(f(M(x, y))) \le f(M(f(x), f(y))) \le M(f^2(x), f^2(y)),$$

and further, by induction, for all $x, y \in I$ and $n \in \mathbb{N}$,

$$f^n(M(x, y)) = f(f^{n-1}(M(x, y))) \le f(M(f^{n-1}(x), f^{n-1}(y))) \le M(f^n(x), f^n(y)).$$

Hence, since f^m is M-affine for an $m \in \mathbb{N}$, i.e.

$$f^m(M(x, y)) = M(f^m(x), f^m(y)), \qquad x, y \in I,$$

we obtain, for all $x, y \in I$,

$$f^m(M(x, y)) = f(M(f^{m-1}(x), f^{m-1}(y))) = M(f^m(x), f^m(y)). \tag{2.1}$$

Now, if f is one-to-one, from the first of these equalities we get

$$f^{m-1}(M(x, y)) = M(f^{m-1}(x), f^{m-1}(y)), \qquad x, y \in I,$$

which means that f^{m-1} is an M-affine function. Repeating this procedure $m - 2$ times we obtain the M-affinity of f. Now assume that f is onto I. If $m = 1$ there is nothing to prove. Assume that $m \ge 2$. Since f^{m-1} is also onto I, for arbitrary $u, v \in I$ there exist $x, y \in I$ such that $u = f^{m-1}(x)$ and $v = f^{m-1}(y)$. Now, from the second equality in (2.1), we get

$$f(M(u, v)) = M(f(u), f(v)), \qquad u, v \in I,$$

that is, f is M-affine.

As the same argument can be used in the case when f is M-concave, the proof is finished. □

Let us introduce the notions of an iteration group and an iteration semigroup.

A family $\{f^t : t \in \mathbb{R}\}$ of homeomorphisms of an interval I is said to be *an iteration group (of function f)*, if $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{R}$ (and $f^1 = f$). An iteration group is called *continuous* if for every $x \in I$ the function $t \to f^t(x)$ is continuous.

Note that f^t is increasing for every $t \in \mathbb{R}$.

A one parameter family $\{f^t : t \geq 0\}$ of continuous one-to-one functions $f^t : I \to I$ such that $f^t \circ f^s = f^{t+s}$, $t, s \geq 0$ is said to be *an iteration semigroup*. If for every $x \in I$ the mapping $t \to f^t(x)$ is continuous then an iteration semigroup is said to be *continuous*.

More information on iteration groups and semigroups can be found, for example, in [3], [4], [8] and [10].

Remark 6 (see [10, Remark 4.1])**.** If $I \subset \mathbb{R}$ is an open interval and there exists at least one element of an iteration semigroup $\{f^t : t \geq 0\}$ without a fixed point and it is not surjective, then this semigroup is continuous.

Remark 7. Every iteration semigroup can be uniquely extended to the relative iteration group (cf. Zdun [9]). Namely, for a given iteration semigroup $\{f^t : t \geq 0\}$ define

$$F^t := \begin{cases} f^t, & t \geq 0, \\ (f^{-t})^{-1}, & t < 0, \end{cases}$$

where $\operatorname{Dom} F^t = I$ and $\operatorname{Dom} F^{-t} = f^t[I]$ for $t > 0$. It is easy to observe that $\{F^t : t \in \mathbb{R}\}$ is a continuous group, i.e. $F^t \circ F^s(x) = F^{t+s}(x)$ for all values of x for which this formula holds. Moreover, if at least one of f^t is a homeomorphism, then $\{F^t : t \in \mathbb{R}\}$ is an iteration group.

In this paper we consider iteration semigroups consisting of (M, N)-convex and (M, N)-concave elements or semigroups consisting of M-convex and M-concave elements. Iteration groups consisting of convex functions were studied earlier, among others, by A. Smajdor [6], [7] and M.C. Zdun [10].

Remark 8. Let $\{f^t : t \geq 0\}$ be a continuous iteration semigroup. If there exists a sequence $(f^{t_n})_{n\in\mathbb{N}}$ of (M, N)-convex functions such that $\lim_{n\to+\infty} t_n = 0$, then $M \leq N$. Similarly, if in a continuous semigroup $\{f^t : t \geq 0\}$ there exists a sequence $(f^{t_n})_{n\in\mathbb{N}}$ of (M, N)-concave elements such that $\lim_{n\to+\infty} t_n = 0$, then $M \geq N$.

Indeed, the continuity of the semigroup implies that f^0, as the limit of a sequence of (M, N)-convex or (M, N)-concave functions, is (M, N)-convex or (M, N)-concave, respectively. Since $f^0 = id$, it follows that $M \leq N$ or $M \geq N$, respectively.

3. Results

We start with an example of an iteration semigroup consisting of (M, N)-concave elements, where $M \neq N$.

Example 3.1. Let $I = (0, \infty)$. For every $t \geq 0$ put $f^t(x) = x^{4^t}$ and let $M(x, y) = x + y$, $N(x, y) = \frac{x+y}{2}$. Since the inequality

$$(x + y)^{4^t} \geq \frac{x^{4^t} + y^{4^t}}{2} \tag{3.1}$$

holds for all $t, x, y > 0$, there are (M, N)-concave elements in the semigroup $\{f^t : t \geq 0\}$. One can use standard calculus methods to prove (3.1).

In [5], Matkowski considered continuous multiplicative iteration groups of homeomorphisms $f^t : (0, \infty) \to (0, \infty)$ such that, for every $t > 0$ the function f^t is M-convex or M-concave, where M is continuous on $(0, \infty) \times (0, \infty)$. The main result of [5] says that if in such a group there are two elements f^r and f^s, $r < 1 < s$, which are both M-convex or both M-concave, then all elements of the group are M-affine. While discussing the possiblility of a generalization of this result it was shown that an analogous theorem with (M, N)-convex or (M, N)-concave functions, where $M \neq N$, is not valid.

Our first result establishes conditions under which the desirable thesis holds.

Theorem 3.1. *Let $M, N : I^2 \to I$ be continuous functions. Suppose that a continuous iteration semigroup $\{f^t : t \geq 0\}$ is such that f^t is (M, N)-convex or (M, N)-concave for every $t > 0$. If there exist $r > s > 0$ such that f^r and f^s are (M, N)-affine, then every element of this semigroup is M-affine and $M = N$ on the set $f^s[I] \times f^s[I]$.*

Proof. Let f^r and f^s be (M, N)-affine. By Remark 2, the function $(f^s)^{-1}$ is (N, M)-affine. It is easy to see that $h := (f^s)^{-1} \circ f^r = f^{r-s}$ is M-affine. Moreover, by the (M, N)-affinity of f^s,

$$N(x, y) = f^s(M((f^s)^{-1}(x), (f^s)^{-1}(y))), \qquad x, y \in f^s[I]. \tag{3.2}$$

The (M, N)-convexity or the (M, N)-concavity of f^u for every $u > 0$, and (3.2) imply that, for every $u > 0$, the function f^u satisfies the inequality

$$f^u(M(x, y)) \leq N(f^u(x), f^u(y)) = f^s(M((f^s)^{-1}(f^u(x)), (f^s)^{-1}(f^u(y))))$$

or the inequality

$$f^u(M(x, y)) \geq N(f^u(x), f^u(y)) = (f^s)(M((f^s)^{-1}(f^u(x)), (f^s)^{-1}(f^u(y)))),$$

for every x and y such that $f^u(x), f^u(y) \in f^s[I]$. Since for $u \geq s$ the inclusion $f^u(x) \in f^s[I]$ holds for every $x \in I$, we hence get, for all $u \geq s$, $x, y \in I$

$$\begin{aligned} f^{u-s}(M(x, y)) &= (f^s)^{-1} \circ f^u(M(x, y)) \\ &\leq M((f^s)^{-1} \circ f^u(x), (f^s)^{-1} \circ f^u(y)) \\ &= M(f^{u-s}(x), f^{u-s}(y)), \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} f^{u-s}(M(x, y)) &= (f^s)^{-1} \circ f^u(M(x, y)) \\ &\geq M((f^s)^{-1} \circ f^u(x), (f^s)^{-1} \circ f^u(y)) \\ &= M(f^{u-s}(x), f^{u-s}(y)), \end{aligned} \tag{3.4}$$

i.e. for every $t := u - s \geq 0$ and all $x, y \in I$,

$$f^t(M(x, y)) \leq M(f^t(x), f^t(y))$$

or

$$f^t(M(x, y)) \geq M(f^t(x), f^t(y)),$$

which means that every element of the semigroup with iterative index $t \geq 0$ is M-convex or M-concave. Define $h^t := \{f^{t(r-s)} : t \geq 0\}$. Since $h^{1/m} = f^{(r-s)/m}$ for $m \in \mathbb{N}$, it is M-convex or M-concave as an element of the semigroup. On the other hand, $h^{1/m}$ is the m-th iterative root of $h = h^1$ which is M-affine. Hence, by Lemma 2.2, the function $h^{1/m}$ is M-affine. It follows that, for all positive integers m, n, the function $h^{n/m}$ is M-affine. Thus the set $\{h^t : t \in \mathbb{Q}^+\}$ consists of M-affine functions. The continuity of the iteration semigroup and the continuity of M imply that, for every $t \geq 0$, the function h^t is M-affine and, consequently, f^t, for all $t \geq 0$, are M-affine. To end the proof take f^s which is both (M, N)-affine and M-affine. Then, for all $x, y \in I$,

$$f^s(M(x, y)) = N(f^s(x), f^s(y))$$

and

$$f^s(M(x,y)) = M(f^s(x), f^s(y)),$$

whence

$$N(f^s(x), f^s(y)) = M(f^s(x), f^s(y)), \qquad x, y \in I.$$

Since f^s is onto $f^s[I]$, $M(x,y) = N(x,y)$ for $x, y \in f^s[I]$. The proof is completed. $\square$

Remark 9. Let us note that if in an iteration group for some $t_0 \in \mathbb{R}$ the function f^{t_0} is M-convex, then the function $(f^{t_0})^{-1}$ is M-concave.

Now we present two results which generalize Matkowski's Theorem 1 ([5]).

Theorem 3.2. *Let $M : I^2 \to I$ be continuous. Suppose that an iteration semigroup $\{f^t : t \geq 0\}$ is continuous. If there exist $r, s > 0$ such that $\frac{r}{s} \notin \mathbb{Q}$, $f^r < id$, $f^s < id$ and f^r is M-convex and f^s is M-concave, then every element of the semigroup is M-affine.*

Proof. Take the relative iteration group $\{F^t : t \in \mathbb{R}\}$ defined as in Remark 7. Assume that f^r is M-convex and f^s is M-concave. Put $g := f^r$ and $h := f^{-s}$. It is obvious that, for each pair (m, n) of positive integers, the functions g^m and h^n are M-convex.

Let $\mathcal{N}(x) := \{(m,n) \in \mathbb{N} \times \mathbb{N} : h^n(x) \in g^m[I]\}$ and $D(x) := \{rm - sn : (m,n) \in \mathcal{N}(x)\}$. Note that if $x < y$, then $\mathcal{N}(x) \subset \mathcal{N}(y)$. Moreover, for every $x \in I$, the set $D(x)$ is dense in $\mathbb{R}$ (see [2]).

Let $x \in I$ be fixed. Take an arbitrary $t \in \mathbb{R}$. By the density of the set $D(x)$, there exists a sequence (m_k, n_k) with terms from $\mathcal{N}(x)$ such that $t = \lim_{k\to+\infty}(m_k r - n_k s)$. Moreover,

$$F^t(x) = \lim_{k\to+\infty} f^{-n_k s} \circ f^{m_k r}(x) = \lim_{k\to+\infty} h^{n_k} \circ g^{m_k}(x).$$

Hence, for every $t \in \mathbb{R}$, the function F^t is M-convex, as it is the limit of a sequence of M-convex functions.

Now let $t > 0$ be fixed. Since F^t and F^{-t} are both M-convex and $F^{-t} \circ F^t = id$, by Lemma 2.1, F^t is M-affine. Consequently, f^t is M-affine for every $t \geq 0$. $\square$

Theorem 3.3. *Let $M : I^2 \to I$ be continuous. Suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup such that f^t is M-convex or M-concave for every $t > 0$. If there exist $r, s > 0$ such that $f^r < id$ is M-convex and $f^s < id$ is M-concave, then f^t is M-affine for every $t > 0$.*

Proof. If $\frac{r}{s} \notin \mathbb{Q}$, then the thesis follows from the previous theorem. Suppose that $\frac{r}{s} \in \mathbb{Q}$. Then there exist $m, n \in \mathbb{N}$ such that $nr = ms$. Thus $(f^r)^n = (f^s)^m$. Put $H := (f^r)^n$. Since $(f^r)^n$ is M-convex and $(f^s)^m$ is M-concave, H is M-affine. By Lemma 2.2, the function f^r is M-affine. Let $n \in \mathbb{N}$ be fixed. As

$$\underbrace{f^{r/n} \circ f^{r/n} \circ \cdots \circ f^{r/n}}_{n \text{ times}} = f^r,$$

by Lemma 2.2, the function $f^{r/n}$ is M-affine. Thus for all $n, m \in \mathbb{N}$, the functions $f^{\frac{m}{n}r} = (f^{r/n})^m$ are M-affine. Let us fix $t > 0$ and take a sequence $(w_n)_{n\in\mathbb{N}}$ of positive rational numbers such that $f^t = \lim_{n\to\infty} f^{w_n r}$. The continuity of M, the continuity of the semigroup and the formula for f^t imply that f^t is M-affine. $\square$

From Theorems 3.2 and 3.3 we obtain the additive version of Matkowski's result [5] which reads as follows.

Corollary 3.4. *Let $M : I^2 \to I$ be continuous and suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup of homeomorphisms $f^t : I \to I$ such that:*

(i) *f^t is M-convex or M-concave for every $t > 0$;*
(ii) *there exist $r, s > 0$ such that f^r is M-convex and f^s is M-concave.*

Then f^t is M-affine for every $t \geq 0$.

Now we prove the following

Theorem 3.5. *Let $M : I^2 \to I$ be a continuous function. If every element of a continuous iteration semigroup $\{f^t : t \geq 0\}$ is M-convex or M-concave and there exists an $s \neq 0$ such that f^s is M-affine, then f^t is M-affine for every $t \geq 0$.*

Proof. Assume that every element of the iteration semigroup is M-convex and $g := f^s$ is M-affine. By Lemma 2.2, for an $m \in \mathbb{N}$ the function $g^{1/m}$ is M-affine. Now the same argument as in the proof of Theorem 3.1 can be repeated. □

Coming back to a group with (M, N)-convex or (M, N)-concave elements, we present:

Theorem 3.6. *Let $M, N : I^2 \to I$ be continuous functions. Suppose that an iteration semigroup $\{f^t : t \geq 0\}$ is continuous and such that, for every $t > 0$, the function f^t is (M, N)-convex or (M, N)-concave.*
Assume moreover that:

(i) *there exists $t_0 > 0$ such that f^{t_0} is (M, N)-affine;*
(ii) *there exist $r, s > t_0$ such that f^r is (M, N)-convex and f^s is (M, N)-concave.*

Then, for every $t \geq 0$, the function f^t is M-affine and $M = N$ on $f^{t_0}[I] \times f^{t_0}[I]$.

Proof. By (i) we obtain equality (3.2) with f^{t_0} instead of f^s. This equality and the (M, N)-convexity of f^r give

$$f^r(M(x, y) \leq N(f^r(x), f^r(y)) = f^{t_0}(M((f^{t_0})^{-1}(f^r(x)), (f^{t_0})^{-1}(f^r(y))))$$

for all $x, y \in I$. The monotonicity of the function $(f^{t_0})^{-1}$ implies that

$$(f^{t_0})^{-1}(f^r(M(x, y))) \leq M((f^{t_0})^{-1}(f^r(x)), (f^{t_0})^{-1}(f^r(y))), \quad x, y \in I,$$

that is, the function f^{r-t_0} is M-convex. Similarly, f^{s-t_0} is M-concave. Moreover, repeating the procedure used in the proof of Theorem 3.1, we have (3.3) or (3.4) with t_0 instead of s for every $u \geq t_0$. Hence for every $t \geq 0$, the function f^t is M-convex or M-concave. Since the semigroup satisfies all the assumptions of Theorem 3.3, we obtain the first part of the thesis. To prove the second part, it is enough to take $f = f^{t_0}$, that is, simultaneously (M, N)-affine and M-affine, and apply the argument used at the end of the proof of Theorem 3.1. □

In the context of the above proof a natural question arises. Is it true that every (M, N)-convex function has to be M-convex? The following example shows that the answer is negative.

Example 3.2. Let $I = (0, \infty)$, $M(x, y) = x + y$, $N(x, y) = \sqrt{xy}$ and put $f^t(x) = \frac{x}{tx+1}$ for every $t > 0$. It is easy to check that $\{f^t : t \geq 0\}$ is a semigroup. The function f^t is (M, N)-concave and M-convex for every $t > 0$.

The proof needs only some standard calculations.

We now present theorems which establish the regularity of the semigroup we deal with. Namely,

Theorem 3.7. *Suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup. If f^t is M-convex or M-concave for every $t > 0$, then in this semigroup either for every $t > 0$ element f^t is M-convex or, contrarily, for every $t > 0$ element f^t is M-concave.*

Proof. Let $A = \{t > 0 : f^t(M(x,y)) \le M(f^t(x), f^t(y)), x, y \in I\}$ and $B = \{t > 0 : f^t(M(x,y)) \ge M(f^t(x), f^t(y)), x, y \in I\}$. The sets A and B are relatively closed subsets of $(0,\infty)$. Moreover, $A \cup B = (0,\infty)$. Let us consider two cases:
(i) $A \cap B = \emptyset$. Then the connectivity of the set $(0,\infty)$ implies that $A = \emptyset$ or $B = \emptyset$;
(ii) $A \cap B \ne \emptyset$. Then there exists $u \in A \cap B$, $u \ne 0$, so f^u is M-affine. Hence all the assumptions of Theorem 3.5 are satisfied and the semigroup consists only of M-affine elements, so the thesis is fulfilled. □

However, for a semigroup with (M,N)-convex or (M,N)-concave elements, we have the following weaker result:

Theorem 3.8. *Suppose that $\{f^t : t \ge 0\}$ is a continuous iteration semigroup. If f^t is (M,N)-convex or (M,N)-concave for every $t > 0$, then there exists $t_0 \ge 0$ such that in this semigroup either for every $t \ge t_0$ the element f^t is (M,N)-convex and for every $0 \le t \le t_0$ the element f^t is (M,N)-concave or, contrarily, for every $t \ge t_0$ the element f^t is (M,N)-concave and for every $0 \le t \le t_0$ the element f^t is (M,N)-convex.*

Proof. Let $A = \{t > 0 : f^t(M(x,y)) \le N(f^t(x), f^t(y)), x, y \in I\}$ and $B = \{t > 0 : f^t(M(x,y)) \ge N(f^t(x), f^t(y)), x, y \in I\}$. The sets A and B are relatively closed subsets of $(0,\infty)$. Moreover, $A \cup B = (0,\infty)$. Now we consider three cases:
(i) $A \cap B = \emptyset$. Then the connectivity of the set $(0,\infty)$ implies that $A = \emptyset$ or $B = \emptyset$;
(ii) $A \cap B \ne \emptyset$ and there exist at least two elements in this set. All the assumptions of Theorem 3.1 are satisfied and the semigroup consists only of (M,N)-affine elements, of course $t_0 = 0$;
(iii) $A \cap B$ is a singleton. Denote $A \cap B = \{u\}$. The function f^u is (M,N)-affine. Hence all the assumptions of Theorem 3.6 are satisfied and the semigroup contains only (M,N)-affine elements. The thesis is thus fulfilled. Of course, f^{t_0} is (M,N)-affine. □

Applying Theorem 3.8, we obtain the following

Corollary 3.9. *Let us assume that a continuous iteration semigroup $\{f^t : t \ge 0\}$ consists only of (M,N)-convex or (M,N)-concave functions and there are $r, s > 0$ such that f^r and f^s are both (M,N)-affine. Then either $M \le N$ or $N \le M$.*
If $M \le N$ and for at least one point $(x_0, y_0) \in I^2$ the strict inequality

$$M(x_0, y_0) < N(x_0, y_0) \tag{3.5}$$

holds, then for every $t > 0$, the functions f^t are (M,N)-convex.

Proof. Assume, on the contrary, that there exists $t_0 > 0$ such that f^{t_0} is (M,N)-concave. By Theorem 3.8, for every $t > 0$, the function f^t is (M,N)-concave. Hence $f^0 = id$ is (M,N)-concave since it is the limit of an (M,N)-concave function. Thus

$$M(x,y) \ge N(x,y) \quad x, y \in I,$$

which contradicts the assumed inequality (3.5). □

In all theorems, according to Remark 6, if at least one function in a semigroup is without a fixed point and not surjective, then the assumption of the continuity of the semigroup can be omitted.

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